

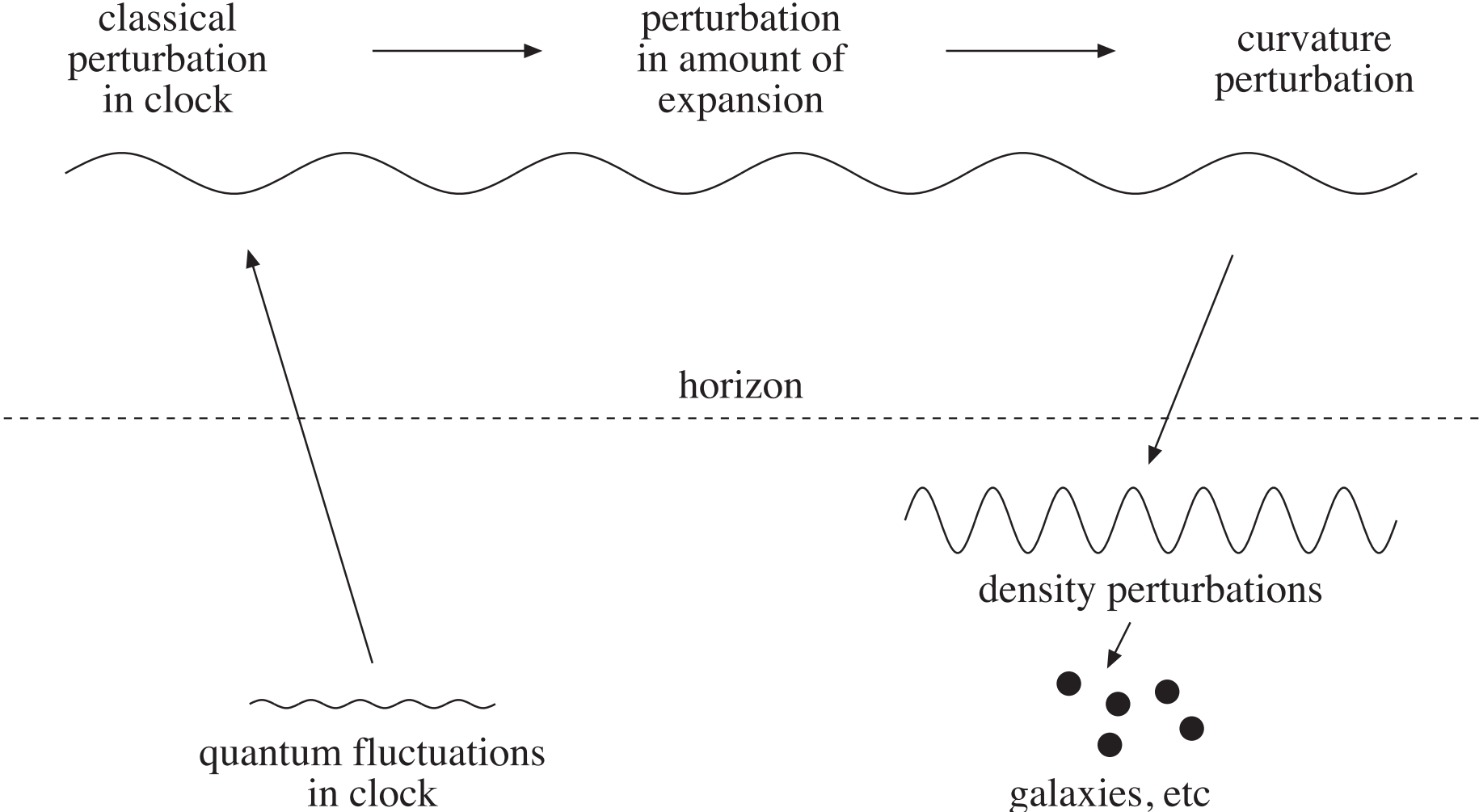
# Cosmological perturbations during inflation

