Reconstructing the potential of the enhanced power spectrum and duality

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Outline

Introduction

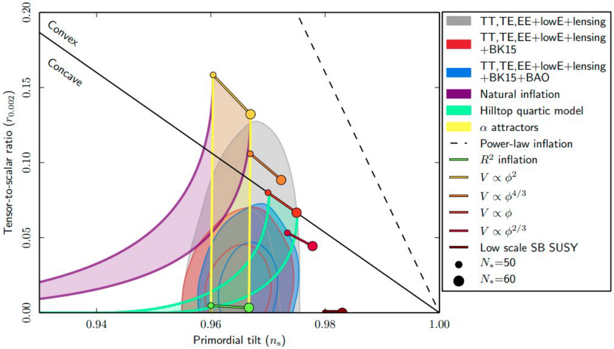


Figure: Planck 2018 results (1807.06211)

The observables $(n_s,\ r,\ n_T,\ f_{nl},\cdots)$ are not enough to discriminate among the suggested inflationary models.

Introduction

Reconstruction of the inflaton potential

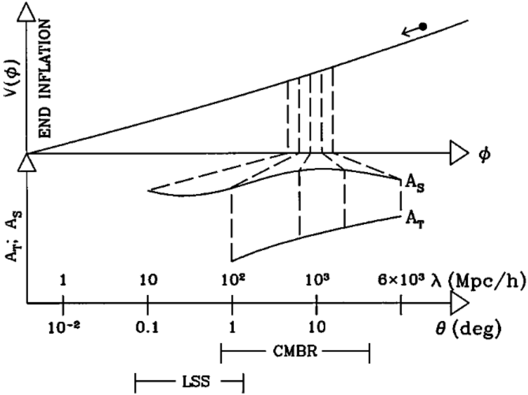


Figure: Lidsely et al, Rev. Mod. Phys (1997)

The spectra A_S of the density perturbations and A_T of the gravitational waves are measured over a range of scales that corresponds to some interval of the underlying potential $V(\phi)$.

Introduction

Reconstruction of the inflaton potential

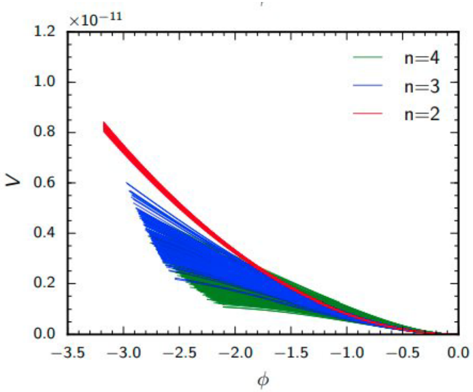


Figure: Planck 2018 results (1807.06211)

The potential is Taylor-expanded at order $n = 2,\ 3$ and 4 and trusted until at the end of inflation, and under the assumption $N_* = 55$ e-folds of inflation.

Outline

Reconstrction of the potential

[Lidsely et al, Rev. Mod Phys (1997)]
Taylor series of the potential about the ϕ_0

$$V(\phi) = V\ \ (\phi_0) + V'(\phi_0)\,\Delta\phi + \frac{1}{2}V''(\phi_0)\,\Delta\phi^2 + \cdots \tag{1}$$

Using the slow-roll conditions

$$V(\phi) = \frac{3m_{pl}^2}{8\pi}H^2(\phi), \tag{2}$$

we can write

$$V(\phi) = \frac{3m_{\rm Pl}^2 H_0^2}{8\pi}\left[1 - (16\pi\epsilon_0)^{1/2}\frac{\Delta\phi}{m_{\rm Pl}} + 4\pi(\epsilon_0 + \eta_0)\frac{(\Delta\phi)^2}{m_{\rm Pl}^2} + \mathcal{O}\left(\frac{(\Delta\phi)^3}{m_{\rm Pl}^3}\right)\right], \tag{3}$$

Reconstrction of the potential

[Lidsely et al, Rev. Mod Phys (1997)]
The slow-roll parameters are defined by

$$\epsilon(\phi) = \frac{m_{\rm Pl}^2}{4\pi}\left(\frac{H'(\phi)}{H(\phi)}\right)^2, \quad \eta(\phi) = \frac{m_{\rm Pl}^2}{4\pi}\frac{H''(\phi)}{H(\phi)}, \tag{4}$$

$$\xi(\phi) = \frac{m_{\rm Pl}^2}{4\pi}\left(\frac{H'(\phi)H'''(\phi)}{H^2(\phi)}\right)^{1/2} \tag{5}$$

and the observable quantities are expressed in terms of the slow-roll parameters

$$n_s - 1 = 2\eta - 4\epsilon, \quad r = 16\epsilon \tag{6}$$

generalized power spectrum

(Stewart, 2002)

