Cosmological perturbations during inflation

– Quantum theory and power spectrum –

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What we had before was…

- We introduced the comoving curvature perturbation, i.e. the perturbation in the spatial curvature evaluated on the comoving hypersurfaces where delta{phi} = 0:

$$\mathcal{R} = \varphi - \frac{H}{\dot{\phi}_0}\delta\phi$$

- We need the equation of motion for R. How can we derive it? This time, we derive it from the perturbation action

Action for Einstein gravity + minimal scalar

- Our starting point is the action for the Einstein gravity (i.e. Einstein-Hilbert term) and a minimally coupled scalar field:

$$S = \int d^4x \sqrt{-g} \left[\frac{m_{\rm Pl}^2}{2} R - \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - V(\phi) \right]$$

- Since we are working for R eventually, we keep only the scalar metric perturbations A, B, varphi and E and the scalar field fluctuations delta{phi}. Further, we keep up to linear order in perturbations, the action from which the equation is derived should be 2$^{\rm nd}$ order. Thus, we expand the action up to quadratic order in perturbations

Quadratic action for perturbations

- The detailed steps are given e.g. Mukhanov, Feldman & Brandenberger (1992), Section 10. In terms of the conformal time d{tau} = adt

$$
\begin{aligned}
S_2^{(s)} = \int d^4x \frac{m_{\rm Pl}^2}{2} a^2 \Big[& -6\varphi'^2 + 12\mathcal{H}A\varphi' - 2\left(\mathcal{H}' + 2\mathcal{H}^2\right) A^2 - 2(2A+\varphi)\delta\phi \\
& + \frac{1}{m_{\rm Pl}^2}\left(\delta\phi'^2 + \delta\phi\Delta\delta\phi - a^2 V_{\phi\phi}\delta\phi^2\right) \\
& + \frac{2}{m_{\rm Pl}^2}\left(-3\phi_0'\varphi'\delta\phi - \phi_0' A\delta\phi' - a^2 V_\phi A\delta\phi\right) \\
& +4\left(\frac{1}{2m_{\rm Pl}^2}\phi_0'\delta\phi + \varphi' - \mathcal{H}A\right)\Delta(B - E')\Big]
\end{aligned}
$$

Structure of perturbation action

- We can see that varphi, delta{phi} and E have time derivatives, while A and B do not. This means A and B are not dynamical and their equations of motion are just **constraints** that should hold all the time
- Thus, we expect varphi, delta{phi} and E can be written in the canonical form with Hamiltonian, while A and B multiplied by their constraints:

$$\mathcal{L}_2^{(s)} = \Pi_\varphi \varphi' + \Pi_{\delta\phi}\delta\phi' + \Pi_E E' - \mathfrak{H}^{(s)} - \mathcal{C}_A A - \mathcal{C}_B B$$

- This is the reason why A and B are solved algebraically: By construction they are not dynamical

Conjugate momenta

- We can easily find the conjugate momenta for varphi, delta{phi} and E

$$\Pi_\varphi \equiv \frac{\delta}{\delta\phi'}\mathcal{L}_2^{(s)} = \frac{m_{\rm Pl}^2}{2}a^2\left[-12\varphi' + 12\mathcal{H}A - \frac{6}{m_{\rm Pl}^2}\phi'\delta\phi + 4\Delta(B-E')\right]$$

$$\Pi_{\delta\phi} \equiv \frac{\delta}{\delta(\delta\phi')}\mathcal{L}_2^{(s)} = a^2\left(\delta\phi' - \phi_0' A\right)$$

$$\Pi_E \equiv \frac{\delta}{\delta E'}\mathcal{L}_2^{(s)} = \frac{m_{\rm Pl}^2}{2}a^2\Delta\left(-4\varphi' + 4\mathcal{H}A - \frac{2}{m_{\rm Pl}^2}\phi_0'\delta\phi\right)$$

- Using these, we can replace B – E’ and delta{phi}’ in terms of momenta, and we end up with the following form:

