BIG BANG & INFLATION

Theory & Observations

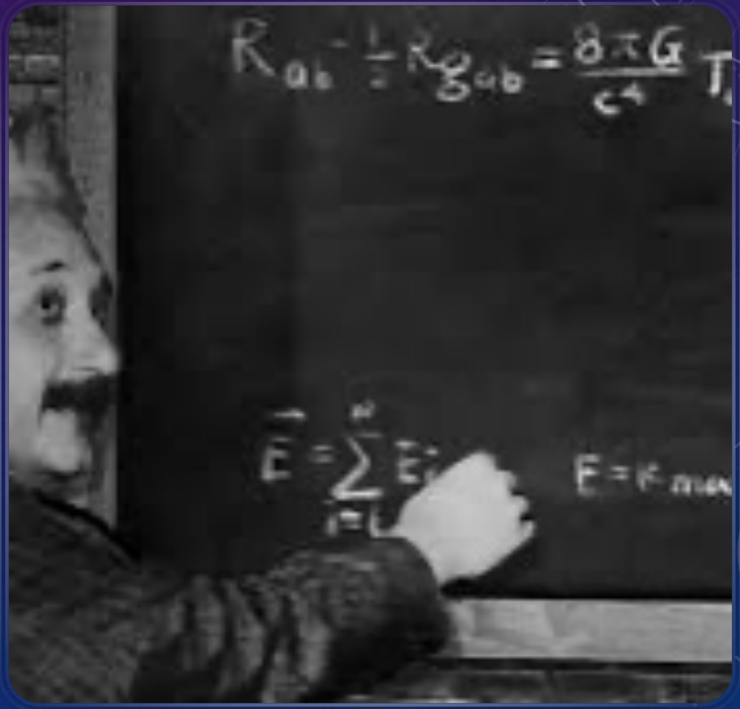
CQUEST GROUP II WORKSHOP 2022
SEP. 28 ~ OCT. 02 2022, ONLINE & JEJU NATIONAL UNIVERSITY
SEOKCHEON LEE (SUNGKYUNKWAN UNIVERSITY)
SEP.28.10:00AM

OUTLINE

- RW metric (CP + Weyl's postulate)
- Big Bang Model
- Problems of Big Bang
- Solutions for problems (Yamaguchi's talk)
- Predictions of Inflation (Gong's talk)
- Stochastic properties
- Observables (Appleby's talk)

COSMOLOGY

- General theory of Relativity (GR) : local theory (Gaussian normal coordinate), geometric theory of gravitation
- 2nd order nonlinear partial DEs : difficult to solve and no general solutions
- Few specific solutions for GR
  - Static (stationary) local solutions : BHs
  - Static global solution : Einstein static univ (assume spatial hom + iso : CP , need Λ to satisfy EFEs)
  - Dynamical sols : de Sitter, RW metric (assume CP+Weyl's postulate) : cosmic (global) time + comoving spatial element + scale factor (information for expansion)
- Cosmology : Solve EFEs by using RW type metric $G_{\mu\nu}$ + Thermodymanics $T_{\mu\nu}$ (adiabaticity)



ROBERTSON-WALKER (RW) METRIC

- Cosmological principle : 3D space is hom + iso at a given cosmic time, t_l (Killing vectors) = same as Einstein's static Univ
- Weyl's postulate : Spacelike hypersurfaces = surfaces of simultaneity w.r.t the t
- Time varying speed of light (VSL in the expanding Univ, out of scope)
- Refer : Islam p.37-45, Ryder p.344-355, Hubson p.355-367, Narlikar p.94-112

- @ a given time t_l (Einstein's static Univ) :

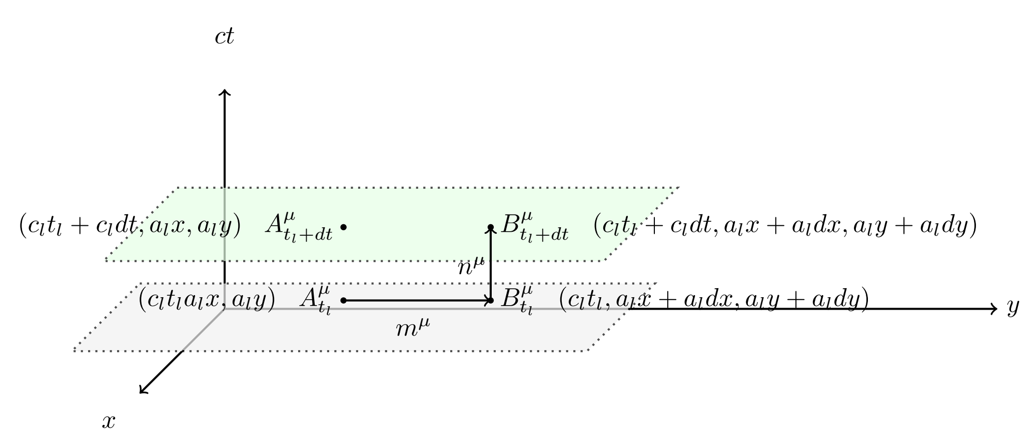
$$ds_l^2 \equiv ds^2(t_l) = -c_l^2 dt^2 + a^2(t_l)\left[\frac{dr^2}{1-Kr^2} + r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi^2\right] \equiv -c_l^2 dt^2 + a^2(t_l) d\sigma^2$$

- null geodesic @ t_1

$$ds_l^2 = 0 \quad \Rightarrow \quad c_1 = \frac{a(t_1)d\sigma}{dt} = \frac{\text{physical distance @ } t_l}{dt}$$

- null geodesic @ t_2

$$ds_l^2 = 0 \quad \Rightarrow \quad c_2 = \frac{a(t_2)d\sigma}{dt} = \frac{\text{physical distance @ } t_2}{dt}$$



symmetric, $a(t)$ scale factor. Friedmann (1922, 1924), Robertson (1935, 1936) Walker (1937), Lemaitre (1931)

Eur. Phys. J. H **39**, 63–85 (2014)
DOI: 10.1140/epjh/e2013-40038-x

THE EUROPEAN PHYSICAL JOURNAL H

Einstein's cosmic model of 1931 revisited: an analysis and translation of a forgotten model of the universe

C. O'Raifeartaigh[a] and B. McCann

The **Friedmann–Einstein universe** is a model of the universe published by Albert Einstein in 1931.[1] The model is of historic significance as the first scientific publication in which Einstein embraced the possibility of a cosmos of time-varying radius.

Description

Interpreting Edwin Hubble's discovery of a linear relation between the redshifts of the galaxies and their radial distance[2] as evidence for an expanding universe, Einstein abandoned his earlier static model of the universe and embraced the dynamic cosmology of Alexander Friedmann. Removing the cosmological constant term from the Friedmann equations on the grounds that it was both unsatisfactory and unnecessary, Einstein arrived at a model of a universe that expands and then contracts, a model that was later denoted the Friedmann–Einstein model of the universe.[3][4]

Kosmologische Betrachtungen zur allgemeinen Relativitätstheorie.

Von A. Einstein.

Einstein's 1917 Static Model of the Universe: A Centennial Review

Cormac O'Raifeartaigh,[a] Michael O'Keeffe,[a] Werner Nahm[b] and Simon Mitton[c]

Einstein's blackboard

In May 1931, Einstein chose the Friedmann–Einstein universe as the topic of his 2nd Rhodes lecture at Oxford University. A blackboard used by Einstein during the lecture, now known as Einstein's Blackboard, has been preserved at the Museum of the History of Science, Oxford. It has been suggested[5] that the source of the numerical errors in the Friedmann–Einstein model can be discerned on Einstein's blackboard.