– Classical theory and gauge issue –

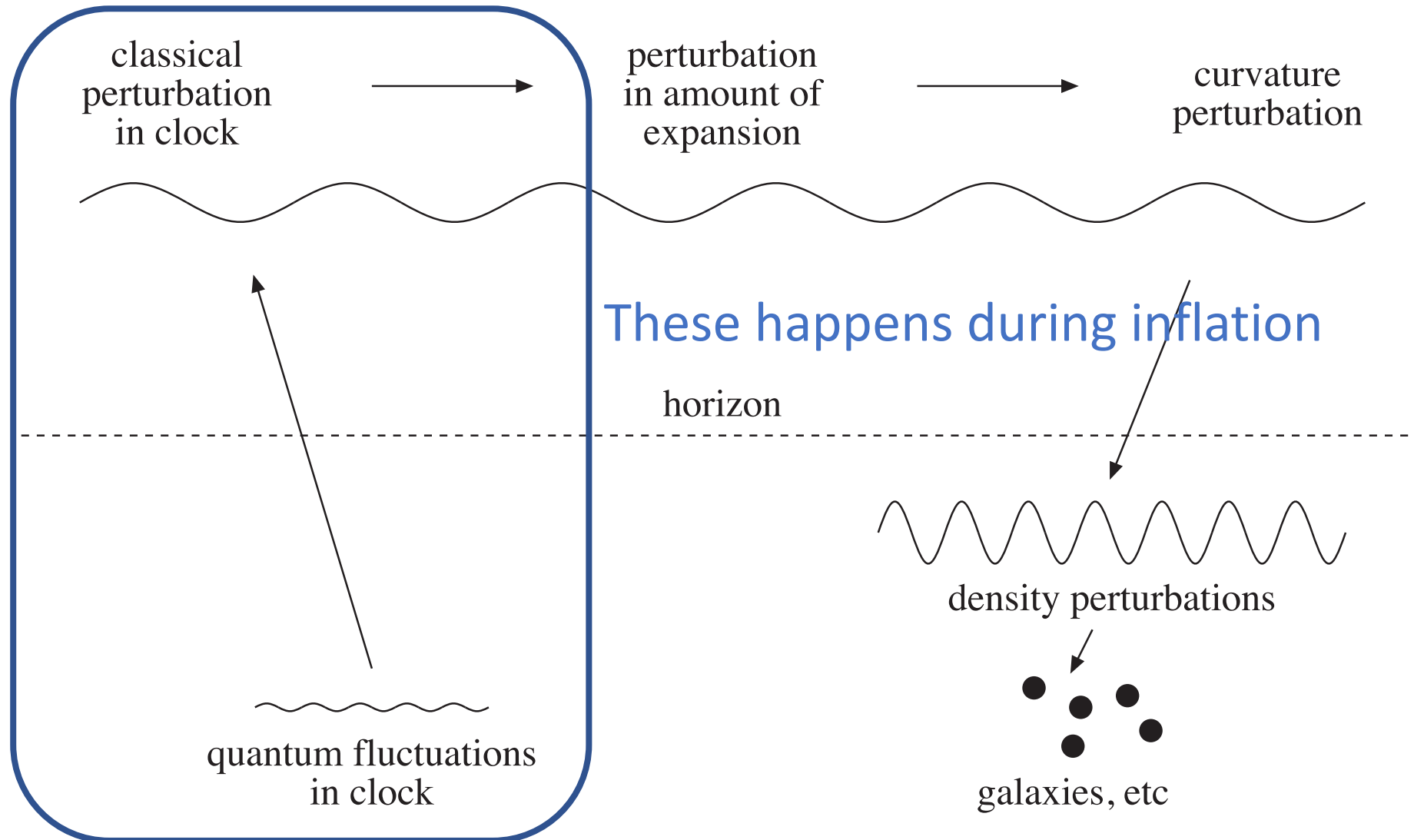
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# Generation and evolution of perturbations



# Generation and evolution of perturbations



# All observable structure from perturbations

- Temperature fluctuations from CMB, galaxies, galaxy clusters...
- Were it not for small deviations from otherwise homogeneous background, what we can observe now would not have been formed
- Thus, by investigating closely the currently observable universe, we can infer important information on the properties of the primordial, seed perturbations and the physics behind them
- It is then not a problem of initial conditions, but a question on the possible dynamical process in the early universe (presumably inflation)

# Universe with perturbations

- For definiteness, we consider GR + minimally coupled scalar field w/ FRW metric
- We can include small deviations from background in the metric as well as the scalar field

$$g_{\mu\nu} \rightarrow g_{\mu\nu}^{(0)} + \delta g_{\mu\nu}$$

$$\phi \rightarrow \phi_0 + \delta\phi$$

# Perturbations in the metric

- We consider flat FRW universe as the background

$$\begin{aligned} g_{\mu\nu} dx^\mu dx^\nu &= \left[ -dt^2 + a^2(t) \delta_{ij} dx^i dx^j \right] + \delta g_{\mu\nu} dx^\mu dx^\nu \\ &= -(1 + 2A) dt^2 + 2a \mathcal{B}_i dt dx^i + a^2 \left[ (1 + 2\varphi) \delta_{ij} + \mathcal{E}_{ij} \right] dx^i dx^j \end{aligned}$$

- Now what to do? We need the dynamics of the perturbations, which we can derive from the perturbed Einstein equation. So our first task is to write the Einstein equation including perturbations. Since they are small, we keep up to linear order in them (**linear perturbation theory**)