First-order form with constraints

$$
\begin{aligned}
\mathcal{L}_2^{(s)} &= \Pi_\varphi \varphi' + \Pi_{\delta\phi}\delta\phi' + \Pi_E E' \\
&\quad - \left\{ \frac{1}{2a^2 m_{\rm Pl}^2}\left[-\Pi_\varphi \Delta^{-2}\Pi_E + \frac{3}{2}\left(\Delta^{-1}\Pi_E\right)^2 + m_{\rm Pl}^2 \Pi_{\delta\phi}^2 \right] - \frac{1}{2m_{\rm Pl}^2}\phi_0' \Pi_\varphi \delta\phi \right. \\
&\qquad \left. + a^2 m_{\rm Pl}^2 \left[\varphi\Delta\varphi - \frac{3}{4m_{\rm Pl}^2}{\phi_0'}^2 \delta\phi^2 - \frac{1}{2m_{\rm Pl}^2}\left(\delta\phi\Delta\delta\phi - a^2 V_{\phi\phi}\delta\phi^2\right)\right]\right\} \\
&\quad - \left[\mathcal{H}\Pi_\varphi + \phi'\Pi_{\delta\phi} + 2a^2 m_{\rm Pl}^2 \Delta\varphi + a^2\left(3\mathcal{H}\phi_0' + a^2 V_\phi\right)\delta\phi\right] A - \Pi_E B
\end{aligned}
$$

- From CB = 0, E disappears and from CA = 0, we can replace replace the conjugate momentum of delta{phi} in terms of the others (or vice versa)

First-order form without constraints (1)

- After plugging back the solutions of the constraints and rearrangement, we find the following:

$$
\begin{aligned}
\mathcal{L}_2^{(s)} = &\left(\Pi_\varphi + \frac{2a^2 m_{\rm Pl}^2}{\phi_0'}\Delta\delta\phi\right)\left(\varphi - \mathcal{H}\frac{\delta\phi}{\phi_0'}\right)' - \frac{2m_{\rm Pl}^2\mathcal{H}}{{\phi_0'}^2}\left(\Pi_\varphi + \frac{2a^2 m_{\rm Pl}^2}{\phi_0'}\Delta\delta\phi\right)\Delta\left(\varphi - \mathcal{H}\frac{\delta\phi}{\phi_0'}\right) \\
&- \frac{\mathcal{H}^2}{2a^2{\phi_0'}^2}\left(\Pi_\varphi + \frac{2a^2 m_{\rm Pl}^2}{\phi_0'}\Delta\delta\phi\right)^2 - \frac{2a^2 m_{\rm Pl}^2}{{\phi_0'}^2}\left[\Delta\left(\varphi - \mathcal{H}\frac{\delta\phi}{\phi_0'}\right)\right]^2 \\
&- a^2 m_{\rm Pl}^2\left(\varphi - \mathcal{H}\frac{\delta\phi}{\phi_0'}\right)\Delta\left(\varphi - \mathcal{H}\frac{\delta\phi}{\phi_0'}\right)
\end{aligned}
$$

First-order form without constraints (2)

- Finally, we redefine the following variables:

$$\mathcal{R} \equiv \varphi - \frac{\mathcal{H}}{\phi_0'}\delta\phi$$

$$\Pi_{\mathcal{R}} \equiv \Pi_\varphi + \frac{2a^2 m_{\mathrm{Pl}}^2}{\phi_0'}\Delta\delta\phi$$

- Then we end up with the following compact form of the Lagrangian:

$$\mathcal{L}_2^{(s)} = \Pi_{\mathcal{R}}\mathcal{R}' - \underbrace{\left[\frac{2a^2 m_{\mathrm{Pl}}^2}{{\phi_0'}^2}\left(\Delta\mathcal{R} + \frac{\mathcal{H}}{2a^2 m_{\mathrm{Pl}}^2}\Pi_{\mathcal{R}}\right)^2 + a^2 m_{\mathrm{Pl}}^2 \mathcal{R}\Delta\mathcal{R}\right]}_{\equiv \mathfrak{H}_{\mathcal{R}}}$$