See also

- Einstein–de Sitter universe

References

1. ^ *Einstein, Albert (1931). "Zum kosmologischen Problem der allgemeinen Relativitätstheorie". Sitzungs. König. Preuss. Akad.: 235–237.*

BIG BANG MODEL

- Einstein's Field equation : FLRW equations
- Solutions for FLRW eqs : Big Bang Models
- BB = Decelerated expanding Univ with ordinary materials (radiation + baryon)
- FLRW Univ assumes CP but it can't explain it : horizon problem
- BB shows that early Univ was made of many causally disconnected regions of space : horizon problem

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho + \frac{\Lambda c^2}{3} - \frac{Kc^2}{a^2} \equiv H^2$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho + \frac{3P}{c^2}\right) + \frac{\Lambda c^2}{3} \equiv -qH^2$$

PROBLEMS OF BIG BANG UNIVERSE

- Flatness problem (= oldness problem, fine-tuning) : even closer to 1 in the past

$$H^2 = \frac{8\pi G}{3}\rho - \frac{kc^2}{a^2}$$

$$1 = \frac{8\pi G}{3H^2}\rho - \frac{kc^2}{a^2H^2} \equiv \Omega + \Omega_k$$

$$\Omega = \frac{1}{1 + \frac{(\Omega_0^{-1}-1)}{(1+z)^{1+3\omega}}} \simeq 1 - \frac{(\Omega_0^{-1}-1)}{(1+z)^{1+3\omega}}$$

$$\Omega_0^{-1} = \frac{1}{1-\Omega_{k0}} \overset{\Omega_{k0}\to 0}{\simeq} 1 + \Omega_{k0}$$

$$\Omega_{k0} = 0.0007^{+0.0037}_{-0.0037} \sim 10^{-3} \quad \text{Planck 18}$$

$$-\frac{10^{-3}}{(1+10^3)} \le 1 - \Omega(z=10^3) \le \frac{10^{-3}}{(1+10^3)} \quad \text{for MD}$$

$$-\frac{10^{-3}}{(1+10^{10})^2} \le 1 - \Omega(z=10^{10}) \le \frac{10^{-3}}{(1+10^{10})^2} \quad \text{for RD}$$

- Horizon problem (CMB isotropy)

$$d_{\rm H} = a(t)\int_0^t \frac{cdt'}{a(t')} = a\int_0^a \frac{cda'}{a'^2H(a')}$$

$$= \frac{2ca}{H_0\Omega_{m0}}\left(\sqrt{\Omega_{m0}a + \Omega_{r0}} - \sqrt{\Omega_{r0}}\right)$$

$$d_{\rm A} = a(t)\int_t^{t_0} \frac{cdt'}{a(t')} = a\int_a^1 \frac{cda'}{a'^2H(a')}$$

$$= \frac{2ca}{H_0\Omega_{m0}}\left(\sqrt{\Omega_{m0} + \Omega_{r0}} - \sqrt{\Omega_{m0}a + \Omega_{r0}}\right)$$

$$\frac{d_{\rm H}}{d_{\rm A}}(a_{\rm ls}) = \frac{\sqrt{\Omega_{m0}a_{\rm ls} + \Omega_{r0}} - \sqrt{\Omega_{r0}}}{\sqrt{\Omega_{m0} + \Omega_{r0}} - \sqrt{\Omega_{m0}a_{\rm ls} + \Omega_{r0}}}$$

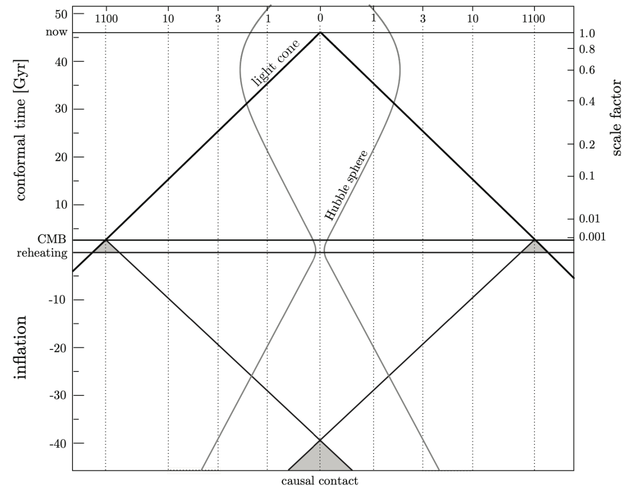
$$\approx 0.018(\text{rad}) = 1.03^\circ$$

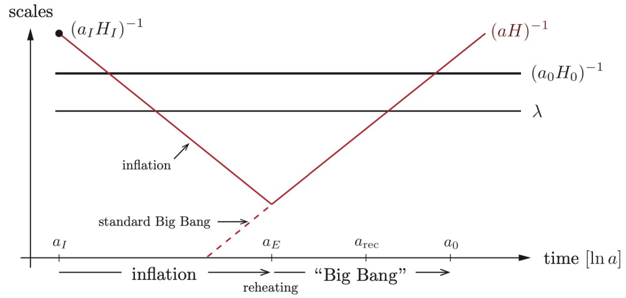
$$\text{Roughly } \frac{4\pi}{(0.018)^2} \approx 10^4$$

causally disconnected regions in the sky

REASON FOR BB PROBLEMS AND SOLUTIONS

- Both problems are due to the increasement of Hubble radius ($\frac{c}{aH} = \frac{c}{\dot{a}}$) during decelerating expansion
- Require shrinking Hubble radius (i) accelerating expansion , ii) larger c , iii) bouncing (Novello & Bergliaffa 08 or Brandenberger & Peter 17)) at early Univ
- (Refer Yamaguchi's talk for similar approaches in different areas)
- Early Univ = before hot BB





HOW MUCH INFLATION DO WE NEED?

- At least, we need the larger Hubble radius at the beginning of the inflation than that of today :

$$\frac{c}{a_I H_I} \geq \frac{c}{a_0 H_0}$$

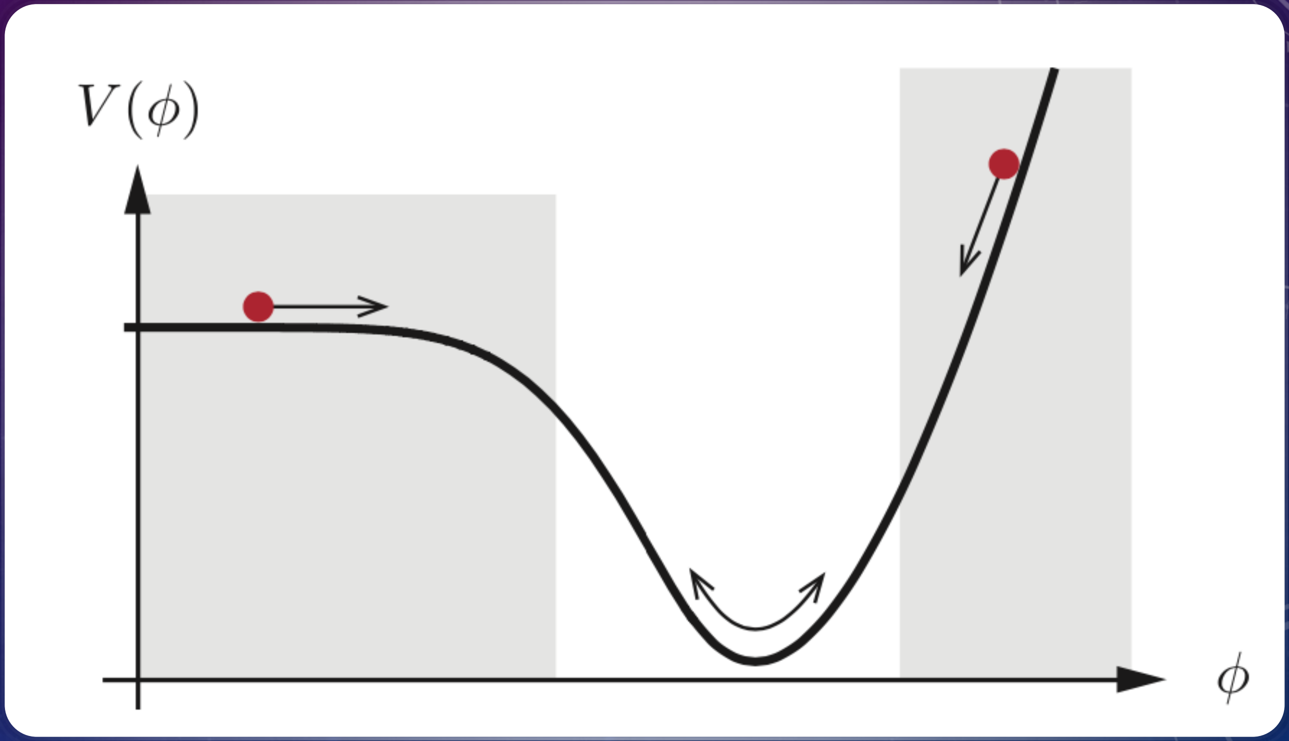
$$\frac{a_0 H_0}{a_{\mathrm{E}} H_{\mathrm{E}}} \overset{a\sim\sqrt{t}}{\approx} \frac{a_0}{a_{\mathrm{E}}}\left(\frac{a_{\mathrm{E}}}{a_0}\right)^2 = \frac{a_{\mathrm{E}}}{a_0} = \frac{T_{\gamma 0}}{T_{\gamma \mathrm{E}}} \sim \frac{2.7\mathrm{K}}{10^{28}\mathrm{K}}$$