Sasaki-Mukhanov equation for the canonical single field models

$$v'' + \left(k^2 - \frac{z''}{z}\right) v = 0 \tag{7}$$

with the asymptotic conditions

$$v_k \longrightarrow \begin{cases} \frac{1}{\sqrt{2k}} e^{-ik\eta} & \text{as} \quad -k\eta \to \infty \\ A_k z & \text{as} \quad -k\eta \to 0 \end{cases} \tag{8}$$

where

$$z = \frac{a\dot{\phi}}{H}, \quad v = z\mathcal{R} \tag{9}$$

The power spectrum is given by

$$\mathcal{P}_{\mathcal{R}_c} = \left(\frac{k^3}{2\pi^2}\right) \lim_{-k\eta \to 0} \left|\frac{\varphi_k}{z}\right|^2 = \frac{k^3}{2\pi^2} |A_k|^2 \tag{10}$$

generalized power spectrum

(Stewart, 2002)
Defining

$$y = \sqrt{2k} v_k, \quad x = -k\eta, \quad z = \frac{1}{x} f(\ln x) \tag{11}$$

the Sasaki-Mukhanov equation becomes

$$\frac{d^2 y}{dx^2} + \left(1 - \frac{2}{x^2}\right) y = \frac{1}{x^2} g(\ln x) y \tag{12}$$

where

$$g = \frac{f'' - 3f'}{f}, \quad ' = \frac{d}{d\eta} \tag{13}$$

If we define the slow-roll parameters

$$\epsilon_H \equiv -\frac{\dot{H}}{H^2}, \quad \delta_H \equiv -\frac{\ddot{\phi}}{H\dot{\phi}}, \quad \eta_H \equiv \frac{\dot{\epsilon}_H}{H\epsilon_H}, \quad \zeta_H \equiv \frac{\dot{\eta}_H}{H\eta_H} \tag{14}$$

we have

$$g = -2 + \xi^2 a^2 H^2 \left(2 - \epsilon_H + \frac{3}{2}\eta_H - \frac{1}{2}\epsilon_H \eta_H + \frac{1}{4}\eta_H^2 + \frac{1}{2}\zeta_H \eta_H\right)$$

generalized power spectrum

(Stewart, 2002)

The homogeneous part of the equation (12) is the de Sitter and the source terms imply the deviation from the de Sitter limit. Using the Green function method, the solution of (12) with the boundary condition (8) is

$$y(x) = y_0(x) + \frac{i}{2}\int_x^\infty du \frac{1}{u^2} g(\ln u) y(u) \left[y_0^*(u) y_0(x) - y_0^*(x) y_0(u)\right] \tag{16}$$

where the homogeneous solution is

$$y_0(x) = \left(1 + \frac{i}{x}\right) e^{ix} \tag{17}$$

The power spectrum is

$$\mathcal{P}_{\mathcal{R}_\mathcal{C}} = \left(\frac{k}{2\pi}\right)^2 \lim_{x \to 0} \left|\frac{xy}{f}\right|^2$$

generalized power spectrum

(Stewart, 2002)
If we assume that $g \ll 1$, the power spectrum up to leading order of $\mathcal{O}(g)$ is

$$\ln \mathcal{P}_{\mathcal{R}}(k) = \int_0^\infty \frac{d\xi}{\xi} \left[-k\xi W'(k\xi)\right] \left[\ln\left(\frac{1}{f^2}\right) + \frac{2}{3}\frac{f'}{f} + \mathcal{O}\left(g^2\right)\right] \tag{18}$$

where $W(X)$ is the Window function

$$W(x) = \frac{3\sin(2x)}{2x^3} - \frac{3\cos(2x)}{x^2} - \frac{3\sin(2x)}{2x} - 1 \tag{19}$$

This equation is an integral transformation between $\mathcal{P}_{\mathcal{R}}$ and $f(\ln\xi)$ via the window function $W(x)$.