Quadratic action for comoving curvature pert

- Thus, the Lagrangian is of the canonical form without constraints, so we are left with a single physical dof: comoving curvature perturbation
- We can rewrite in a more familiar form by eliminating the conjugate momentum in favour of R':

$$\frac{\delta \mathfrak{H}_{\mathcal{R}}}{\delta \Pi_{\mathcal{R}}} = \mathcal{R}' \;\to\; \frac{\mathcal{H}}{2a^2 m_{\mathrm{Pl}}^2}\Pi_{\mathcal{R}} = \frac{{\phi_0'}^2}{2m_{\mathrm{Pl}}^2\mathcal{H}}\mathcal{R}' - \Delta\mathcal{R}$$

$$\therefore\; S_2^{(s)} = \int d\tau d^3x \frac{1}{2}\left(\frac{a\phi_0'}{\mathcal{H}}\right)^2\left[\mathcal{R}'^2 - (\nabla\mathcal{R})^2\right] = \int dt d^3x a^3 \epsilon m_{\mathrm{Pl}}^2\left[\dot{\mathcal{R}}^2 - \frac{(\nabla\mathcal{R})^2}{a^2}\right]$$

Equation from quadratic action

- Therefore the equation for R follows immediately from the action:

$$\frac{1}{a^3\epsilon}\frac{d}{dt}\left(a^3\epsilon\dot{\mathcal{R}}\right)-\frac{\Delta}{a^2}\mathcal{R}=0$$

- Thus, in all 3 approaches, we can end up with the same equation of motion for the comoving curvature perturbation R
- Especially, the action approach provides us a passage to quantum theory of cosmological perturbation R

Harmonic oscillator action

- To proceed further, we introduce the following:

$$z \equiv \frac{a\phi_0'}{\mathcal{H}} \quad \text{and} \quad u \equiv z\mathcal{R} = a\left(\delta\phi - \frac{\phi_0'}{\mathcal{H}}\varphi\right)$$

- This transforms the quadratic action for R into the following form:

$$S_2 = \int d^4x \frac{1}{2}\left[u'^2 - (\nabla u)^2 + \frac{z''}{z}u^2\right]$$

- This action is exactly the same form as that of harmonic oscillator with time-varying frequency, so we can apply the standard wisdom for harmonic oscillator: Promotion to operators satisfying canonical commutation relations!

Decomposition using standard operators (1)

1. Promote the canonical conjugate pair to operators that satisfy the standard equal-time commutation relation:

$$\Pi_u = \frac{\delta \mathcal{L}}{\delta u'} = u' \;\rightarrow\; \left[\widehat{u}(\tau, \boldsymbol{x}), \widehat{\Pi}_u(\tau, \boldsymbol{y})\right] = i\delta^{(3)}(\boldsymbol{x} - \boldsymbol{y})$$

2. Decompose hat{u} in terms of the creation and annihilation operators that satisfy the standard commutation relations:

$$\widehat{u}(\tau, \boldsymbol{x}) = \int \frac{d^3k}{(2\pi)^3} e^{i\boldsymbol{k}\cdot\boldsymbol{x}} \widehat{u}(\tau, \boldsymbol{k}) = \int \frac{d^3k}{(2\pi)^3} e^{i\boldsymbol{k}\cdot\boldsymbol{x}} \left[a_{\boldsymbol{k}} u_k(\tau) + a^\dagger_{-\boldsymbol{k}} u^*_k(\tau)\right]$$

$$\left[a_{\boldsymbol{k}}, a^\dagger_{\boldsymbol{q}}\right] = (2\pi)^3 \delta^{(3)}(\boldsymbol{k} - \boldsymbol{q}) \,, \text{ otherwise zero}$$

Decomposition using standard operators (2)