$$(a_{\mathrm{I}} H_{\mathrm{I}})^{-1} \geq (a_0 H_0)^{-1} \sim 10^{28}\,(a_{\mathrm{E}} H_{\mathrm{E}})^{-1}$$

$$\Rightarrow \frac{a_{\mathrm{E}}}{a_{\mathrm{I}}} \geq 10^{28} \quad N_{\text{e-folding}} = \log_{10}\frac{a_{\mathrm{E}}}{a_{\mathrm{I}}} \geq 64$$

PREDICTIONS OF INFLATION
(REFER GONG'S TALK FOR DETAILS)

- Inflation models provide a testable prediction of PS of primordial fluctuations (CMB anisotropy & LSS)
- Cosmology starts to predict observational results
- Pioneering works by Starobinsky (79), Guth (81), Linde (82), Albrecht & Steinhardt (82)
- Single scalar field slow-roll inflation



SINGLE SCALAR FIELD SLOW-ROLL INFLATION

- Equations

$$\mathcal{L} = \frac{1}{2} g^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi + V(\varphi)$$

$$\rho_\varphi = -T^0{}_0 = \frac{1}{2}\dot{\varphi}^2 + V(\varphi)$$

$$P_\varphi = \frac{1}{3}\delta^i{}_j T^j{}_i = \frac{1}{2}\dot{\varphi}^2 - V(\varphi)$$

$$\ddot{\varphi} + 3H\dot{\varphi} + V_{,\varphi} = 0$$

$$H^2 = \frac{8\pi G}{3}\left[\frac{1}{2}\dot{\varphi}^2 + V(\varphi)\right]$$

$$\frac{\ddot{a}}{a} = -\frac{8\pi G}{3}\left[\dot{\varphi}^2 - V(\varphi)\right]$$

- Slow-roll conditions & parameters

$$\frac{|\dot{H}|}{H^2} \ll 1$$

$$\dot{\varphi}^2 \ll V(\varphi)$$

$$P_\varphi \approx -\rho_\varphi \approx -V(\varphi) \approx \text{constant}$$

$$\epsilon \equiv -\frac{\dot{H}}{H^2} = \frac{d}{dt}\left(\frac{1}{H}\right)$$

$$\dot{\epsilon} = \frac{2\dot{H}^2}{H^3} - \frac{\ddot{H}}{H^2}$$

$$\dot{\epsilon} = 2H\epsilon^2 + \frac{8\pi G\dot{\varphi}}{H^2}\ddot{\varphi}$$

$$= 2H\epsilon^2 - 2\frac{\dot{H}\ddot{\varphi}}{H^2\dot{\varphi}} = 2H\epsilon^2 + 2\epsilon\frac{\ddot{\varphi}}{\dot{\varphi}}$$

$$\equiv 2H\epsilon(\epsilon - \eta) \quad \text{where } \eta \equiv -\frac{1}{H}\frac{\ddot{\varphi}}{\dot{\varphi}}$$

$$\epsilon \approx \frac{1}{16\pi G}\left(\frac{V_{,\varphi}}{V}\right)^2 \equiv \epsilon_V$$

$$\eta + \epsilon \approx \frac{1}{8\pi G}\frac{V_{,\varphi\varphi}}{V} \equiv \eta_V$$

$$V(\varphi) = V_0 + \frac{1}{2} V_{,\varphi\varphi}\big|_{\varphi=\varphi_0} (\varphi - \varphi_0)^2 + \dots$$
$$\approx \frac{1}{2} m_\varphi^2 (\varphi - \varphi_0)^2$$
$$\dot{\rho}_\varphi + 3H\rho_\varphi(1 + w_\varphi) = -\Gamma\rho_\varphi$$
$$\dot{\rho}_\mathrm{r} + 3H\rho_\mathrm{r}(1 + w_\mathrm{r}) = \Gamma\rho_\varphi$$

REHEATING

- Need to terminate inflation
- Slow-roll parameters attain values of order unity (breaking slow-roll conditions)
- Potential around its minimum becomes harmonic oscillations
- Propose coupling of inflaton to other fluids (put by hand , need to be solved)
- How? (parametric resonant, etc…)

PRODUCTION OF GWS DURING INFLATION

- Metric perturbation : $g_{\mu\nu} \equiv a^2(\eta)\,(\eta_{\mu\nu} + h_{\mu\nu})$
- Scalar perturbations (Newtonian gauge) : $g_{00} = -a^2\,(1 + 2\Psi[\eta, \mathbf{x}])$, $g_{0i} = 0$, $g_{ij} = a^2\,(1 + 2\Phi[\eta, \mathbf{x}])$
- Tensor perturbations : $g_{00} = -a^2$, $g_{0i} = 0$, $g_{ij} = a^2\left(\delta_{ij} + h^T_{ij}\right)$

$$h^{T''}_{ij} + 2\mathcal{H}h^{T'}_{ij} + k^2 h^T_{ij} = 16\pi G a^2 \pi^T_{ij}$$

where $h^T_{ii} = 0$, $\quad \partial^j h^T_{ij} = 0$ traceless and transverse

$$\pi^T_{ij} = \left(\delta^l{}_i - \hat{k}^l\hat{k}_i\right)\left(\delta^m{}_j - \hat{k}^m\hat{k}_j\right)\pi_{lm} + \frac{1}{2}\hat{k}^l\hat{k}^m\pi_{lm}\left(\delta_{ij} - \hat{k}_i\hat{k}_j\right)$$

$\pi^T_{ij} = 0$ for a scalar field

$\hat{k}^i h^T_{ij} = 0$ divergenceless condition

$\gamma^{ij}e_{a,i}\hat{k}_j = 0$, $\quad \gamma^{ij}e_{a,i}e_{b,j} = \delta_{ab}$ where $\{\hat{e}_1, \hat{e}_2\} \perp \hat{k}$

$$h^T_{ij}(\mathbf{k}) = (e_{1,i}e_{1,j} - e_{2,i}e_{2,j})(\hat{k})h_+(\mathbf{k}) + (e_{1,i}e_{2,j} + e_{2,i}e_{1,j})(\hat{k})h_\times(\mathbf{k})$$

- In absence of quadrupole moments $h''_{+,\times} + 2\mathcal{H}h'_{+,\times} + k^2 h_{+,\times} = 0$
- If we choose $\hat{k} = \hat{z}$ (*i.e.*, the propagation direction of a GW along z-direction) and $\hat{e}_1 = \hat{x}$, $\hat{e}_2 = \hat{y}$, then

$$h^T_{ij}(k\hat{z}) = \begin{pmatrix} h_+ & h_\times & 0 \\ h_\times & -h_+ & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

- Prefer to use combinations $h_+ \mp i h_\times \equiv h$ because these have helicity ± 2
- Tensor quantum perturbations are Gaussian with power spectrum (PS) : $P_h(\eta, k) \propto |h(\eta, k)|^2$
- Dimensionless PS :

$$\Delta^2_h(\eta, k) \equiv \frac{k^3 P_h(\eta, k)}{2\pi^2} = \frac{8\pi G}{\pi^2}H^2_\Lambda\left[1 + (k^2\eta^2)\right] = \frac{H^2_\Lambda}{\pi^2 M^2_{\rm Pl}}\left[1 + (k^2\eta^2)\right] \quad \text{only for } k|\eta_i| \gg 1$$

