How to add new degree of freedom in modified gravity : case of field transformation

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$c = \hbar = M_G^2 = 1/(8\pi G) = 1$

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Introduction

Necessity of the extension of standard model of particle physics and gravity

$$\begin{aligned} S &= S_{\mathrm{g}} + S_{\mathrm{m}} \\ &= \int d^4x \sqrt{-g}\, \frac{1}{2} M_G^2 R + \int d^4x \sqrt{-g}\, \mathcal{L}_{\mathrm{SM}} \end{aligned}$$

$(M_G^2 = 1/(8\pi G))$

Einstein-Hilbert action (GR)

Action for SM of particle physics

These actions might not be able to account for DE(inflation) and/or DM (in addition to the non-zero neutrino masses, baryon asymmetry, etc.)

Extension of GR and/or SM of particle physics

Add new d.o.f. responsible for DE(inflation) and/or DM.

Standard way of the extension of GR and/or (SM of particle physics)

$$\begin{aligned} S &= S_{\mathrm{g}} + S_{\mathrm{m}} \\ &= \int d^4x\sqrt{-g}\,\frac{1}{2}M_G^2 R + \int d^4x\sqrt{-g}\,\mathcal{L}_{\mathrm{SM}} \end{aligned}$$

$$(M_G^2 = 1/(8\pi G))$$

Einstein-Hilbert action (GR) **Action for SM of particle physics**

For example →

$$\begin{aligned} \tilde{S} &= \tilde{S}_{\mathrm{g}} + \tilde{S}_{\mathrm{m}} \\ &= \int d^4x\sqrt{-g}\,\mathcal{L}_{\mathrm{Horndeski}} + \int d^4x\sqrt{-g}\,\mathcal{L}_{\mathrm{MSSM}} \end{aligned}$$

$$\left\{ \begin{aligned} \mathcal{L}_{\mathrm{Horndeski}} &= \mathcal{L}_2 + \mathcal{L}_3 + \mathcal{L}_4 + \mathcal{L}_5, \\ \mathcal{L}_2 &= K(\phi, X), \\ \mathcal{L}_3 &= -G_3(\phi, X)\Box\phi, \\ \mathcal{L}_4 &= G_4(\phi, X)R + G_{4X}\left[(\Box\phi)^2 - (\nabla_\mu\nabla_\nu\phi)^2\right], \\ \mathcal{L}_5 &= G_5(\phi, X)G_{\mu\nu}\nabla^\mu\nabla^\nu\phi \\ &\quad -\frac{1}{6}G_{5X}\left[(\Box\phi)^3 - 3\,(\Box\phi)\,(\nabla_\mu\nabla_\nu\phi)^2 + 2\,(\nabla_\mu\nabla_\nu\phi)^3\right]. \end{aligned} \right.$$

Another way of the extension

$$\begin{aligned} S &= S_{\mathrm{g}} + S_{\mathrm{m}} \\ &= \int d^4x \sqrt{-g}\, \frac{1}{2} M_G^2 R + \int d^4x \sqrt{-g}\, \mathcal{L}_{\mathrm{SM}} \end{aligned}$$

$(M_G^2 = 1/(8\pi G))$

Einstein-Hilbert action (GR)

Action for SM of particle physics

⇒ **Extension through field transformation**

$$g_{\mu\nu} = g_{\mu\nu}\left(h_{\sigma\tau}, \phi\right)$$

N.B. No new d.o.f. appears as long as a transformation is regular and invertible.
However, if a transformation is singular, this might not be the case.

Singular transformation

Transformation : $y_i = f_i(x_j)$

Jacobian matrix :

$$\mathcal{J}_{ij} = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots \\ \cdots & \cdots & \cdots \end{pmatrix}$$

Singular ⟺ $\det \mathcal{J}_{ij} = 0$ ⟺ **An eigenvalue vanishes**

(Regular ⟺ $\det \mathcal{J}_{ij} \neq 0$ ⟺ **No eigenvalue vanishes)**

Mimetic gravity (dark matter)

(Chamseddine & Mukhanov 2013)