# Perturbed Einstein tensor (1)

- First we need the inverse metric, which can be found up to the desired order in perturbations as we want

$$g^{\mu\rho} g_{\rho\nu} = \delta^{\mu}_{\nu}$$

- For example,  $g^{00}$ : Assume  $g^{00} = -1 + \alpha$  and  $g^{0i} = \beta^i$ , then

$$g^{0\mu} g_{\mu 0} = 1 = g^{00} g_{00} + g^{0i} g_{i0} = (-1 + \alpha)(-1 - 2A) + \beta^i a \mathcal{B}_i$$

$$\approx 1 + 2A - \alpha$$

$$\therefore \alpha = 2A$$

# Perturbed Einstein tensor (2)

- In exactly the same way, we can derive the inverse metric up to linear order in perturbations

$$\delta g_{00} = -2A, \quad \delta g_{0i} = a\mathcal{B}_i, \quad \delta g_{ij} = 2a^2 (\varphi\delta_{ij} + \mathcal{E}_{ij})$$
$$\delta g^{00} = 2A, \quad \delta g^{0i} = a^{-1}\mathcal{B}^i, \quad \delta g^{ij} = 2a^{-2} (-\varphi\delta^{ij} - \mathcal{E}^{ij})$$

(indices of  $\mathcal{B}_i$  and  $\mathcal{E}_{ij}$  raised / lowered by  $\delta_{ij}$ )

- Then we can straightly calculate the perturbed Einstein tensor up to linear order in perturbations



# Perturbed Einstein tensor (3)

$$\begin{aligned}
 \delta G_{00} &= 6H\dot{\varphi} + \frac{2}{a^2} \left\{ -\Delta\varphi - aH\mathcal{B}^i{}_{,i} + a^2 \left[ H\dot{\mathcal{E}}^i{}_i + \frac{1}{2} (\mathcal{E}^{ij}{}_{,ij} - \Delta\mathcal{E}^i{}_i) \right] \right\} \\
 \delta G_{0i} &= 2(-\dot{\varphi} + HA)_{,i} - a(2\dot{H} + 3H^2)\mathcal{B}_i + \frac{1}{2a}(\mathcal{B}^j{}_{,ij} - \Delta\mathcal{B}_i) + (\dot{\mathcal{E}}^j{}_{i,j} - \dot{\mathcal{E}}^j{}_{j,i}) \\
 \delta G_{ij} &= a^2 \left( -2\ddot{\varphi} + 2H(\dot{A} - 3\dot{\varphi}) + 2(2\dot{H} + 3H^2)(A - \varphi) \right. \\
 &\quad \left. + \frac{1}{a^2} \left\{ \Delta(A + \varphi) + a^2 (\dot{\mathcal{B}}^k + 2H\mathcal{B}^k)_{,k} - a^2 \left[ \ddot{\mathcal{E}}^k{}_k + 3H\dot{\mathcal{E}}^k{}_k + (\mathcal{E}^{kl}{}_{,kl} - \Delta\mathcal{E}^k{}_k) \right] \right\} \right) \\
 &\quad + \left\{ -(A + \varphi)_{,ij} - a \left[ \frac{\dot{\mathcal{B}}_{i,j} + \dot{\mathcal{B}}_{j,i}}{2} + H(\mathcal{B}_{i,j} + \mathcal{B}_{j,i}) \right] \right. \\
 &\quad \left. + a^2 \left[ \ddot{\mathcal{E}}_{ij} + 3H\dot{\mathcal{E}}_{ij} - 2(2\dot{H} + 3H^2)\mathcal{E}_{ij} + \frac{1}{a^2} (\mathcal{E}^k{}_{j,ik} + \mathcal{E}^k{}_{i,jk} - \mathcal{E}^k{}_{k,ij} - \Delta\mathcal{E}_{ij}) \right] \right\}
 \end{aligned}$$

(Here, Delta = spatial Laplacian)

# Decomposing perturbations

- It is convenient to decompose the vector and tensor components  $B_i$  and  $E_{ij}$  into pure scalar, transverse vector and transverse and traceless tensor component

$$\mathcal{B}_i = \underbrace{B}_{,i} + \underbrace{S}_i$$

$$\mathcal{E}_{ij} = \underbrace{E}_{,ij} + \frac{1}{2} \underbrace{(F_{i,j} + F_{j,i})}_{\text{Pure vector}} + \frac{1}{2} \underbrace{h_{ij}}_{\text{Pure tensor}}$$

Pure scalar

Pure vector

Pure tensor

$$\begin{aligned} S^i{}_{,i} &= F^i{}_{,i} = 0 \\ h^i{}_i &= h^i{}_{j,i} = 0 \end{aligned}$$

# Number of degrees of freedom

- For scalar, we have 4 functions:  $A$ ,  $B$ ,  $\varphi$  and  $E$
- For vector, we have 2 ( $S_i$  and  $F_i$ ) so there are 3 dof each (spatial vector), **but** the transverse conditions remove 1 each so there are in total 4 dof
- For tensor, we have a symmetric  $3 \times 3$  matrix  $h_{ij}$  so there are 6 dof, **but** the transverse and traceless conditions remove 4 so there are in total 2 dof
- So, in total there are  $4$  (scalar) +  $4$  (vector) +  $2$  (tensor) =  $10$  dof
- This is consistent with the fact that  $4 \times 4$  symmetric matrix  $g_{\mu\nu}$  have 10 independent components