This means that direct measure of GW background determines the energy scale of inflation (*i.e.*, H)

- Horizon crossing condition $k = \mathcal{H}(\eta_k) = 1/[(1-\epsilon)|\eta_k|]$ which at first order in ϵ gives $k|\eta_k| = 1 + \epsilon$
- Thus, PS evaluated at horzion crossing

$$\Delta^2_h(k) = \frac{1}{\pi^2 M^2_{\rm Pl}}\frac{\mathcal{H}(\eta_k)^2}{a^2(\eta_k)} = \left.\frac{H^2}{\pi^2 M^2_{\rm Pl}}\right|_{k=aH}$$

- Spectral index :

$$\Delta^2_T \equiv 2\Delta^2_h \equiv \frac{k^3 P_h(k)}{\pi^2} = \left.\frac{2H^2}{\pi^2 M^2_{\rm Pl}}\right|_{k=aH} \equiv A_T\left(\frac{k}{k_*}\right)^{n_T(k)}$$

PRODUCTION OF SCLAR PERTURBATIONS DURING INFLATION

- Comoving curvature perturbation (or Lukash variable 80) :

$$\mathcal{R} = \Phi - \frac{\mathcal{H}}{\bar{\varphi}'}\delta\varphi$$

- Evolution eq for $\mathcal{R}$ (Mukhanov-Sasaki eq 85, 86) :

$$\mathcal{R}'' + 2\frac{z'}{z}\mathcal{R}' + k^2\mathcal{R} = 0 \quad \text{where} \quad z \equiv \frac{a\bar{\varphi}'}{\mathcal{H}}$$

- Scalar PS :

$$P_{\mathcal{R}} = \left.\frac{H^2}{4M_{\rm Pl}^2\epsilon k^3}\right|_{k=aH} \quad , \quad \Delta^2_{\mathcal{R}} = \left.\frac{H^2}{8\pi^2 M_{\rm Pl}^2\epsilon}\right|_{k=aH}$$

- Spectral index :

$$\Delta^2_S \equiv \Delta^2_{\mathcal{R}} \equiv \frac{k^3 P_{\mathcal{R}}(k)}{2\pi^2} = \left.\frac{H^2}{8\pi^2 M_{\rm Pl}^2\epsilon}\right|_{k=aH} \equiv A_S\left(\frac{k}{k_*}\right)^{n_S(k)-1}$$

LINK TO OBSERVATIONS

•**Planck 2018 results. X : Constraints on inflation**

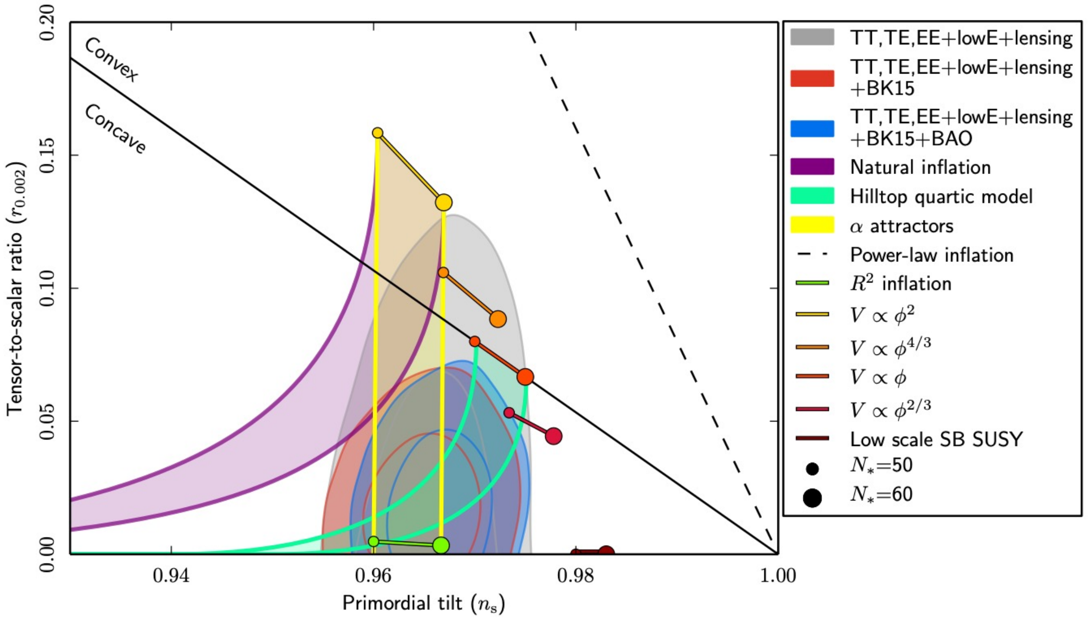


Fig. 8. Marginalized joint 68 % and 95 % CL regions for n_s and r at $k = 0.002\,\mathrm{Mpc}^{-1}$ from *Planck* alone and in combination with BK15 or BK15+BAO data, compared to the theoretical predictions of selected inflationary models. Note that the marginalized joint 68 % and 95 % CL regions assume $dn_s/d\ln k = 0$.

Cosmological model ΛCDM+r	Parameter	*Planck* TT,TE,EE +lowEB+lensing	*Planck* TT,TE,EE +lowE+lensing+BK15	*Planck* TT,TE,EE +lowE+lensing+BK15+BAO
	r	< 0.11	< 0.061	< 0.063
	$r_{0.002}$	< 0.10	< 0.056	< 0.058
	n_s	0.9659 ± 0.0041	0.9651 ± 0.0041	0.9668 ± 0.0037
	r	< 0.16	< 0.067	< 0.068
	$r_{0.002}$	< 0.16	< 0.065	< 0.066
$+dn_s/d\ln k$	n_s	0.9647 ± 0.0044	0.9639 ± 0.0044	0.9658 ± 0.0040
	$dn_s/d\ln k$	-0.0085 ± 0.0073	-0.0069 ± 0.0069	-0.0066 ± 0.0070

- k_* : pivot scale 0.002 Mpc^{-1} or 0.05 Mpc^{-1} (Plnack)
- A_S (A_T) : scalar (tensor) spectral amplitude
- $n_S(k)(n_T(k))$: scalar (tensor) spectral index
- Dimensionless spectra including running of spectra indices (*i.e.*, $d/d\ln k$) :

$$\ln\frac{\Delta_S^2}{A_S} = \left[n_S - 1 + \frac{1}{2}\frac{dn_S}{d\ln k}\ln\frac{k}{k_*} + \frac{1}{6}\frac{d^2 n_S}{d(\ln k)^2}\left(\ln\frac{k}{k_*}\right)^2 + \cdots\right]\ln\frac{k}{k_*},$$
$$\ln\frac{\Delta_T^2}{A_T} = \left[n_T + \frac{1}{2}\frac{dn_T}{d\ln k}\ln\frac{k}{k_*} + \cdots\right]\ln\frac{k}{k_*}$$

- spectral indices

$$n_S - 1 = \frac{d\ln\Delta_S^2}{d\ln k} = \left.\frac{d\ln(H^2/\epsilon)}{d\ln k}\right|_{aH=k} = -4\epsilon + 2\eta = -6\epsilon_V + 2\eta_V$$
$$n_T = \frac{d\ln\Delta_T^2}{d\ln k} = 2\left.\frac{k}{H}\frac{dH}{dk}\right|_{aH=k} = -2\epsilon$$

- Tensor-to-scalar ratio :

$$r_* \equiv \frac{\Delta_T^2(k_*)}{\Delta_S^2(k_*)} = \frac{A_T}{A_S} = 16\epsilon = -8n_T$$

- Energy scale of inflation

$$H_*^2 = \frac{\pi^2 M_{\rm Pl}^2}{2}\Delta_T^2(k_*) = \frac{\pi^2 M_{\rm Pl}^2}{2} r_*\Delta_S^2(k_*) = \frac{\pi^2 M_{\rm Pl}^2}{2} r_* A_S$$
$$\Rightarrow V_* = \frac{3\pi^2 M_{\rm Pl}^4}{2} r_* A_S$$