Seed system : $g_{\mu\nu}$ and matter field (Ψ_M) with a seed action,

$$S_{\text{seed}}[g, \Psi_M]$$

Singular transformation (with conformal (Weyl) invariance)

$$g_{\mu\nu} = g_{\mu\nu}\left(h_{\sigma\tau}, \phi\right) \qquad \left(g_{\mu\nu} \to g_{\mu\nu} \text{ for } h_{\mu\nu} \to \omega^2 h_{\mu\nu}\right)$$

Transformed system : $h_{\mu\nu}$ and matter field (Ψ_M) with

new d.o.f (φ) (constrained by conformal inv.)

$$S_{\text{dis}}[h, \phi, \Psi_M] = S_{\text{seed}}[g(h, \phi), \Psi_M]$$

$$\Rightarrow \quad \frac{\delta S_{\text{dis}}[h, \phi, \Psi_M]}{\delta h_{\mu\nu}} = 0, \qquad \frac{\delta S_{\text{dis}}[h, \phi, \Psi_M]}{\delta \phi} = 0,$$

give gravitational eq. including new d.o.f and its eq.

Concrete (original) example of mimetic gravity

(Chamseddine & Mukhanov 2013)

Seed system : $g_{\mu\nu}$ and matter field (Ψ_M) with a seed action,

e.g. $$S_{\mathrm{seed}}[g, \Psi_M] = \int d^4x \sqrt{-g} R + S_{\mathrm{matter}}$$

Singular transformation with conformal invariance

$$g_{\mu\nu} = \left(h^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi\right) h_{\mu\nu} \equiv Y h_{\mu\nu}$$

$$\left(g_{\mu\nu} \to g_{\mu\nu} \text{ for } h_{\mu\nu} \to \omega^2 h_{\mu\nu}\right)$$

(Dis)transformed system : $h_{\mu\nu}$ and matter field (Ψ_M) with new d.o.f ϕ (constrained by conformal inv.)

$$S_{\mathrm{dis}}[h, \phi, \Psi_M] = S_{\mathrm{seed}}[g(h, \phi), \Psi_M]$$

$$\begin{cases} \delta S_{\mathrm{seed}}[g, \Psi_M] = \int d^4x \sqrt{-g} \left(G^{\mu\nu}(g) - T^{\mu\nu}\right) \delta g_{\mu\nu} \\ \delta g_{\mu\nu} = Y \delta h_{\sigma\rho} \left(\delta^\sigma_\mu \delta^\rho_\nu - g_{\mu\nu} g^{\sigma\alpha} g^{\rho\beta} \partial_\alpha\phi\partial_\beta\phi\right) + 2 g_{\mu\nu} g^{\sigma\rho} \partial_\sigma \delta\phi \partial_\rho\phi \end{cases}$$

$$\Rightarrow \quad \frac{\delta S_{\mathrm{dis}}[h, \phi, \Psi_M]}{\delta h_{\mu\nu}} = 0, \qquad \frac{\delta S_{\mathrm{dis}}[h, \phi, \Psi_M]}{\delta \phi} = 0$$

Concrete (original) example of mimetic gravity II

(Chamseddine & Mukhanov 2013)

$$S_{\mathrm{dis}}[h,\phi,\Psi_M] = S_{\mathrm{seed}}[g(h,\phi),\Psi_M] = \int d^4x\sqrt{-g}R + S_{\mathrm{matter}}.$$

$$g_{\mu\nu} = \left(h^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi\right)h_{\mu\nu} \equiv Y h_{\mu\nu} \qquad \left(g_{\mu\nu}\to g_{\mu\nu} \text{ for } h_{\mu\nu}\to\omega^2 h_{\mu\nu}\right)$$

$$\left(\Longrightarrow g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi = \frac{h^{\mu\nu}}{Y}\partial_\mu\phi\partial_\nu\phi = 1\right) \qquad \underbrace{(G(g)-T)}_{\neq 0}\left(1-g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\right)=0.$$

Trace part is trivially satisfied.

$$\left\{\begin{array}{l} \delta S_{\mathrm{seed}}[g,\Psi_M] = \int d^4x\sqrt{-g}\left(G^{\mu\nu}(g)-T^{\mu\nu}\right)\delta g_{\mu\nu} \\ \delta g_{\mu\nu} = Y\delta h_{\sigma\rho}\left(\delta^\sigma_\mu\delta^\rho_\nu - g_{\mu\nu}g^{\sigma\alpha}g^{\rho\beta}\partial_\alpha\phi\partial_\beta\phi\right) + 2g_{\mu\nu}g^{\sigma\rho}\partial_\sigma\delta\phi\partial_\rho\phi \end{array}\right.$$