# Perturbed Einstein tensor reprocessed

$$\delta G^0_0 = 6H(-\dot{\varphi} + HA) - 2\frac{\Delta}{a^2} \left[ -\varphi - H(aB - a^2\dot{E}) \right]$$

$$\delta G^0_i = -2(-\dot{\varphi} + HA)_{,i} + \frac{\Delta}{2a} S_i - \frac{\Delta}{2} \dot{F}_i$$

$$\delta G^i_i = 6 \left[ \frac{d}{dt} (-\dot{\varphi} + HA) + 3H(-\dot{\varphi} + HA) + \dot{H}A + \frac{\Delta}{3a^2} D \right] \longrightarrow \text{Trace component (i = j)}$$

$$\delta G^{T^i}_j = \delta G^i_j - \frac{1}{3} \delta^i_j \delta G^k_k$$

$$= -\frac{1}{a^2} \left( \partial^i \partial_j - \delta^i_j \frac{\Delta}{3} \right) \left[ (A + \varphi) + \frac{d}{dt} (aB - a^2\dot{E}) + H(aB - a^2\dot{E}) \right]$$

$$+ \left[ -\frac{1}{2a} (\dot{S}^i + 2HS^i) + \frac{1}{2} (\ddot{F}^i + 3H\dot{F}^i) \right]_{,j} + \left[ -\frac{1}{2a} (\dot{S}_j + 2HS_j) + \frac{1}{2} (\ddot{F}_j + 3H\dot{F}_j) \right]^{,i}$$

$$+ \frac{1}{2} (\ddot{h}^i_j + 3H\dot{h}^i_j - \Delta h^i_j) \longrightarrow \text{Traceless component (if i = j, vanishes identically)}$$

# Energy-momentum tensor

- We consider a minimally coupled scalar field  $\phi$

$$S_m = \int d^4x \sqrt{-g} \left[ -\frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - V(\phi) \right]$$

- Energy-momentum tensor follows by perturbing w.r.t. metric

$$T_{\mu\nu} \equiv -\frac{2}{\sqrt{-g}} \frac{\delta \mathcal{L}_m}{\delta g^{\mu\nu}} = \partial_\mu \phi \partial_\nu \phi - g_{\mu\nu} \left[ \frac{1}{2} g^{\rho\sigma} \partial_\rho \phi \partial_\sigma \phi + V(\phi) \right]$$

# Perturbed energy-momentum tensor

- Including perturbations in both scalar field and metric, we can find straightly each component of the perturbed energy-momentum tensor

$$\delta T^0_0 = - \left[ \dot{\phi} \left( \delta\dot{\phi} - \dot{\phi}A \right) + V' \delta\phi \right]$$

$$\delta T^0_i = -\dot{\phi} \delta\phi_{,i}$$

$$\delta T^i_i = \frac{1}{3} \left[ \dot{\phi} \left( \delta\dot{\phi} - \dot{\phi}A \right) - V' \delta\phi \right]$$

$$\delta T^{T^i}_j = 0$$

# Perturbed Einstein equation

$$00 \text{ component: } 3H(-\dot{\varphi} + HA) - \frac{\Delta}{a^2} \left[ -\varphi - H(aB - a^2\dot{E}) \right] = -\frac{1}{2m_{\text{Pl}}^2} \left[ \dot{\phi} (\delta\dot{\phi} - \dot{\phi}A) + V'\delta\phi \right]$$

$$\text{Scalar } 0i \text{ component: } -\dot{\varphi} + HA = \frac{1}{2m_{\text{Pl}}^2} \dot{\phi} \delta\phi,$$

$$\text{Vector } 0i \text{ component: } S_i - a\dot{F}_i = 0,$$

$$\begin{aligned} \text{Trace } ij \text{ component: } & \frac{d}{dt} (-\dot{\varphi} + HA) + 3H(-\dot{\varphi} + HA) + \dot{H}A \\ & + \frac{\Delta}{3a^2} \left[ (A + \varphi) + \frac{d}{dt} (aB - a^2\dot{E}) + H(aB - a^2\dot{E}) \right] \\ & = \frac{1}{2m_{\text{Pl}}^2} \left[ \dot{\phi} (\delta\dot{\phi} - \dot{\phi}A) - V'\delta\phi \right], \end{aligned}$$

$$\text{Traceless scalar } ij \text{ component: } (A + \varphi) + \frac{d}{dt} (aB - a^2\dot{E}) + H(aB - a^2\dot{E}) = 0,$$

$$\text{Traceless vector } ij \text{ component: } \frac{d}{dt} (S^i - a\dot{F}^i) + 2H(S^i - a\dot{F}^i) = 0,$$

$$\text{Traceless tensor } ij \text{ component: } \ddot{h}_{ij} + 3H\dot{h}_{ij} - \frac{\Delta}{a^2} h_{ij} = 0$$