3. Plugging the Fourier decomposition into the equal-time commutation relation gives the normalization:

$$\left[\widehat{u}(\tau,\boldsymbol{x}),\widehat{\Pi}_u(\tau,\boldsymbol{y})\right]=\int\frac{d^3k}{(2\pi)^3}\frac{d^3q}{(2\pi)^3}e^{i\boldsymbol{k}\cdot\boldsymbol{x}}e^{i\boldsymbol{q}\cdot\boldsymbol{y}}\left\{\left[a_{\boldsymbol{k}|},a^\dagger_{-\boldsymbol{q}}\right]u_k\frac{du_q^*}{d\tau}-\left[a_{\boldsymbol{q}},a^\dagger_{-\boldsymbol{k}}\right]\frac{du_q}{d\tau}u_k^*\right.$$

$$\left.+\left[a_{\boldsymbol{k}},a_{\boldsymbol{q}}\right]u_k\frac{du_q}{d\tau}+\left[a^\dagger_{-\boldsymbol{k}},a^\dagger_{-\boldsymbol{q}}\right]u_k^*\frac{du_q^*}{d\tau}\right\}$$

$$=\int\frac{d^3k}{(2\pi)^3}e^{i\boldsymbol{k}\cdot(\boldsymbol{x}-\boldsymbol{y})}\left(u_k\frac{du_k^*}{d\tau}-\frac{du_k}{d\tau}u_k^*\right)$$

$$\therefore\ u_k\frac{du_k^*}{d\tau}-\frac{du_k}{d\tau}u_k^*=i$$

Vacuum state

- We worked out the standard commutation relations for operators, but still we need to determine the mode function u(tau,k). This amounts to fix the vacuum state |0> defined by

$$a_{\boldsymbol{k}}|0\rangle_{\mathrm{vac}} = 0 \quad \text{for all } k$$

- Because the background is time-evolving, it is not clear when we define the vacuum (we will be back to this point later). In fact we only have 1 situation where time dependence is not relevant: k >> aH, where frequency becomes time-independent. Then it is straight to compute the mode function that minimizes the Hamiltonian

Mode function in sub-horizon limit (1)

$$\mathcal{L} = \frac{1}{2}\left[u'^2 - (\nabla u)^2\right]$$

$$\widehat{\mathfrak{H}} = \int d^3x \frac{1}{2}\left[\widehat{\Pi}_u^2 + (\nabla \widehat{u})^2\right]$$

$$= \int \frac{d^3k}{(2\pi)^3}\left\{a_{\boldsymbol{k}}a_{-\boldsymbol{k}}\left(\widehat{u}_k'^{\,2} + k^2\widehat{u}_k^2\right) + c.c. + \left[2a_{\boldsymbol{k}}^\dagger a_{\boldsymbol{k}} + (2\pi)^3\delta^{(3)}(0)\right]\left(|\widehat{u}_k'|^2 + k^2|\widehat{u}_k|^2\right)\right\}$$

- Evaluating the expectation value of the Hamiltonian w.r.t. vacuum |0>_0 at some very early times tau0:

$${}_0\langle 0|\widehat{\mathfrak{H}}|0\rangle_0 = \frac{1}{2}\int d^3k\,\delta^{(3)}(0)\left(|\widehat{u}_k'|^2 + k^2|\widehat{u}_k|^2\right)$$

Mode function in sub-horizon limit (2)

- Since deep inside the horizon the solution is a plane wave, we assume

$$u_k = \psi_k e^{i\theta_k} \text{ with } -2i\psi_k^2\theta_k' = i \quad \text{(from normalization)}$$

- Then the expectation value of the Hamiltonian becomes

$$|\widehat{u}_k'|^2 + k^2|\widehat{u}_k|^2 = \psi_k'^{\,2} + k^2\psi_k^2 + \frac{1}{4\psi_k^2}$$

- This is minimized when psik and thetak are give by

$$\psi_k = \frac{1}{\sqrt{2k}} \text{ and } \theta_k = -k\tau \quad \rightarrow \quad u_k = \frac{1}{\sqrt{2k}} e^{-ik\tau}$$

Massless scalar field solution!