➡

$$\left\{\begin{array}{lll} \dfrac{\delta S_{\mathrm{dis}}[h,\phi,\Psi_M]}{\delta h_{\mu\nu}} = 0 & \Longrightarrow & G^{\mu\nu}(g) - T^{\mu\nu} - (G(g)-T)g^{\mu\sigma}g^{\nu\rho}\partial_\sigma\phi\partial_\rho\phi = 0 \\ & \Longleftrightarrow & G^{\mu\nu}(g) = T^{\mu\nu} + \tilde{T}^{\mu\nu} \quad \text{with} \quad \overset{(g)}{\nabla}_\mu \tilde{T}^\mu_\nu = 0 \\ \dfrac{\delta S_{\mathrm{dis}}[h,\phi,\Psi_M]}{\delta\phi} = 0 & \Longrightarrow & \overset{(g)}{\nabla}_\alpha\left((G(g)-T)\partial^\alpha\phi\right) = 0 \end{array}\right.$$

Fix ε

$$\tilde{T}^{\mu\nu} \equiv (\epsilon+p)u^\mu u^\nu - pg^{\mu\nu}, \quad \epsilon = G(g)-T,\ p=0,\ u^\mu = g^{\mu\alpha}\partial_\alpha\phi \quad \textbf{(dust)}$$

$$\left(u^\mu u_\mu = g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi = 1\right)$$

Important extension and applications

Depending on seed action and/or transformation, one can obtain different features of mimetic matters.

- **Change seed actions:**
  - **Chamseddine, Mukhanov, Vikman 2014**
  - **Mirzagholi, Vikman 2015**
  - **Arroja, Bartolo, Karmakar, Matarrese 2015**
  - **Takahashi, Kobayashi 2017**
  - **Gorji, Mansoori, Firouzjahi 2018**
  - **many many important others…**

- **Change transformations:**
  - **Gorji, Mukohyama, Firouzjahi, Mansoori 2018**
  - **Gorji, Mukohyama, Firouzjahi 2019**
  - **Jirousek, Vikman 2019**
  - **Hammer, Jirousek, Vikman 2020**
  - **many many important others..**

In this talk, we will take another direction.

The following questions arise !!

- What’s the essence of mimetic gravity ?
- What determines the form of a mimetic matter ?
- How important is conformal invariance ?
- Is there any relation between conformal invariance and the form of mimetic matter ?
- Do we need to impose conformal invariance on transformation “a priori” ?

These are the questions we would like to address in this talk.

What’s the essence of mimetic gravity ?

(Pavel Jiroušek, Keigo Shimada, Alexander Vikman, MY, arXiv: 2207.12611)

Singular
(disformal) transformation

Disformal transformation

(Bekenstein 1992)

$$g_{\mu\nu} = C(Y,\phi)h_{\mu\nu} + D(Y,\phi)\partial_\mu\phi\partial_\nu\phi, \qquad Y = h^{\sigma\tau}\partial_\sigma\phi\partial_\tau\phi$$

$$\left(g^{\mu\nu} = \frac{1}{C}\left(h^{\mu\nu} - \frac{D}{C+DY}\partial^\mu\phi\partial^\nu\phi\right)\right)$$

To check the invertibility, we consider Jacobian, its eigenvalues and eigenvectors.