STOCHASTIC COSMOLOGICAL PERTURBATIONS

- Cosmological perturbations ⟺ statistics
- Set of DEs for δs with ICs (from spatial (scale) dependences : $\delta(\eta, \vec{x})$)
- Ex : Evolution eqs depend only on magnitude k : $|\vec{x}|$ not $\vec{k}$: $\delta_c' + kV_c + 3\Phi' = 0, V_c' + \mathcal{H}V_c - k\Psi = 0$
- IC for CDM : $\delta_c(\vec{k}) = \frac{15}{15+4R_\nu}\zeta(\vec{k})$
- IC from quantum origin = probability feature (random variables)
- If probability distribution is Gaussian (all information in its variance = PS)
- Descriptive statistics (Due to signals from past light cone)

RANDOM FIELDS

- A function $G(\vec{x})$: a random field (RF) with g a certain value
- Probability distribution function $p_1(g_1) = \frac{dF_1(g_1)}{dg_1}$ where F is a cumulative probability : $F_1(-\infty) = 0, F_1(\infty) = 1$
- In cosmology, $G(\vec{x})$ is a perturbative quantity, such as $\delta(\overrightarrow{x})$
- Ensemble average = an expectation value of random field : $< G(\overrightarrow{x_1}) > \equiv \int_\Omega g_1 p_1(g_1) dg_1$ where Ω denotes ensemble
- For a statistically homogeneous random field, ensemble average becomes independent of $\vec{x}$: $< G \geq \int_\Omega gp(g)dg$

TWO-POINT CORRELATION FUNCTION

- Probability of $G(\overrightarrow{x_1})$ and $G(\overrightarrow{x_2})$ being g_1 and g_2 : $p_{12}(g_1, g_2)dg_1\, dg_2$: can be written as derivatibe of distribution function of F_{12}
- In general, $p_{12}(g_1, g_2) \neq p_1(g_1)p_2(g_2)$
- $p_{12}(g_1, g_2) = p_1(g_1)p_2(g_2)$: when realisations are independent (Poissonian random process)
- Two-point correlation function (2-pt CF) : $\xi(x_1, x_2) =< G(x_1)G(x_2) >\equiv \int_\Omega g_1 g_2\, p_{12}(g_1, g_2)dg_1\, dg_2$
- One can generalize to N-point CFs
- For a statistically homogeneous RF, 2-pt CF becomes $\xi(x_1, x_2)= \xi(x_1 - x_2)$
- Statistically isotropic : $p_1(g_1\ ) = P_{R1}(g_{R1})$ where $x_{R1} = R(x_1)$ for a rotation matrix R
- If a RF is statistically hom + iso, then 2-pt CF becomes $\xi(x_1, x_2)= \xi(x_1 - x_2)= \xi(r_{12}\ )$: depends only on the distance btw two points

ERGODIC THEOREM

- Assuming statistical homogeneity :

$$\langle X^2 \rangle = \frac{1}{V^2}\int_V d^3\mathbf{x}_1 \int_V d^3\mathbf{x}_2\, \xi(\mathbf{x}_1 - \mathbf{x}_2) - \langle G \rangle^2$$
$$= \frac{1}{V}\int_V d^3\mathbf{r}\, \xi(\mathbf{r}) - \langle G \rangle^2$$

- Ergodic theorem : $\langle X^2 \rangle \to 0$ if $V \to \infty$
- Cosmic variance : In practice, V is finite and thus $\langle X^2 \rangle \neq 0$ in general
- Assuming statistical isotropy and using a spherical volume, then

$$\langle X^2 \rangle = \frac{3}{R^3}\int_0^R dr r^2 \xi(r) - \langle G \rangle^2$$

- Application of random fields to cosmology
- These are done in configuration space whereas cosmological perturbations are done in Fourier modes
- Assume that FT of a RF is also a RF

- Observation : probe a realization in a finite $V = L^3$
- FT is defined as a Fourier series :

$$G(\mathbf{x}) = \frac{1}{L^3}\sum_n G_n e^{i\mathbf{k}_n \cdot \mathbf{x}}$$

where

$$G_n = \int d^3\mathbf{x} G(\mathbf{x}) e^{-i\mathbf{k}_n \cdot \mathbf{x}} \quad , \quad \mathbf{k}_n = \frac{2\pi}{L}\mathbf{n}$$

- If $L \to \infty$, then

$$G(\mathbf{x}) = \int \frac{d^3\mathbf{k}}{(2\pi)^3}\tilde{G}(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{x}} \quad , \quad \tilde{G}(\mathbf{k}) = \int d^3\mathbf{x} G(\mathbf{x}) e^{-i\mathbf{k}\cdot\mathbf{x}}$$

- If $G(\mathbf{x})$ is a real field, then $\tilde{G}(-\mathbf{k}) = \tilde{G}^*(\mathbf{k})$

- 2-pt CF for FT of $G(\mathbf{x})$:

$$\langle \tilde{G}(\mathbf{k})\tilde{G}^*(\mathbf{k}')\rangle = \int d^3\mathbf{x} \int d^3\mathbf{x}' \langle G(\mathbf{x})G(\mathbf{x}')\rangle e^{-i\mathbf{k}\cdot\mathbf{x}} e^{i\mathbf{k}'\cdot\mathbf{x}'}$$

- Assuming statistical homogeneity

$$\begin{aligned}\langle \tilde{G}(\mathbf{k})\tilde{G}^*(\mathbf{k}')\rangle &= \int d^3\mathbf{x} \int d^3\mathbf{x}' \xi_G(\mathbf{x}'-\mathbf{x}) e^{-i\mathbf{k}\cdot\mathbf{x}} e^{i\mathbf{k}'\cdot\mathbf{x}'} \\ &= \int d^3\mathbf{x}\, e^{-i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{x}} \boxed{\int d^3\mathbf{z}\; \xi_G(\mathbf{z}) e^{i\mathbf{k}'\cdot\mathbf{z}}} \\ &\equiv (2\pi)^3 \delta^{(3)}(\mathbf{k}-\mathbf{k}') P_G(\mathbf{k}')\end{aligned}$$

- **Power spectrum (PS)** $\equiv$ FT of the 2-pt CF
- If statistical isotropy is assumed, then

$$\begin{aligned}P_G(k) &= 2\pi \int_0^\infty dr\, r^2\, \xi_G(r) \int_{-1}^{1} du\, e^{-ikru} \quad \text{where} \quad \mu \equiv \cos(\mathbf{k}\cdot\mathbf{x}) \\ &= 4\pi \int_0^\infty dr\, r^2\, \xi_G(r) \frac{\sin(kr)}{kr}\end{aligned}$$

- Gaussianity implies statistical homogeneity
- Expecation value and all the odd power correlators are vanishing

$$\langle \tilde{G}(\mathbf{k})\rangle = \langle \tilde{G}(\mathbf{k}_1)\tilde{G}(\mathbf{k}_2)\tilde{G}(\mathbf{k}_3)\rangle = \cdots = 0$$

- All even order correlators can be written in terms of the second-order correlators (PS)

$$\begin{aligned}\langle \tilde{G}(\mathbf{k}_1)\tilde{G}(\mathbf{k}_2)\tilde{G}(\mathbf{k}_3)\tilde{G}(\mathbf{k}_4)\rangle &= \langle \tilde{G}(\mathbf{k}_1)\tilde{G}(\mathbf{k}_2)\rangle\langle \tilde{G}(\mathbf{k}_3)\tilde{G}(\mathbf{k}_4)\rangle \\ &+ \langle \tilde{G}(\mathbf{k}_1)\tilde{G}(\mathbf{k}_3)\rangle\langle \tilde{G}(\mathbf{k}_2)\tilde{G}(\mathbf{k}_4)\rangle + \langle \tilde{G}(\mathbf{k}_1)\tilde{G}(\mathbf{k}_4)\rangle\langle \tilde{G}(\mathbf{k}_2)\tilde{G}(\mathbf{k}_3)\rangle\end{aligned}$$

- Fourier modes are uncorrelated, their superposition is Gaussian-distributed

$$p(g) = \frac{1}{\sqrt{2\pi\sigma_g}} e^{-g^2/2\sigma_g^2}$$

where $\langle g\rangle = 0$ and use g instead of $G(\mathbf{x})$ due to statistical homogeneity