Mode function at some other time

- This mode function (or vacuum) does not remain valid, since the frequency does not remain simply k all the time. The Hamiltonian is time-dependent, thus the mode function /vacuum that minimizes it also varies
- Let tau1 > tau0, and the Fourier expansion i.t.o. new operators and mode function:

$$\widehat{u}(\tau, \boldsymbol{k}) = b_{\boldsymbol{k}} v_k(\tau) + b^{\dagger}_{-\boldsymbol{k}} v_k^*(\tau) \quad \text{with} \quad b_{\boldsymbol{k}}|0\rangle_1 = 0$$

Bogoliubov transformation

- In general, the new mode function vk is related to another uk via a linear transformation, "**Bogoliubov transformation**":

$$v_k = \alpha_k u_k + \beta_k u_k^* \quad \text{with} \quad |\alpha_k|^2 - |\beta_k|^2 = 1$$

- Thus, the vacuum state does not remain the same:

$$|0\rangle_0 \neq |0\rangle_1$$

Generation of perturbations

- Let's consider the expectation value of the number operator of the b-particle w.r.t. tau0:

$$ {}_0\langle 0|N_k^{(b)}|0\rangle_0 = {}_0\langle 0|\left(\alpha_k a_{\boldsymbol{k}}^\dagger - \beta_k a_{-\boldsymbol{k}}\right)\left(\alpha_k^* a_{\boldsymbol{k}} - \beta_k^* a_{-\boldsymbol{k}}^\dagger\right)|0\rangle_0 = (2\pi)^3|\beta_k|^2\delta^{(3)}(0) $$

- That is, even if we have started from vacuum at tau0, at a later time tau1 we find that the vacuum at tau0 contains a non-vanishing number of b-particles

We have something out of nothing

General mode function equation

- Now let us return to the quadratic action for u: The equation of motion for the mode function is

$$u_k'' + \left(k^2 - \frac{z''}{z}\right) u_k = 0$$

- Here, the time-dependent part of the frequency z''/z is **exactly** given:

$$\frac{z''}{z} = 2a^2H^2\left(1 - \frac{\epsilon}{2} + \frac{3}{4}\eta - \frac{\epsilon\eta}{4} + \frac{\eta^2}{8} + \frac{\dot{\eta}}{4H}\right)$$

← eta = dot{H}/(Hepsilon), another slow-roll parameter

$$\equiv \frac{1}{\tau^2}\left(\nu^2 - \frac{1}{4}\right) \quad \text{with} \quad \nu^2 = \frac{9}{4} + 3\epsilon + \frac{3}{2}\eta + \cdots$$

Asymptotic behavior of mode function

- We can think of 2 limiting cases for the frequency: Either k^2 is dominant over z''/z ~ (aH)^2 or the other way round

$$u_k'' + \left(k^2 - \frac{z''}{z}\right) u_k \longrightarrow \begin{cases} u_k'' + k^2 u_k = 0 & \text{for } k \gg aH \\ u_k'' - \dfrac{z''}{z} u_k = 0 & \text{for } k \ll aH \end{cases}$$

- k versus aH: Is the typical length scale for a mode 1/k much smaller or larger than the comoving Hubble horizon 1/(aH)? Super- (k << aH) or sub- (k >> aH) horizon limits
- For sub-horizon mode, the solution is a plane wave (we have seen!)
- For super-horizon mode, u is proportional to z. What does this mean?

Constancy of comoving curvature pert

- On super-horizon scales, the proportionality of u to z means:

$$\mathcal{R}_k = \frac{u_k}{z} = \text{constant}$$

- Thus, R (in the Fourier mode) is **conserved**
- This is another reason why we consider R: it is conserved on very large scales during inflation, until it reenters the horizon after inflation (other perturbations, e.g. delta{phi}, keep evolving on super-horizon)

General solution of mode function

- What is the mode function solution, with the given normalization condition and boundary condition?
- The general solution can be written i.t.o. Bessel functions, but for later convenience we use Hankel functions:

$$u_k(\tau) = \sqrt{-\tau}\left[c_1(k)H_\nu^{(1)}(-k\tau) + c_2(k)H_\nu^{(2)}(-k\tau)\right]$$

(Here nu is assumed to be constant)

- To fix the coefficients c1 and c2, we consider the boundary condition at tau = tau0 where we find the plane wave solution for k >> z''/z