(Zumalacárregui & García-Bellido 2014)

$$\mathcal{J}^{\mu\nu}_{\sigma\rho} = \frac{\delta g_{\sigma\rho}}{\delta h_{\mu\nu}} = C\delta^\mu_\sigma\delta^\nu_\rho - C_Y h_{\sigma\rho}\partial^\mu\phi\partial^\nu\phi - D_Y\partial_\sigma\phi\partial_\rho\phi\partial^\mu\phi\partial^\nu\phi$$

$$\mathcal{J}^{\mu\nu}_{\sigma\rho}\xi^a_{\mu\nu} = \lambda_a\xi^a_{\sigma\rho}, \quad \zeta_a^{\sigma\rho}\mathcal{J}^{\mu\nu}_{\sigma\rho} = \lambda_a\zeta_a^{\mu\nu}.$$

$$\left(C_Y \equiv \frac{\partial C}{\partial Y}, \; D_Y \equiv \frac{\partial D}{\partial Y}\right)$$

- **(9) eigenvalues, eigenvectors, dual-eigenvectors :**

$$\lambda_C = C, \quad \xi^C_{\mu\nu} = \phi^\perp_{\mu\nu}, \quad \zeta_C^{\mu\nu} = \phi_\top^{\mu\nu}$$

$$\left(\phi^\perp_{\mu\nu}\partial^\mu\phi\partial^\nu\phi = 0, \quad \phi_\top^{\mu\nu}\xi^D_{\mu\nu} = 0\right)$$

- **(1) eigenvalue, eigenvector, dual-eigenvector :**

$$\lambda_D = C - C_Y Y - D_Y Y^2, \quad \xi^D_{\mu\nu} = C_Y h_{\mu\nu} + D_Y\partial_\mu\phi\partial_\nu\phi, \quad \zeta_D^{\mu\nu} = \partial^\mu\phi\partial^\nu\phi$$

$\lambda_C = 0$ and/or $\lambda_D = 0$ ⇔ Singular transformation

Consequences of singular transformation

$$S_{\text{dis}}[h,\phi,\Psi_M] = S_{\text{seed}}[g(h,\phi),\Psi_M]$$

⇒ $$\delta S_{\text{dis}} = \int d^4x\, \frac{\delta S_{\text{seed}}}{\delta g_{\sigma\rho}}\, \mathcal{J}^{\mu\nu}_{\sigma\rho}\, \delta h_{\mu\nu}$$

(i) $\mathcal{J}^{\mu\nu}_{\sigma\rho}\left(=\frac{\delta g_{\sigma\rho}}{\delta h_{\mu\nu}}\right)$ **: regular** ⇒ $\frac{\delta S_{\text{dis}}}{\delta h_{\mu\nu}} = 0 \Longleftrightarrow \frac{\delta S_{\text{seed}}}{\delta g_{\sigma\rho}} = 0.$

(ii) $\mathcal{J}^{\mu\nu}_{\sigma\rho}\left(=\frac{\delta g_{\sigma\rho}}{\delta h_{\mu\nu}}\right)$ **: singular (λa = 0)**

⇒ $$\delta S_{\text{dis}} = \int d^4x \left(\frac{\delta S_{\text{seed}}}{\delta g_{\sigma\rho}} - \rho\zeta_a^{\sigma\rho}\right) \mathcal{J}^{\mu\nu}_{\sigma\rho}\, \delta h_{\mu\nu}$$

$\left(\zeta_a^{\sigma\rho}\mathcal{J}^{\mu\nu}_{\sigma\rho} = \lambda_a\zeta_a^{\mu\nu},\ \lambda_a = 0\right)$

$\left(\frac{\delta S_{\text{dis}}}{\delta h_{\mu\nu}} = 0\right)$ ⇒ $\frac{\delta S_{\text{seed}}}{\delta g_{\sigma\rho}} = \rho\zeta_a^{\sigma\rho}$

In original case (C=Y≠0, D=0)

⇒ $$G_{\mu\nu} - T_{\mu\nu} = \underset{(G-T)}{\underset{||}{\tilde{\rho}}}\,\partial_\mu\phi\partial_\nu\phi = \tilde{T}_{\mu\nu}$$

$\left(\tilde{\rho} = \frac{(C+DY)^2}{\sqrt{-g}}\rho\right)$

$\left(\lambda_D = C - C_Y Y - D_Y Y^2 = 0\right)$

The first important message :

The property of mimetic matter
is determined by the eigenvector with
zero eigenvalue.

Why conformal symmetry ?