Inverse formula

(Joy, Stewart, Gong, Lee, 2005)
Given the power spectrum $\mathcal{P}_{\mathcal{R}}(k)$, one can find a formal inverse formula, which is valid to the leading order in general slow roll (GSR) formalism

$$\ln\left(\frac{1}{f^2}\right) = \int_0^\infty \frac{dk}{k} m(k\xi) \ln \mathcal{P}_{\mathcal{R}}(k)$$

where

$$m(x) = \frac{2}{\pi}\left[\frac{1}{x} - \frac{\cos(2x)}{x} - \sin(2x)\right]$$

The following identity is used to derive the inverse formula:

$$\int_0^\infty \frac{dk}{k} m(k\zeta) W(k\xi) = \left(\frac{\zeta^3}{\xi^3} - 1\right)\theta(\xi - \zeta). \tag{20}$$

Inverse formula: example

(Choi, Gong, Gang, Raveendran, 2021)
Example 1: Power-law spectrum

$$\mathcal{P}_{\mathcal{R}}(k) = A_s \left(\frac{k}{k_*}\right)^{n_s - 1} \tag{21}$$

Using the inverse formula, we obtain

$$\log\left(\frac{1}{f^2}\right) = \log A_s + (n_s - 1)\left[\alpha - \log\left(k_* \xi\right)\right] \tag{22}$$

From the equation $\dot{H} = -\frac{\dot{\phi}^2}{2m_{\mathrm{Pl}}^2}$ and $f(\log\xi) = \frac{2\pi a \xi \dot{\phi}}{H}$, we obtain

$$V(\phi) = \frac{3m_{\mathrm{Pl}}^2 H_i^2 \beta}{\beta - (k_* \xi_i)^{n_s - 1}} \frac{1 - \frac{1}{6}(1 - n_s)\tanh^2\left[\sqrt{1 - n_s}\frac{\phi - \phi_0}{2m_{\mathrm{Pl}}}\right]}{1 + \sinh^2\left[\sqrt{1 - n_s}\frac{\phi - \phi_0}{2m_{\mathrm{Pl}}}\right]} \tag{23}$$

Inverse formula: example

(Choi, Gong, Gang, Raveendran, 2021)
Exampel 2: Featured spectrum

$$\log \mathcal{P}_{\mathcal{R}}(k) = \log \mathcal{P}^0_{\mathcal{R}}(k) + \mathcal{I}_0(k) + \log\left[1 + \mathcal{I}_1^2(k)\right] \qquad (24)$$

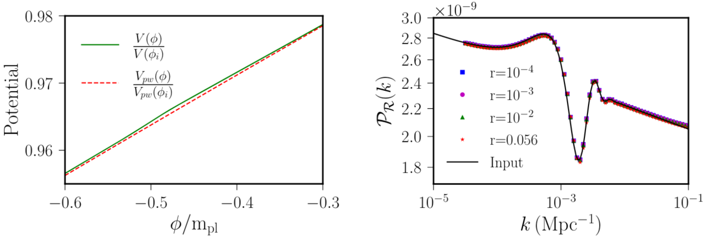


Figure: From 2109.14241

Inverse formula: example

(Choi, Gong, Gang, Raveendran, 2021)
Example 3: Power spectrum with a peak

$$\mathcal{P}_{\mathcal{R}}(k) = \mathcal{P}^0_{\mathcal{R}}(k)\left\{1 + A_p \exp\left[-\left(\frac{\log_{10}(k/k_c)}{\Delta}\right)^2\right]\right\} \tag{25}$$

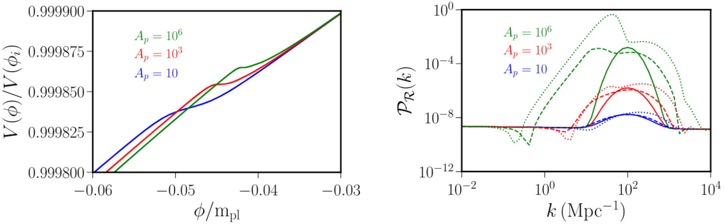


Figure: From 2109.14241: $|g| = \mathcal{O}(10)$

new movement:

How can we improve the reconstructing the potential with the power spectrum with a peak?

Formation of the primordial black holes

(Che, Koh, Tumurtushaa, 2021)

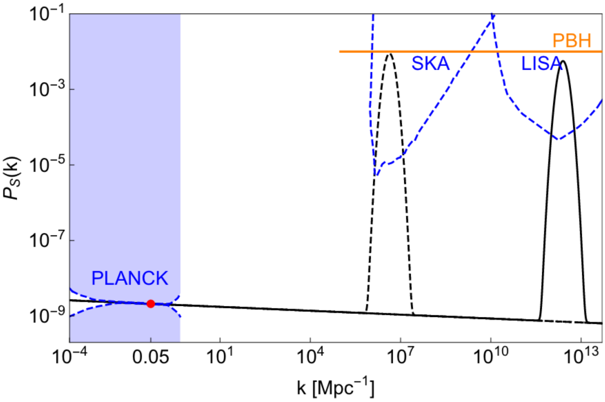
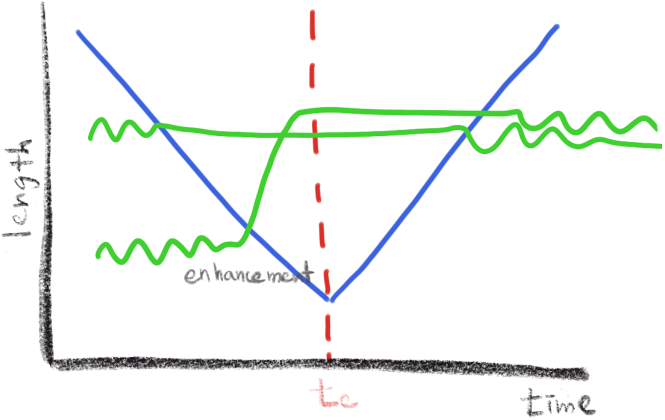