Fixing coefficients and correct solution

- For very large argument (-ktau ~ k/aH >> 1)

$$H_\nu^{(1)}(z) \underset{z\gg 1}{\longrightarrow} \sqrt{\frac{2}{\pi z}} e^{i(z-\pi\nu/2-\pi/4)}$$

$$\therefore \ c_1(k) = \frac{\sqrt{\pi}}{2} e^{i(\nu+1/2)\pi/2} \ \text{ and } \ c_2(k) = 0$$

- Especially, for nu = 3/2, i.e. perfect de Sitter case, the solution is the 1st kind of Hankel function with order 3/2:

$$u_k(\tau) = \frac{1}{\sqrt{2k}} \left(1 - \frac{i}{k\tau}\right) e^{-ik\tau}$$

Power spectrum of comoving curvature pert

- The power spectrum, or the Fourier transform of the 2-pt correlation function, is defined by (expectation value w.r.t. vacuum state)

$$\langle \mathcal{R}(\boldsymbol{k})\mathcal{R}(\boldsymbol{q})\rangle \equiv (2\pi)^3\delta^{(3)}(\boldsymbol{k}+\boldsymbol{q})\frac{2\pi^2}{k^3}\mathcal{P}_{\mathcal{R}}(k)$$

- Using u = zR, the power spectrum is written as the mode function:

$$\mathcal{P}_{\mathcal{R}}(k) = \frac{k^3}{2\pi^2}\left|\frac{u_k}{z}\right|^2$$

Evaluating the power spectrum

- Since R(k) is time-dependent, so is the power spectrum. Then when should we evaluate the power spectrum?
- R becomes constant when k >> aH, or -k tau → 0 and maintains its value until the end of inflation. Thus, we evaluate the spectrum then

$$H_\nu^{(1)}(z) \xrightarrow[z \ll 1]{} \sqrt{\frac{2}{\pi}} e^{-i\pi/2} 2^{\nu-3/2} \frac{\Gamma(\nu)}{\Gamma(3/2)} z^{-\nu}$$

alpha = 2 – log2 – EulerGamma = 0.729637…

$$\therefore \ \mathcal{P}_\mathcal{R}(k) = \lim_{-k\tau \to 0} 2^{2\nu-3} \left[\frac{\Gamma(\nu)}{\Gamma(3/2)}\right]^2 (1+\epsilon)^{1-2\nu} \left(\frac{H}{2\pi}\right)^2 \left(\frac{H}{\dot{\phi}_0}\right)^2 \left(\frac{k}{aH}\right)^{3-2\nu}$$

$$= \lim_{-k\tau \to 0} [1 + 2(\alpha - 1)\epsilon + \alpha\eta] \left(\frac{H}{2\pi}\right)^2 \left(\frac{H}{\dot{\phi}_0}\right)^2 \left(\frac{k}{aH}\right)^{3-2\nu}$$

1. Amplitude of power spectrum

1. The spectrum can be evaluated at any time after “horizon crossing” k = aH, but usually it is evaluated at the moment of horizon crossing. Then, up to 0^{th} order in slow-roll parameters (i.e. leading order)

$$\mathcal{P}_{\mathcal{R}}(k) = \left(\frac{H}{2\pi}\right)^2 \left(\frac{H}{\dot{\phi}_0}\right)^2 \Bigg|_{k=aH}$$

This is the celebrated result for the inflationary power spectrum!

2. Tilt of power spectrum

- Another important property of the power spectrum is how it scales on different scales, e.g. if it has larger amplitudes on larger or smaller length scales. As a simple ansatz we take a power-law form:

$$\mathcal{P}_{\mathcal{R}} \propto k^{n_{\mathcal{R}}-1}$$

"Spectral index"

$$\therefore \; n_{\mathcal{R}} - 1 \equiv \frac{d\log \mathcal{P}_{\mathcal{R}}}{d\log k} = 3 - 2\nu = -2\epsilon - \eta\big|_{k=aH}$$

Observational constraints

- Most recent observational efforts by Planck says, on the reference scale k = 0.05/Mpc, the amplitude of the power spectrum AR and the spectral index are constrained by

$$A_{\mathcal{R}} = 2.0968^{+0.0296}_{-0.0292} \times 10^{-9}$$
$$n_{\mathcal{R}} = 0.9652 \pm 0.0042$$