Eigenvector as generator of symmetry transformation

Consider the infinitesimal change of $h_{\mu\nu}$ such that

$$\delta_\epsilon h_{\mu\nu} = \epsilon\, \xi_{\mu\nu}$$

small parameter **eigenvector of Jacobian**

$$\Rightarrow \delta_\epsilon g_{\mu\nu} = \epsilon\, \mathcal{J}_{\mu\nu}^{\sigma\rho} \xi_{\sigma\rho} = \epsilon\, \lambda \xi_{\mu\nu}$$

$$\Rightarrow \lambda = 0 \iff \delta_\epsilon g_{\mu\nu} = 0$$

This infinitesmal change of h becomes symmetry transformation of g.

In original case $\left(C = Y \neq 0, D = 0 \implies \lambda_D = C - C_Y Y - D_Y Y^2 = 0\right)$

$$\delta_\epsilon h_{\mu\nu} = \epsilon\, \xi_{\mu\nu}^D = \epsilon h_{\mu\nu}$$ **: conformal trans.** $\iff \delta_\epsilon g_{\mu\nu} = 0$

$\left(\xi_{\mu\nu}^D = C_Y h_{\mu\nu} + D_Y \partial_\mu \phi \partial_\nu \phi\right)$

**Conformal symmetry appears
as the result of singular transformation.**

**Do we need to impose conformal invariance
on transformation “a priori” ?**

Singular behavior of disformal transformation

$$g_{\mu\nu} = C(Y,\phi)h_{\mu\nu} + D(Y,\phi)\partial_\mu\phi\partial_\nu\phi, \qquad Y = h^{\sigma\tau}\partial_\sigma\phi\partial_\tau\phi$$

Jacobian matrix, its eigenvalues and eigenvectors.

$$\mathcal{J}^{\mu\nu}_{\sigma\rho} = \frac{\delta g_{\sigma\rho}}{\delta h_{\mu\nu}} = C\delta^\mu_\sigma\delta^\nu_\rho - C_Y h_{\sigma\rho}\partial^\mu\phi\partial^\nu\phi - D_Y\partial_\sigma\phi\partial_\rho\phi\partial^\mu\phi\partial^\nu\phi\,, \qquad \mathcal{J}^{\mu\nu}_{\sigma\rho}\xi^a_{\mu\nu} = \lambda_a\xi^a_{\sigma\rho},$$

(1) eigenvalue, eigenvector:

$$\lambda_D = C - C_Y Y - D_Y Y^2 = -Y^2\,\partial_Y\left(\frac{C}{Y} + D\right), \quad \xi^D_{\mu\nu} = C_Y h_{\mu\nu} + D_Y\partial_\mu\phi\partial_\nu\phi,$$

- **$\lambda_D = 0$ as a function (for all configurations of φ)** (Deruelle & Rua 2014)

$$\Leftrightarrow \quad D(Y,\phi) = -\frac{C(Y,\phi)}{Y} + c(\phi)$$

← arbitrary function

- **$\lambda_D = 0$ for some configuration for φ** ← **this talk**

$$\Leftrightarrow \quad C = C_Y Y + D_Y Y^2$$

interpreted as a non-trivial equation of motion which may be used to determine the behavior of φ.

In fact, we do NOT need to impose
conformal invariance
on transformation "a priori".

Examples without conformal invariance on transformation "a priori".

Example 1 $\left(D(Y,\phi) \neq \frac{C(Y,\phi)}{Y} + c(\phi)\right)$

Seed action : $S_{\text{seed}}[g,\Psi_M] = \int d^4x\sqrt{-g}R$ (no matter for simplicity)

Transformation : $g_{\mu\nu} = e^{Y-1}h_{\mu\nu}$ $\quad (Y = h^{\sigma\tau}\partial_\sigma\phi\partial_\tau\phi)$

No conformal invariance: $g_{\mu\nu} \nRightarrow g_{\mu\nu}$ for $h_{\mu\nu} \Rightarrow \omega^2 h_{\mu\nu}$

EOM for h

⇒ $$\frac{\delta S_{\text{seed}}}{\delta h_{\mu\nu}} = \frac{\delta S_{\text{seed}}}{\delta g_{\sigma\rho}}\mathcal{J}^{\mu\nu}_{\sigma\rho} = e^{Y-1}\left(G^{\mu\nu}(g) - G^{\rho\sigma}(g)h_{\rho\sigma}\partial^\mu\phi\partial^\nu\phi\right) = 0$$