# What we can see from these equations

- Scalar, vector and tensor perturbations evolve separately
- Only scalar perturbations are sourced (RHS  $\neq 0$ ), but vector and tensor perturbations are unsourced (RHS = 0)
- No anisotropic stress, so traceless scalar  $ij$  component = 0



# Perturbations and background

- The notion of “perturbations” is meaningful only if we can define the corresponding background values
- But we have the freedom to choose the coordinate system, and this introduces ambiguity in defining perturbations. For example, consider 2 perturbations,  $a$  and  $b$ . We may choose one coordinate system where  $a$  vanishes but  $b$  is not. We can choose the other way round. So, just saying “perturbations are such and such” is not enough. We need to specify the coordinate system in describing perturbations

# Example: Scalar perturbation

- Consider for example a scalar quantity  $\hat{\phi}$  at a space-time point  $p$  in the perturbed universe  $\hat{U}$ . “Perturbation” means the difference its value from the corresponding background value  $\phi_0$ . What’s that?
- To specify the background value  $\phi_0$ , we need a coordinate system, or “mapping”  $x^\mu$ , such that each point in the perturbed universe  $\hat{U}$  is associated with the corresponding point in the (fictitious) background universe  $U$ . The  $\phi_0$  is the value of  $\phi$  at  $x^\mu$
- Only then the following perturbation becomes meaningful:

$$\delta\phi(p) = \hat{\phi}(p) - \phi_0(x^\mu)$$

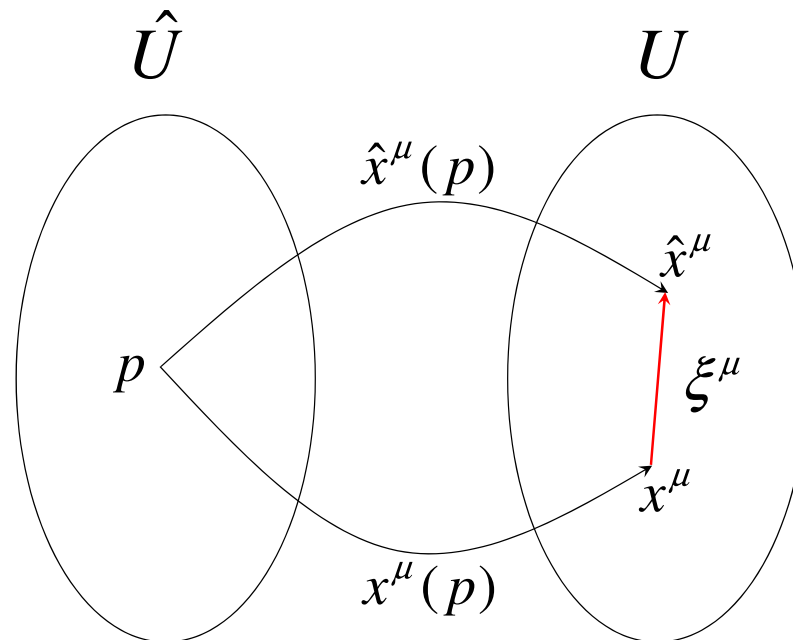
# From one coordinate system to another

- But there is no a priori natural mapping (or coordinate system), so one is as good (or bad) as another. Thus we need to know how one mapping is related to another
- IMPORTANT – Any change induced by a change in mapping is not physical: It is just a transformation since we change the coordinate, or “gauge”, to describe the same thing
- In this sense, this non-physical change is called **gauge transformation**

# Example: Scalar perturbation again (1)

- Suppose 2 coordinate systems,  $x^\mu$  and  $\hat{x}^\mu$ , which map  $p$  in  $\hat{U}$  to the corresponding **different** points in  $U$ , are related by

$$\hat{x}^\mu(p) = x^\mu(p) + \xi^\mu(x^\nu(p))$$



# Example: Scalar perturbation again (2)

- If the transformation is infinitesimal

$$\begin{aligned}\widehat{\delta\phi}(p) &= \widehat{\phi}(p) - \phi_0(\widehat{x}^\mu(p)) \\ &= \delta\phi(p) - [\phi_0(\widehat{x}^\mu(p)) - \phi_0(x^\mu(p))] \\ &= \delta\phi(p) - \xi^\nu \frac{\partial\phi_0}{\partial x^\nu}(x^\mu(p)) \\ &= \delta\phi(p) - \dot{\phi}_0(t(p))\xi^0(x^\mu(p))\end{aligned}$$

← BG is homogeneous and isotropic, so only  $x^0 = t$  acts on the BG quantity  $\phi_0$