GAUSSIAN RFS & POWER SPECTRUM

OBSERVATIONS & COSMIC VARIANCE

- Theoretical prediction : $P_G(k)$, Observational measurement : $\xi_G(r)$
- 2-pt CF :

$$\xi_G(r) = \langle G(\mathbf{x})G(\mathbf{x}+\mathbf{r})\rangle = \int \frac{d^3\mathbf{k}}{(2\pi)^3}\int \frac{d^3\mathbf{k}'}{(2\pi)^3}\langle \tilde{G}(\mathbf{k})\tilde{G}^*(\mathbf{k}')e^{i\mathbf{k}\cdot\mathbf{x}-i\mathbf{k}'\cdot(\mathbf{x}+\mathbf{r})}\rangle$$
$$= \int \frac{d^3\mathbf{k}}{(2\pi)^3}\int d^3\mathbf{k}' P_G(k)\delta^{(3)}(\mathbf{k}-\mathbf{k}')e^{i\mathbf{k}\cdot\mathbf{x}-i\mathbf{k}'\cdot(\mathbf{x}+\mathbf{r})} = \int \frac{d^3\mathbf{k}}{(2\pi)^3}P_G(k)e^{-i\mathbf{k}\cdot\mathbf{r}}$$

ξ : dimensionless , P_G : dimension of volume

$$\xi_G(r) = \int \frac{dk\, k^2}{(2\pi)^2}P_G(k)\int_{-1}^{1} du\; e^{-ikru} = \int_0^\infty dk\, \frac{k^2 P_G(k)}{2\pi^2}\frac{\sin(kr)}{kr}$$
$$= \int_0^\infty \frac{dk}{k}\,\Delta_G^2(k)\frac{\sin(kr)}{kr} \quad \text{where} \quad \Delta_G^2(k) \equiv \frac{k^3 P_G(k)}{2\pi^2}$$

- Suppose none of quantities in the integrand of the above Eq is a stochastic variable and $\langle\cdots\rangle$ is a spatial average

$$\hat{\xi}_G(r) = \frac{1}{V}\int_V d^3\mathbf{x}\sum_{n,m}\frac{1}{V^2}G_n G_m^* e^{i\mathbf{k}_n\cdot\mathbf{x}-i\mathbf{k}_m\cdot(\mathbf{x}+\mathbf{r})}\ , \quad G_n = \tilde{G}(\mathbf{k}_n)\ , \quad \mathbf{k}_n = \frac{2\pi}{L}\mathbf{n}$$
$$= \sum_n \frac{1}{V^2}|G_n|^2 e^{-i\mathbf{k}_n\cdot\mathbf{r}} \quad \text{where} \quad P_n \equiv P(\mathbf{k}_n) = \frac{|G_n|^2}{V}$$

where $V = L^3$: survey volume and use the fact that spatial integration is

$$\frac{1}{L^3}\int_V d^3\mathbf{x}\; e^{i(\mathbf{k}_n-\mathbf{k}_m)\cdot\mathbf{x}} = \delta_{nm}$$

- Cosmic variance of PS

$$\frac{\sigma_P(k)}{P(k)} \simeq \frac{1}{\sqrt{N_k}} = \begin{cases} \frac{1}{r_k^{1/2}(kL)^{3/2}} \\ \frac{1}{kL} \end{cases} \qquad \text{where} \quad N_k = \begin{cases} \frac{1}{2\pi^2}(kL)^3 r_k & \text{for square} \\ \frac{(kL)^2}{\pi} & \text{for sphere} \end{cases}$$

$r_k \equiv dk/k$: resolution of the survey

NON-GAUSSIANITY

- For non-Gaussian perturbations, odd-order correlatpors are non-zero
- Bispectrum $B_G(k_1, k_2, k_3)$ is defined by a FT of a 3-point CF

$$\langle \tilde{G}(\mathbf{k}_1)\tilde{G}(\mathbf{k}_2)\tilde{G}(\mathbf{k}_3)\rangle = (2\pi)^3\delta^{(3)}(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3)B_G(k_1, k_2, k_3)$$

where the relation between bispectrum and reduced bispectrum $\mathcal{B}_G(k_1, k_2, k_3)$ is given by

$$B_G(k_1, k_2, k_3) = \mathcal{B}_G(k_1, k_2, k_3)\left[P_G(k_1, k_2) + P_G(k_2, k_3) + P_G(k_1, k_3)\right]$$

- This formula can be obtained using following expansion (local type non-Gaussianity : square of a Guassian RF is not Gaussian)

$$\begin{aligned} G(\mathbf{x}) &= G_G(\mathbf{x}) + f_{NL}\left(G_G^2(\mathbf{x}) - \langle G_G^2(\mathbf{x})\rangle\right) + \dots \\ &= G_G(\mathbf{x}) + f_{NL}\left(G_G^2(\mathbf{x}) - \sigma_G^2\right) + \dots \end{aligned}$$

- $f_{\rm NL} = 2.5 \pm 5.7$: Planck 16
- This is a primordial non-Gaussianity. Non-Gaussianity naturally arises in non-linear regime of evolution

OBSERVABLES
(REFER APPLEBY'S TALK FOR DETAILS)

"the devil is in the detail" look simple

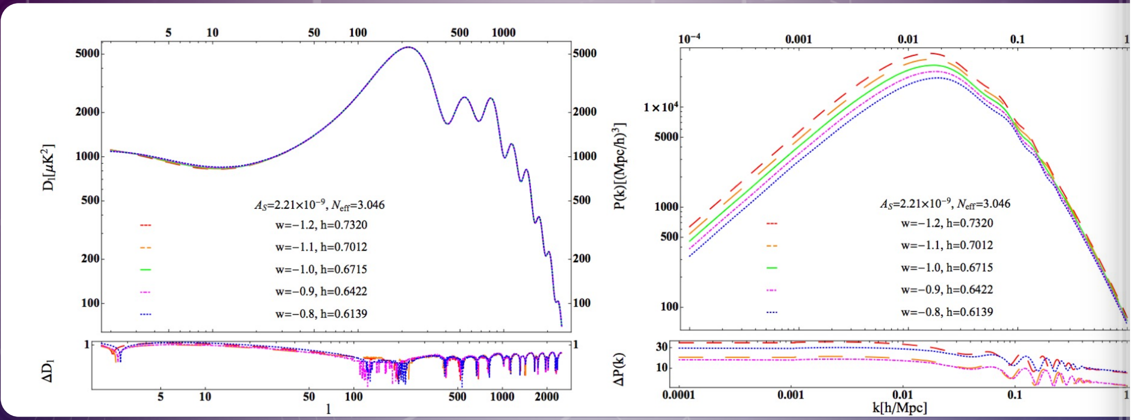
$$\underbrace{\mathcal{O}_k^X(\tau)}_{\text{link to Measurement}} = T_{\mathcal{O}}^X(k,\tau,\tau_*) \underbrace{\mathcal{R}_k(\tau_*)}_{\text{Curvature perturbation}} \xrightarrow{inflaton} T_{\mathcal{O}}^X(k,\tau,\tau_*) \underbrace{\frac{\mathcal{H}}{\bar{\phi}'}\delta\phi_k}_{\text{no evolution}}$$

$$\langle \mathcal{O}_k^X \rangle = 0 \ : \text{Quantum fluctuation} \rightarrow \text{random distribution}$$