Trace w.r.t. g (contracting with $\xi^D_{\mu\nu} = g_{\mu\nu}$)

⇒ $$G(g)\,\underbrace{e^{Y-1}(1-Y)}_{(=\lambda_D)} = 0.$$

(G(g) = 0 : (standard) GR branch)

λD = 0 (Y=1) branch

⇒ $$G_{\mu\nu} - \underbrace{G\partial_\mu\phi\partial_\nu\phi}_{\text{Mimetic DM}} = 0.$$

$$\frac{\delta S_{\text{dis}}}{\delta\phi} = 0 \quad \Leftrightarrow \quad \overset{(g)}{\nabla}_\mu\left(G\partial^\mu\phi\right) = 0.$$

Fix G

N.B.

- ● G (=ρ) is **not fixed** because the trace part is compensated by Y=1 ⇔ X=1.
- ● On Y=1 ⇔ X=1,
  $g_{\mu\nu} = h_{\mu\nu}$: **regular** trans.

Example 2

$\left(D(Y,\phi) \neq \frac{C(Y,\phi)}{Y} + c(\phi)\right)$

Seed action : $S_{\mathsf{seed}}[g, \Psi_M] = \int d^4x\sqrt{-g}R$

(no matter for simplicity)

Transformation : $g_{\mu\nu} = h_{\mu\nu} + Y\partial_\mu\phi\partial_\nu\phi \qquad (Y = h^{\sigma\tau}\partial_\sigma\phi\partial_\tau\phi)$

No conformal invariance: $g_{\mu\nu} \not\Rightarrow g_{\mu\nu}$ for $h_{\mu\nu} \Rightarrow \omega^2 h_{\mu\nu}$

EOM for h

$$\Rightarrow \frac{\delta S_{\mathsf{seed}}}{\delta h_{\mu\nu}} = \frac{\delta S_{\mathsf{seed}}}{\delta g_{\sigma\rho}}\mathcal{J}^{\mu\nu}_{\sigma\rho} = G^{\mu\nu}(g) - G^{\sigma\rho}(g)\partial_\sigma\phi\partial_\rho\phi\partial^\mu\phi\partial^\nu\phi = 0$$

Contracting with $\xi^D_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi$

$$\Rightarrow G^{\mu\nu}(g)\partial_\mu\phi\partial_\nu\phi\underbrace{\left(1 - Y^2\right)}_{(=\lambda_D)} = 0.$$

($G_{\phi\phi} \equiv G^{\mu\nu}(g)\partial_\mu\phi\partial_\nu\phi = 0$**: GR branch)**

λD = 0 (Y=±1) branch

$$\Rightarrow G_{\mu\nu} - 4G_{\phi\phi}\partial_\mu\phi\partial_\nu\phi = G_{\mu\nu} \pm \underbrace{2G\partial_\mu\phi\partial_\nu\phi}_{\text{Mimetic DM}} = 0.$$

$$\left(G = \pm 2G_{\phi\phi}\right)$$

$$\frac{\delta S_{\mathsf{dis}}}{\delta\phi} = 0 \Leftrightarrow \partial_\mu\left(G\partial^\mu\phi\right) = 0.$$

N.B. Multiple solutions of λD=0.
Time-like one and space-like one.

Singular but invertible transformation

(Pavel Jiroušek, Keigo Shimada, Alexander Vikman, MY, arXiv: 2208.05951)

N.B. A regular and invertible transformation gives the physically same dynamics (and the same number of d.o.f.) because it is nothing but relabelling.

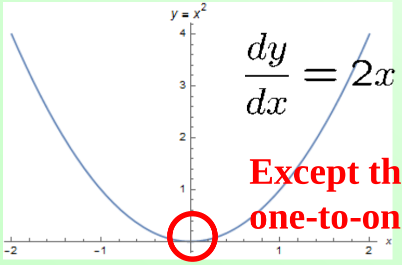
Inverse function theorem

Transformation : $y_i = f_i(x_j)$

Jacobian matrix :

$$\mathcal{J}_{ij} = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots \\ \cdots & \cdots & \cdots \end{pmatrix}$$

Regular ⟺ $\det \mathcal{J}_{ij} \neq 0$ ⟺ **No eigenvalue vanishes**



$$\frac{dy}{dx} = 2x$$

Except the origin, one-to-one.