# Gauge x-form of metric perturbations (1)

- Mathematically, gauge transformation of a generic tensor quantity is given by the exponential map of the Lie derivative along the transformation vector  $\xi^\mu$
- But regarding the metric perturbations, we can follow a simpler approach by demanding that  $ds^2$  remains invariant under gauge transformation: With the coordinate transformation

$$\left. \begin{aligned} t &\rightarrow \hat{t} = t + \xi^0(t, \mathbf{x}) \\ x^i &\rightarrow \hat{x}^i = x^i + \xi^i(t, \mathbf{x}) \end{aligned} \right\} \longrightarrow \left\{ \begin{aligned} a(\hat{t}) &\equiv \hat{a} = (1 + H\xi^0) a(t) \\ \hat{d}t &= (1 + \dot{\xi}^0) dt + \xi^0_{,i} dx^i \\ \hat{d}x^i &= dx^i + \dot{\xi}^i dt + \xi^i_{,j} dx^j \end{aligned} \right.$$

## Gauge x-form of metric perturbations (2)

$$\begin{aligned}
 \widehat{ds}^2 &= - \left(1 + 2\widehat{A}\right) \widehat{dt}^2 + 2\widehat{a}\widehat{\mathcal{B}}_i \widehat{dt}\widehat{dx}^i + \widehat{a}^2 \left[ (1 + 2\widehat{\varphi}) \delta_{ij} + 2\widehat{\mathcal{E}}_{ij} \right] \widehat{dx}^i \widehat{dx}^j \\
 &= - \left[ 1 + 2 \left( \widehat{A} + \dot{\xi}^0 \right) \right] dt^2 + 2a \left( \widehat{\mathcal{B}}_i - \frac{\xi^{0,i}}{a} + a\dot{\xi}_i \right) dt dx^i \\
 &\quad + a^2 \left\{ \left[ 1 + 2 \left( \widehat{\varphi} + H\xi^0 \right) \right] \delta_{ij} + 2 \left( \widehat{\mathcal{E}}_{ij} + \frac{\xi_{i,j} + \xi_{j,i}}{2} \right) \right\} dx^i dx^j \\
 &= - (1 + 2A) dt^2 + 2a\mathcal{B}_i dt dx^i + a^2 \left[ (1 + 2\varphi) \delta_{ij} + 2\mathcal{E}_{ij} \right] dx^i dx^j
 \end{aligned}$$

$$\therefore \widehat{A} = A - \dot{\xi}^0, \quad \widehat{\mathcal{B}}_i = \mathcal{B}_i + \frac{\xi^{0,i}}{a} - a\dot{\xi}_i, \quad \widehat{\varphi} = \varphi - H\xi^0, \quad \widehat{\mathcal{E}}_{ij} = \mathcal{E}_{ij} - \frac{\xi_{i,j} + \xi_{j,i}}{2}$$

# Gauge x-form of metric perturbations (3)

- Further, decomposing the spatial transformation vector  $\xi^i$  as

$$\xi^i = \delta^{ij} \xi_{,j} + \xi^{(\text{tr})i}$$

- Then the scalar, vector and tensor components of  $B_i$  and  $E_{ij}$  transform

$$\hat{B} = B + \frac{\xi^0}{a} - a\dot{\xi}, \quad \hat{S}_i = S_i - a\dot{\xi}_i^{(\text{tr})}, \quad \hat{E} = E - \xi, \quad \hat{F}_i = F_i - \xi_i^{(\text{tr})}, \quad \hat{h}_{ij} = h_{ij}$$

- Tensor perturbation  $h_{ij}$  and the combination  $S_i - a \dot{F}_i$  remain the same under gauge transformation, i.e. they are **gauge-invariant**



# Gauge x-form of metric perturbations (4)

- In the scalar Einstein equations, all 4 scalar perturbations appear but B and E appear only in the combination  $aB - a^2\dot{E}$

$$\begin{aligned} a\hat{B} - a^2\hat{E} &= a \left( B + \frac{\xi^0}{a} - a\dot{\xi} \right) - a^2 \frac{d}{dt} (E - \xi) \\ &= aB - a^2\dot{E} + \xi^0 \end{aligned}$$

- While B and E depend on both  $x^0$  and  $x^i$ , in practice  $x^0$  (time translation) matters

# Constructing gauge-invariant scalar pert

- There are 2 different ways to treat the gauge ambiguity:
  1. Fix the gauge in such a way that the perturbation in some physical quantity of interest to vanish, e.g. the perturbation in  $\phi$  vanish:

$$\widehat{\delta\phi}(p) = 0 = \delta\phi(p) - \dot{\phi}_0(t(p))\xi^0(x^\mu(p)) \longrightarrow \xi^0 = \frac{\delta\phi}{\dot{\phi}_0} \equiv \xi_{\delta\phi}^0$$

The necessary time translation ( $\xi^0$ ) from an arbitrary hypersurface to the one where  $\delta\phi = 0$ . In this choice, constant-time hypersurface in  $\widehat{U} = \text{constant-}\phi$  hypersurface