Figure: The dashed blue lines are the CMB constraints on the $P_S(k)$ and GW limits from SKA and LISA, where the shaded regions are excluded. The orange line at $P_S(k_{PBH})102$ indicates the required amplitude of the power spectra to form PBHs

Outline

enhancement of the curvature perturbation



Duality in cosmological perturbations

Under the transformation of z

$$\tilde{z} = Cz(\tau) + Dz(\tau) \int^{\tau} \frac{d\tau'}{z^2(\tau')},$$

or

$$\frac{z''}{z} = \frac{\tilde{z}''}{\tilde{z}},$$

the Master equation is invariant

$$\tilde{u}_k + \left(k^2 - \frac{\tilde{z}''}{\tilde{z}}\right) \tilde{u}_k = 0.$$

The power spectrum could be enhanced by the factor of $(z/\tilde{z})^2$ if

$$\tilde{\mathcal{P}}_{\tilde{\mathcal{R}}} = \left(\frac{z}{\tilde{z}}\right)^2 \mathcal{P}_{\mathcal{R}} \qquad (26)$$

SL(2,R) symmetry

Given function f, if $S[f]$ is defined by

$$S[f] = \frac{f'''}{f''} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2,$$

it is invariant $S[f] = S[\tilde{f}]$ under the transformation

$$\tilde{f} = \frac{af+b}{cf+d}, \quad ad - bc \neq 0.$$

If we define

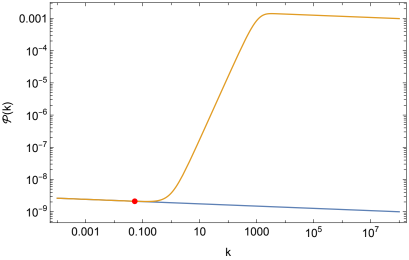
$$z = \frac{1}{\sqrt{f'}},$$

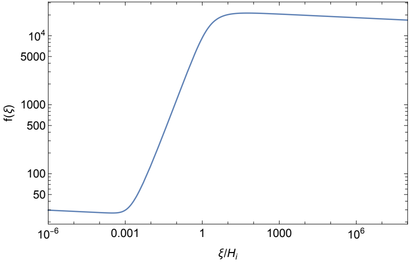
we have

$$\frac{z''}{z} = -\frac{1}{2}S[f].$$

example: Reconstruct potential for the PBH formation

$tanh$-type power spectrum





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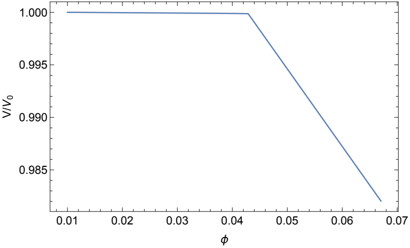
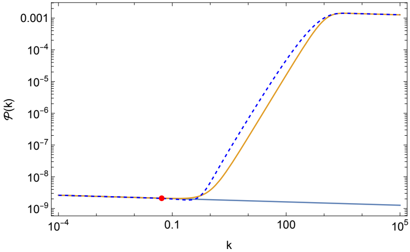
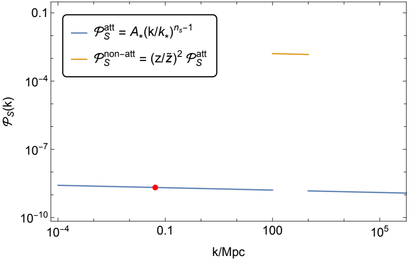
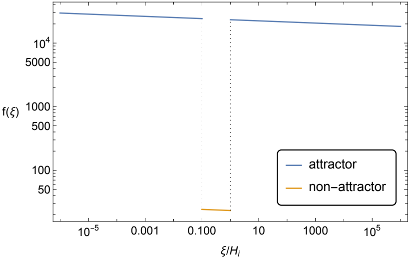




example: Reconstruct potential for the PBH formation

step power spectrum





example: Reconstruct potential for the PBH formation

step power spectrum