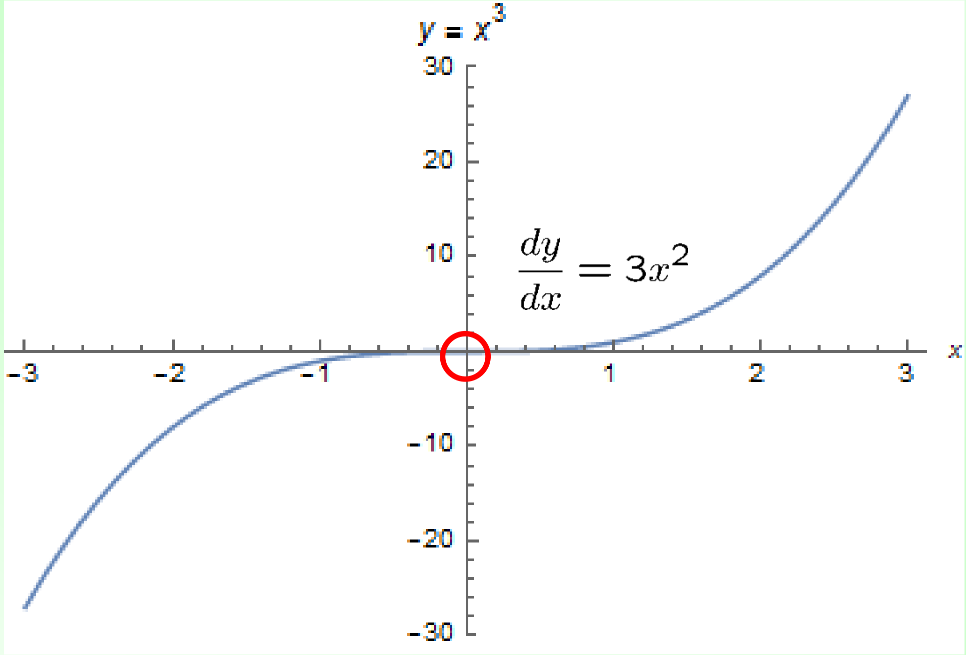
Inverse function theorem :

If the transformation is regular at some point, there is one-to-one correspondence (invertible) locally around that point.

But, the opposite is not necessarily true, that is, even if the transformation is singular at some point, it can be (locally) invertible.

Singular ⟺ $\det \mathcal{J}_{ij} = 0$ ⟺ **An eigenvalue vanishes**

Example of singular but invertible transformation



The transformation is singular at the origin.

But, the transformation is invertible !!

Regular $\Longleftrightarrow$ $\det \mathcal{J}_{ij} \neq 0$ $\Longrightarrow$ (locally) invertible

(Inverse function theorem)

Singular $\Longleftrightarrow$ $\det \mathcal{J}_{ij} = 0$ $\Longrightarrow$ invertible or non-invertible

Appearance of new d.o.f from singular but invertible transformation

$$S\,[q] = \frac{1}{2}\int_{t_1}^{t_2} dt\,\dot{q}^2 \quad \Longrightarrow \quad \ddot{q} = 0$$

$$\Longrightarrow \quad q(t) = q_0 + v_0\,t$$

(2 initial conditions: q_0 (initial position), v_0 (initial velocity) ➔ 1 d.o.f.)

(Singular but) invertible transformation of the variable Q ⬄ q :

$$q(Q,\dot{\phi}) = Q^3 + \dot{\phi} \quad \Longleftrightarrow \quad Q(q,\dot{\phi}) = \sqrt[3]{q-\dot{\phi}}$$

(one-to-one correspondence)

$$J = \frac{\partial q}{\partial Q} = 3Q^2$$

singular at Q = 0

Appearance of new d.o.f from singular but invertible transformation II

$$q(Q,\dot\phi) = Q^3 + \dot\phi \qquad \left(J = \frac{\partial q}{\partial Q} = 3Q^2 \right)$$

$$S\left[Q,\phi\right] = \frac{1}{2}\int_{t_1}^{t_2} dt\, \left(\ddot\phi + 3Q^2\dot Q\right)^2 \Longleftarrow S\left[q\right] = \frac{1}{2}\int_{t_1}^{t_2} dt\, \dot q^2$$

$$\Longrightarrow \quad \frac{\delta S}{\delta Q} = -\ddot q\, J = 0 \;, \qquad \frac{\delta S}{\delta \phi} = \frac{d}{dt}\ddot q = 0 \;.$$

- **Regular** branch with **J ≠ 0** $\Longrightarrow \ddot q = 0 \Longrightarrow q(t) = q_0 + v_0\, t$

(2 constants, 1 d.o.f.)