2. Just use gauge-invariant quantities like  $h_{ij}$  and  $S_i - a \dot{F}_i$ , e.g.

$$\mathcal{R} = \varphi - \frac{H}{\dot{\phi}_0} \delta\phi$$

This quantity is gauge-invariant:

$$\begin{aligned} \widehat{\mathcal{R}} &= \widehat{\varphi} - \frac{H}{\dot{\phi}_0} \widehat{\delta\phi} \\ &= (\varphi - H\xi^0) - \frac{H}{\dot{\phi}_0} (\delta\phi - \xi^0 \dot{\phi}_0) \\ &= \varphi - \frac{H}{\dot{\phi}_0} \delta\phi = \mathcal{R} \end{aligned}$$

- The physical meaning of  $\mathcal{R}$  is clear: Using the necessary time translation to the constant- $\phi$  hypersurface

$$\mathcal{R} = \varphi - H \underbrace{\frac{\delta\phi}{\phi}}_{=\xi_{\delta\phi}^0} \equiv \varphi \delta\phi$$

- That is,  $\mathcal{R}$  is the perturbation in the spatial curvature (perturbation proportional to the BG metric  $\delta_{ij}$ ) on the constant- $\phi$  hypersurface
- If  $\delta\phi = 0$ ,  $T^i_0 = 0$ , which means no momentum flux. In this sense, this gauge is called “comoving gauge”, and correspondingly  $\mathcal{R}$  is “comoving curvature perturbation”
- $\mathcal{R}$  can be evaluated on arbitrary hypersurfaces since it is gauge-invariant

# Eq for comoving curvature perturbation (1)

- Now we find the equation for R in 3 different ways (the reason why R, not any other perturbation, is 3-fold, 2 of them being obvious soon)
1. Choose a different gauge and find the equation for R
    - Let's choose the Newtonian gauge:  $B = E = 0$ . From the traceless  $ij$  scalar component, we further find  $A = -\varphi = \Phi$

$$ds^2 = -(1 + 2\Phi)dt^2 + a^2(1 - 2\Phi)\delta_{ij}dx^i dx^j$$

(weak field limit w/  $\Phi =$  Newtonian gravitational potential)

(also  $aB - a^2E$ , shear, vanishes so it's called zero-shear gauge)

- 00, 0i and ij components of the Einstein equation

$$3H \left( \dot{\Phi} + 3H\Phi \right) - \frac{\Delta}{a^2} \Phi = -\frac{1}{2m_{\text{Pl}}^2} \left[ \dot{\phi} \left( \delta\dot{\phi} - \dot{\phi}\Phi \right) + V' \delta\phi \right]$$

$$\dot{\Phi} + 3H\Phi = \frac{1}{2m_{\text{Pl}}^2} \dot{\phi} \delta\phi$$

$$\frac{d}{dt} \left( \dot{\Phi} + 3H\Phi \right) + 3H \left( \dot{\Phi} + 3H\Phi \right) + \dot{H}\Phi = \frac{1}{2m_{\text{Pl}}^2} \left[ \dot{\phi} \left( \delta\dot{\phi} - \dot{\phi}\Phi \right) - V' \delta\phi \right]$$

- 2 variables are left, Phi and delta{phi}, and 3 eqs, thus 1 eq is redundant: We pick 00 and 0i components, since they are simpler

- From the 00 equation

$$\begin{aligned}
 3H \left( \dot{\Phi} + 3H\Phi \right) - \frac{\Delta}{a^2} \Phi &= -\frac{1}{2m_{\text{Pl}}^2} \left[ \dot{\phi}_0 \delta \dot{\phi} - \dot{\phi}_0^2 \Phi - \left( \ddot{\phi}_0 + 3H\dot{\phi}_0 \right) \delta \phi \right] \\
 &= -\frac{1}{2m_{\text{Pl}}^2} \left( \frac{\dot{\phi}_0 \delta \dot{\phi} - \ddot{\phi}_0 \delta \phi}{\dot{\phi}_0^2} \dot{\phi}_0^2 - \dot{\phi}_0^2 \Phi - 3H\dot{\phi}_0 \delta \phi \right) \\
 &= -\frac{\dot{\phi}_0^2}{2m_{\text{Pl}}^2} \left[ \frac{d}{dt} \left( \frac{\delta \phi}{\dot{\phi}_0} \right) - \Phi \right] + 3H \left( \dot{\Phi} + 3H\Phi \right)
 \end{aligned}
 \left. \vphantom{\begin{aligned} 3H \left( \dot{\Phi} + 3H\Phi \right) - \frac{\Delta}{a^2} \Phi \end{aligned}} \right\} \frac{d}{dt} \left( \frac{\delta \phi}{\dot{\phi}_0} \right) = \Phi - \frac{1}{\dot{H}} \frac{\Delta}{a^2} \Phi$$