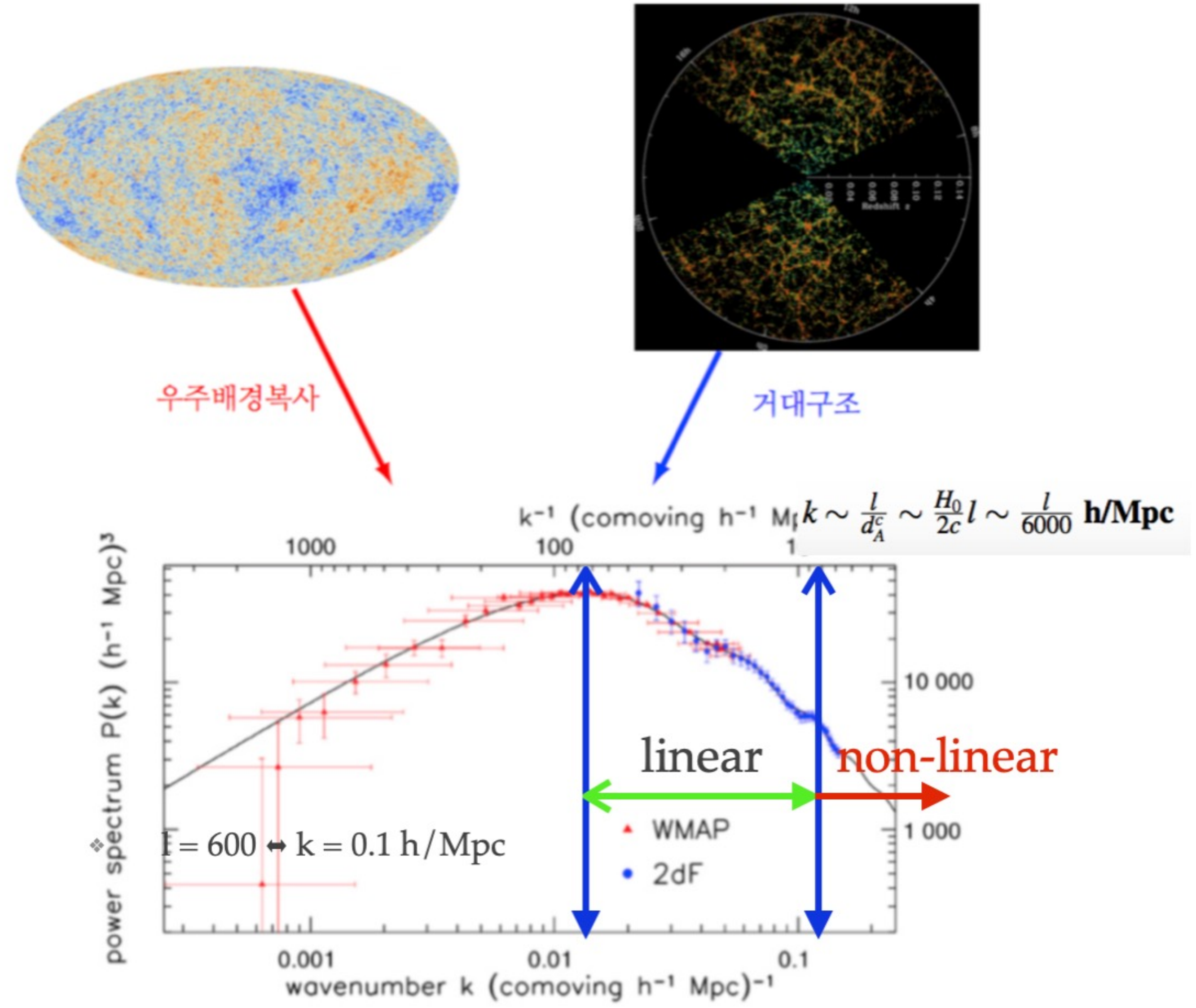
$$P_k^X \equiv \langle \mathcal{O}_k^X \mathcal{O}_k^X \rangle \ : \text{measurements, X : CMB (T, E, B), LSS 測量}$$

$$B_k^X \equiv \langle \mathcal{O}_{k_1}^X \mathcal{O}_{k_2}^X \mathcal{O}_{k_3}^X \rangle = 0 \ : \text{Gaussianity}$$