- **Singular** branch with **J = 0 at Q = 0**,

$$\Longrightarrow \quad \frac{\delta S}{\delta \phi} = \frac{d^4}{dt^4}\phi = 0$$

$$\Longrightarrow \quad \phi(t) = c_0 + c_1 t + c_2\frac{t^2}{2} + c_3\frac{t^3}{6}$$

$$\Longrightarrow \quad q(Q,\dot\phi)|_{Q=0} = \dot\phi = c_1 + c_2 t + c_3\frac{t^2}{2}$$

4 constants, 2 d.o.f.

New d.o.f. appeared !!

Appearance of new "dynamics" from singular but invertible transformation

$$S[q,\phi] = \frac{1}{2}\int_{t_1}^{t_2} dt\,\left(\dot{q}^2 + \dot{\phi}^2\right) \Longrightarrow \ddot{q} = 0,\;\; \ddot{\phi} = 0.$$

$$\Longrightarrow q(t) = q_0 + v_0\,t,\;\; \phi(t) = \phi_0 + u_0\,t.$$

(**4 initial conditions**: **q_0, v_0, φ_0, u_0** → **2 d.o.f.**)

(Singular but) invertible transformation of the variables :

$$\begin{cases} q = Q^3 + \dot{\phi}\,, \\ \phi = \phi\,. \end{cases}$$

Appearance of new “dynamics” from singular but invertible transformation II

$$q = Q^3 + \dot{\phi} \ , \quad \phi = \phi \ .$$

$$S\left[Q,\phi\right] = \frac{1}{2}\int_{t_1}^{t_2} dt \left[\left(\ddot{\phi} + 3Q^2\dot{Q}\right)^2 + \dot{\phi}^2\right] \Leftarrow S[q,\phi] = \frac{1}{2}\int_{t_1}^{t_2} dt \left(\dot{q}^2 + \dot{\phi}^2\right)$$

$$\Rightarrow \frac{\delta S}{\delta Q} = -3\ddot{q}\,Q^2 = 0 \ , \quad \frac{\delta S}{\delta \phi} = \frac{d}{dt}\left(\ddot{q} - \dot{\phi}\right) = 0 \ .$$

- ● **Regular branch with Q ≠ 0** ⇒ $\ddot{q} = 0, \ \ddot{\phi} = 0.$ **(4 constants, 2 d.o.f.)**
- ● **Singular branch with Q = 0,**

$$\Rightarrow \frac{\delta S}{\delta \phi} = \frac{d}{dt}\left(\ddddot{\phi} - \phi\right) = 0$$

$$\Rightarrow \phi(t) = c_0 + c_1 t + c_2 \sinh(t) + c_3 \cosh(t)$$

$$\Rightarrow q(Q,\dot{\phi})|_{Q=0} = \dot{\phi} = c_1 + c_2 \sinh(t) + c_3 \cosh(t)$$

4 constants, 2 d.o.f.
The number of d.o.f.
remains unchanged.
But, new dynamics appeared!!

Summary

- **GR and SM of particle physics are very successful theories, but needs to be modified in order to explain DE (inflation), DM, and so on.**
- **Another way to add d.o.f. is to use singular transformation, whose typical example is mimetic gravity.**
- **What's the essence of mimetic gravity ?**

  Singular transformation
- **What determines the form of a mimetic matter ?**

  Eigenvector of Jacobian matrix with zero eigenvalue
- **How important is conformal invariance ?**

  Important but might be not crucial
- **Is there any relation between conformal invariance and the form of mimetic matter ?**

  No
- **Do we need to impose conformal invariance on transformation "a priori" ?**

  No
- **Singular but invertible can change d.o.f as well as change dynamics.**