- In the Newtonian gauge, we can write  $\mathcal{R}$  as

$$\begin{aligned}
 \mathcal{R} = -\Phi - \frac{H}{\dot{\phi}_0} \delta \phi \quad \rightarrow \quad \dot{\mathcal{R}} &= -\dot{\Phi} - \dot{H} \frac{\delta \phi}{\dot{\phi}_0} - H \frac{d}{dt} \left( \frac{\delta \phi}{\dot{\phi}_0} \right) \\
 &= -\dot{\Phi} - \frac{\dot{H}}{\dot{\phi}_0} 2m_{\text{Pl}}^2 \left( \dot{\Phi} + H\Phi \right) - H \left( \Phi - \frac{1}{\dot{H}} \frac{\Delta}{a^2} \Phi \right) \\
 &= \frac{H}{\dot{H}} \frac{\Delta}{a^2} \Phi
 \end{aligned}
 \left. \vphantom{\begin{aligned} \mathcal{R} = -\Phi - \frac{H}{\dot{\phi}_0} \delta \phi \end{aligned}} \right\} \Phi = \frac{\dot{H}}{H} a^2 \Delta^{-1} \dot{\mathcal{R}}$$

- Now plugging everything into the 0i component:

$$(\text{LHS}) = \dot{\Phi} + H\Phi = a^2 \frac{\dot{H}}{H} \Delta^{-1} \left[ \ddot{\mathcal{R}} + \left( 3H + \frac{2\ddot{\phi}_0}{\dot{\phi}_0} - \frac{\dot{H}}{H} \right) \dot{\mathcal{R}} \right]$$

$$(\text{RHS}) = \frac{1}{2m_{Pl}^2} \dot{\phi}_0 \delta\phi = \frac{\dot{H}}{H} (\mathcal{R} + \Phi) = \frac{\dot{H}}{H} \left( \mathcal{R} + a^2 \frac{\dot{H}}{H} \Delta^{-1} \dot{\mathcal{R}} \right)$$

$$= a^2 \frac{\dot{H}}{H} \Delta^{-1} \left( \frac{\Delta}{a^2} \mathcal{R} + \frac{\dot{H}}{H} \dot{\mathcal{R}} \right)$$

$$\therefore \ddot{\mathcal{R}} + \left( 3H + \frac{2\ddot{\phi}_0}{\dot{\phi}_0} - \frac{2\dot{H}}{H} \right) \dot{\mathcal{R}} - \frac{\Delta}{a^2} \mathcal{R} = \frac{1}{a^3 \epsilon} \frac{d}{dt} \left( a^3 \epsilon \dot{\mathcal{R}} \right) - \frac{\Delta}{a^2} \mathcal{R} = 0$$

$$\epsilon \equiv -\frac{\dot{H}}{H^2} = \frac{1}{2m_{Pl}^2} \frac{\dot{\phi}_0^2}{H^2}, \text{ “slow-roll parameter”}$$



# Eq for comoving curvature perturbation (2)

2. Work in the comoving gauge and find the equation for  $\mathcal{R}$ 
  - This is much simpler than the 1<sup>st</sup> approach, but it has some illuminating aspects for the 3<sup>rd</sup> approach so we place this as 2<sup>nd</sup>
  - In the comoving gauge  $\delta\phi = E = 0$  with  $\varphi = \mathcal{R}$ , then  $00$ ,  $0i$  and traceless  $ij$  components become (trace  $ij$  vanishes)

$$3H \left( -\dot{\mathcal{R}} + HA \right) + \frac{\Delta}{a^2} (\mathcal{R} + aHB) = \frac{\dot{\phi}_0^2}{2m_{\text{Pl}}^2} A$$

$$-\dot{\mathcal{R}} + HA = 0$$

$$(A + \mathcal{R}) + \frac{d}{dt}(aB) + aHB = 0$$

- $0i$  and  $00$  components are algebraically solved (this is not a mere coincidence – we will return to this) to give  $A$  and  $B$  in terms of  $R$  as

$$A = \frac{\dot{\mathcal{R}}}{H}, \quad aB = -\frac{\mathcal{R}}{H} + a^2 \epsilon \Delta^{-1} \dot{\mathcal{R}}$$

- Plugging this back into the traceless  $ij$  component (and applying the Laplacian operator) gives

$$\frac{d}{dt} \left( a^2 \epsilon \dot{\mathcal{R}} \right) + a^2 \epsilon H \dot{\mathcal{R}} - \epsilon \Delta \mathcal{R} = a^2 \epsilon \left[ \frac{1}{a^3 \epsilon} \frac{d}{dt} \left( a^3 \epsilon \dot{\mathcal{R}} \right) - \frac{\Delta}{a^2} \mathcal{R} \right] = 0$$

- We reach the same equation for  $R$ !

# Eq for comoving curvature perturbation (3)

3. Work with the perturbation action
  - This is more directly related to the quantum theory of cosmological perturbations, so we will return to this tomorrow!