become non-linear
at small scales
modes mixing

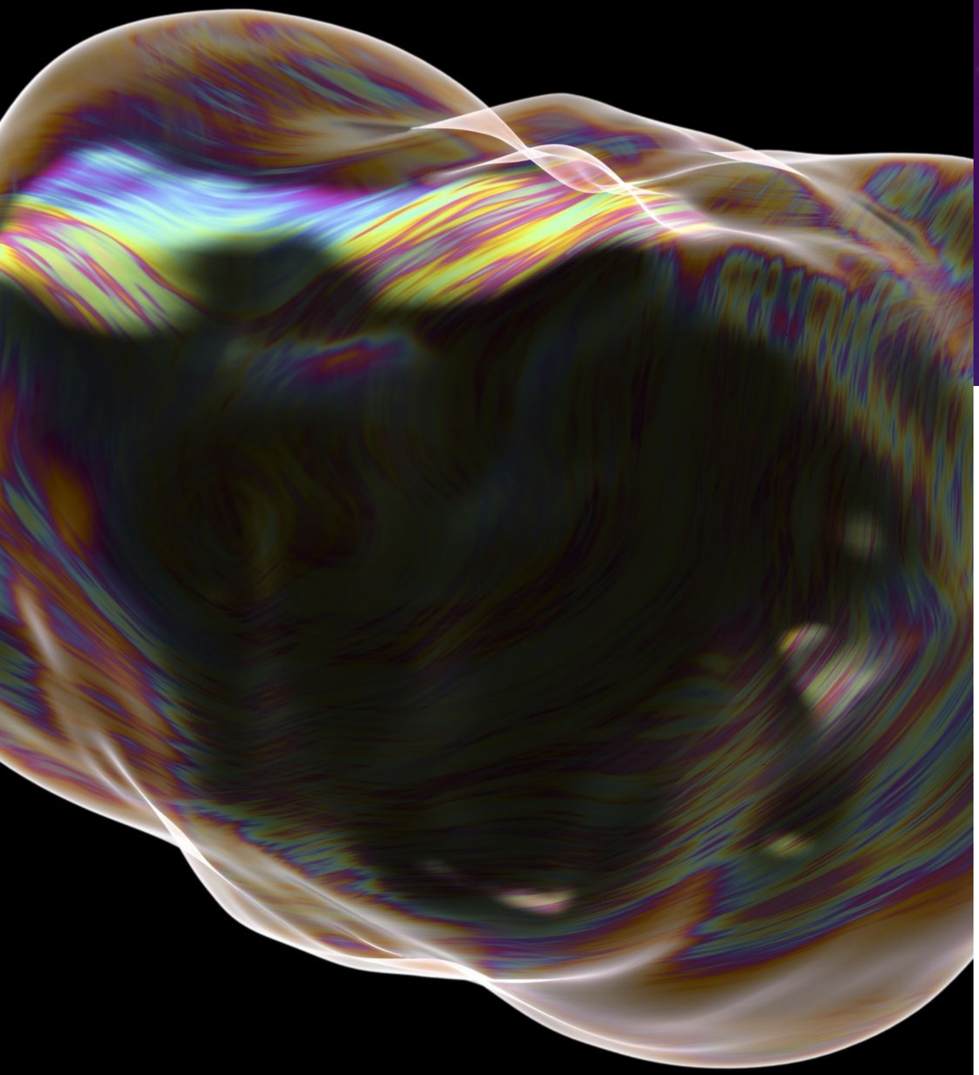


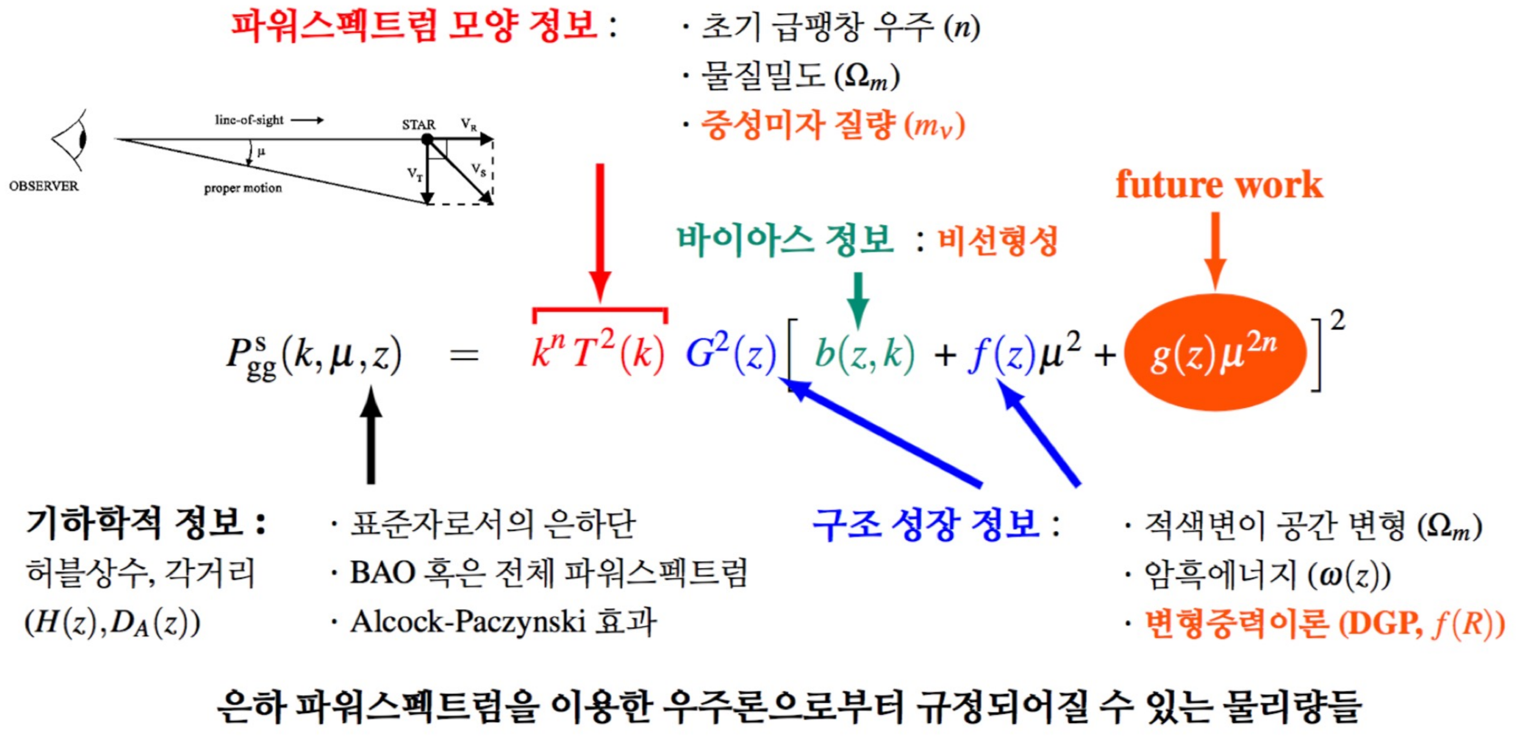
LSS AND CMB
(DETAILS IN THE SECOND REVIEW)



우주배경복사 (CMB)와 은하 거대구조 (LSS)을 이용한 관측량과 이론에서 얻어진 물질 파워스펙트럼

LARGE SCALE STRUCTURE (MATTER POWER SPECTRUM)





COSMIC MICROWAVE BACKGROUND ANISOTROPY

유효온도 : · 물질밀도 (Ω_m) · 복사밀도 (Ω_γ)

중력포텐셜 변화 : · 암흑에너지 (Ω_{DE}) · 암흑물질 (Ω_m)

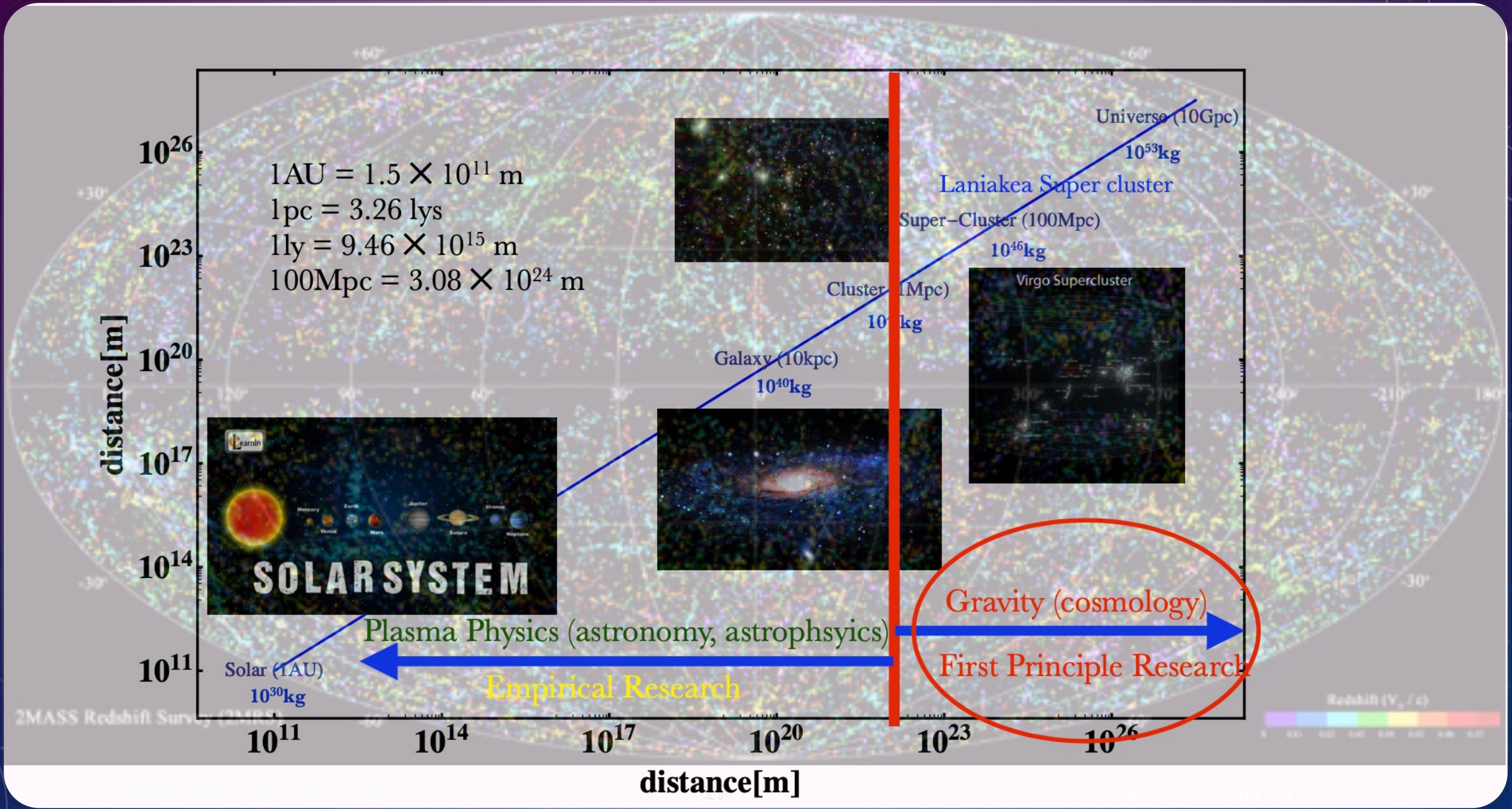
$$C_l^{TT} = \frac{2}{\pi}\int k^2 dk \; P_\zeta(k)\Big[(\Phi + \tfrac{1}{4}\delta_\gamma) j_l(k[\tau_0 - \tau_{rec}]) + v_\gamma j_l'(k[\tau_0 - \tau_{rec}]) + 2\int_{\tau_{rec}}^{\tau_0} d\tau \frac{\Phi'(\tau)}{\Phi'(\tau_0)} j_l(k[\tau_0 - \tau_{rec}])\Big]^2$$

원형 파워스펙트럼 : · 초기 급팽창 우주 (n) · 섭동 크기

도플러 : · 편광정보 · 관측시 제거

우주배경복사의 온도 비등방성 파워스펙트럼을 이용한 물리량들

OBSERVATIONAL LIMITS



OBSERVATIONAL CHALLENGES

(WHEN **Z**, WHERE **K,L**, WHO EARTH, WHAT **Γ,Ν,Δ**, HOW **GROUND, SATELLITE**)

Theories can never be proved, only disproved

Overemphasize on small scale data

TeVeS
from
Rot
Curve

DGP
from
LSS