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Analytic approach to the formation of a three-dimensional black string from a dust cloud

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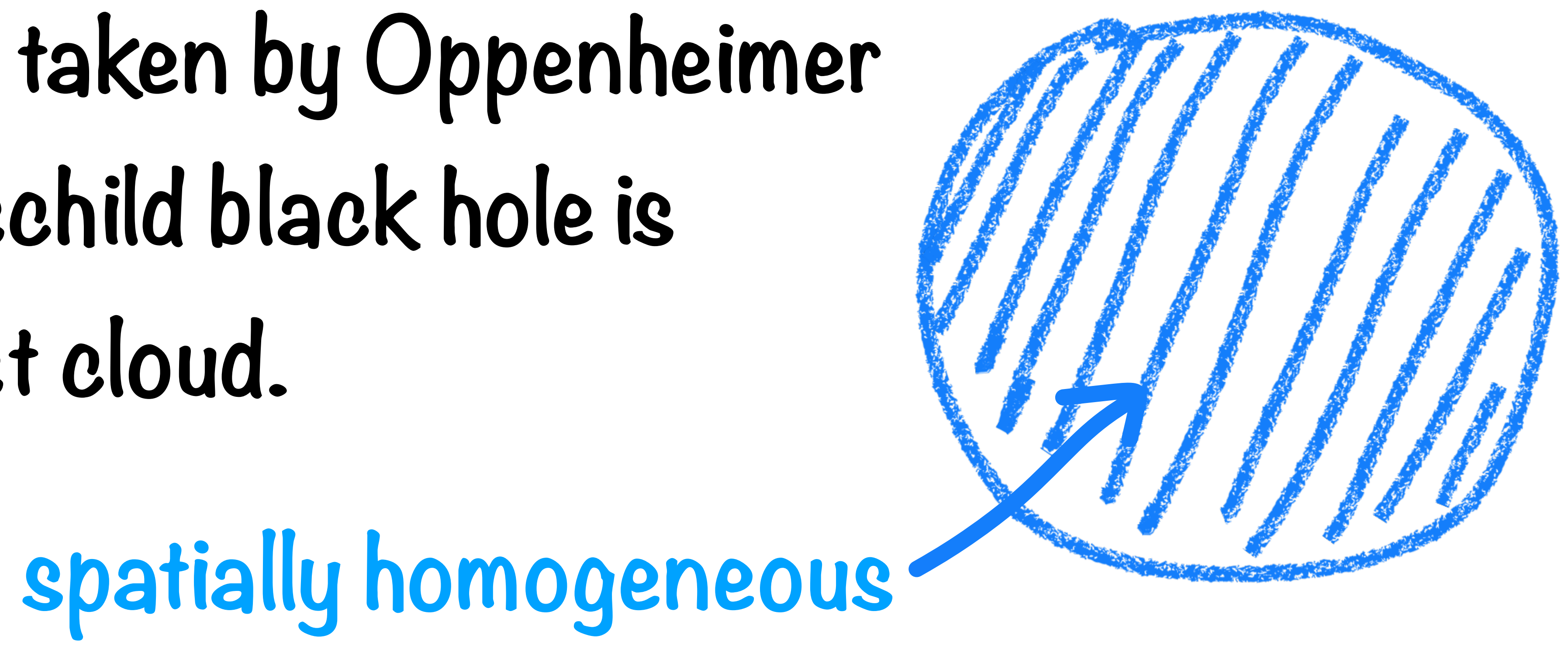
Based on collaboration with Wontae Kim [gr-qc/2202.11327]

Motivation

The first theoretical approach to collapsing stars was taken by Oppenheimer and Snyder in 1939, and they showed that Schwarzschild black hole is formed from the collapse of spherically symmetric dust cloud.

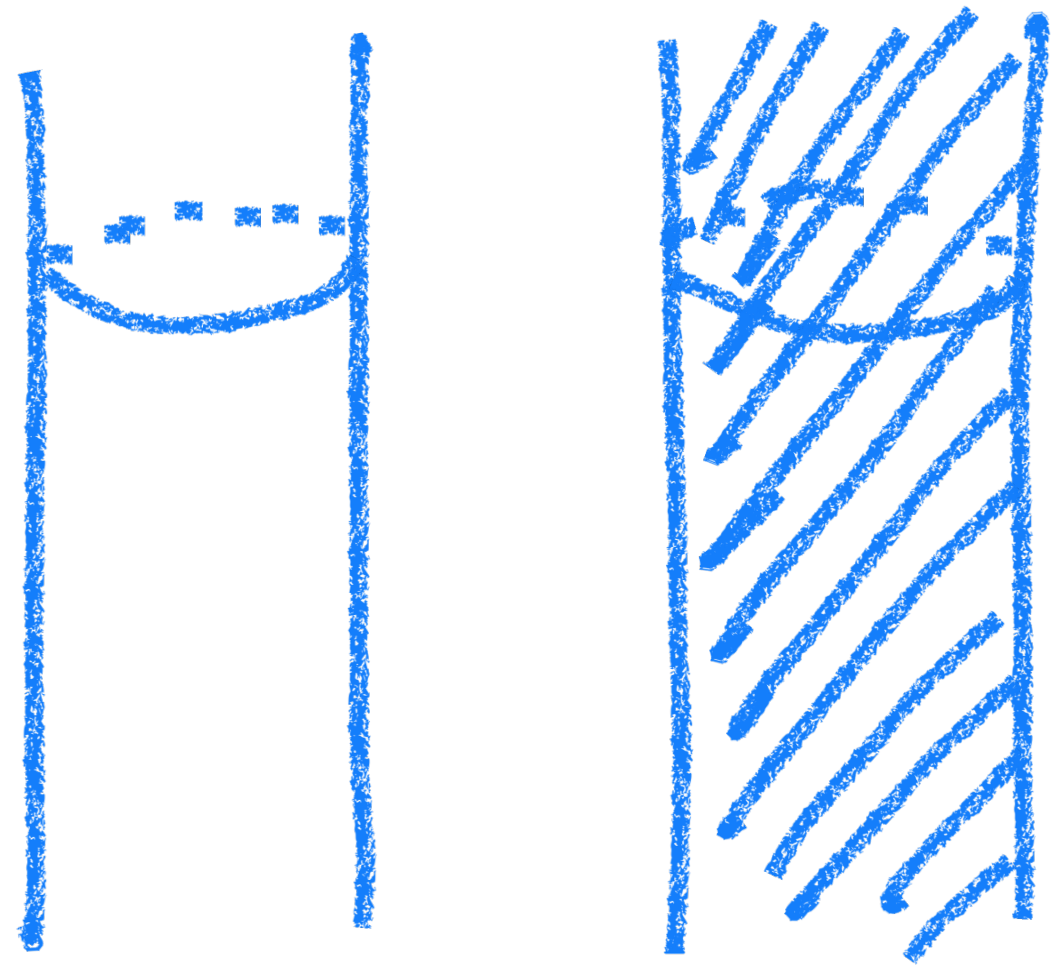
[Oppenheimer, Snyder (1939)]



To generalize the geometry of collapsing objects, cylindrically symmetric spacetime has also been investigated.

- infinitely long cylindrical shell (thin shell of dust) [Thorne (1972)] [Apostolatos, Thorne (1992)] [Echeverria (1994)]
- homogeneous dust cloud [Senovilla,Vera (1997)] [Tod, Mena (2004)]
- fluids, massless scalar fields, so on.

[Di Prisco et el. (2009)]
[Greenwood et el. (2010)]
[Wang (2003)]



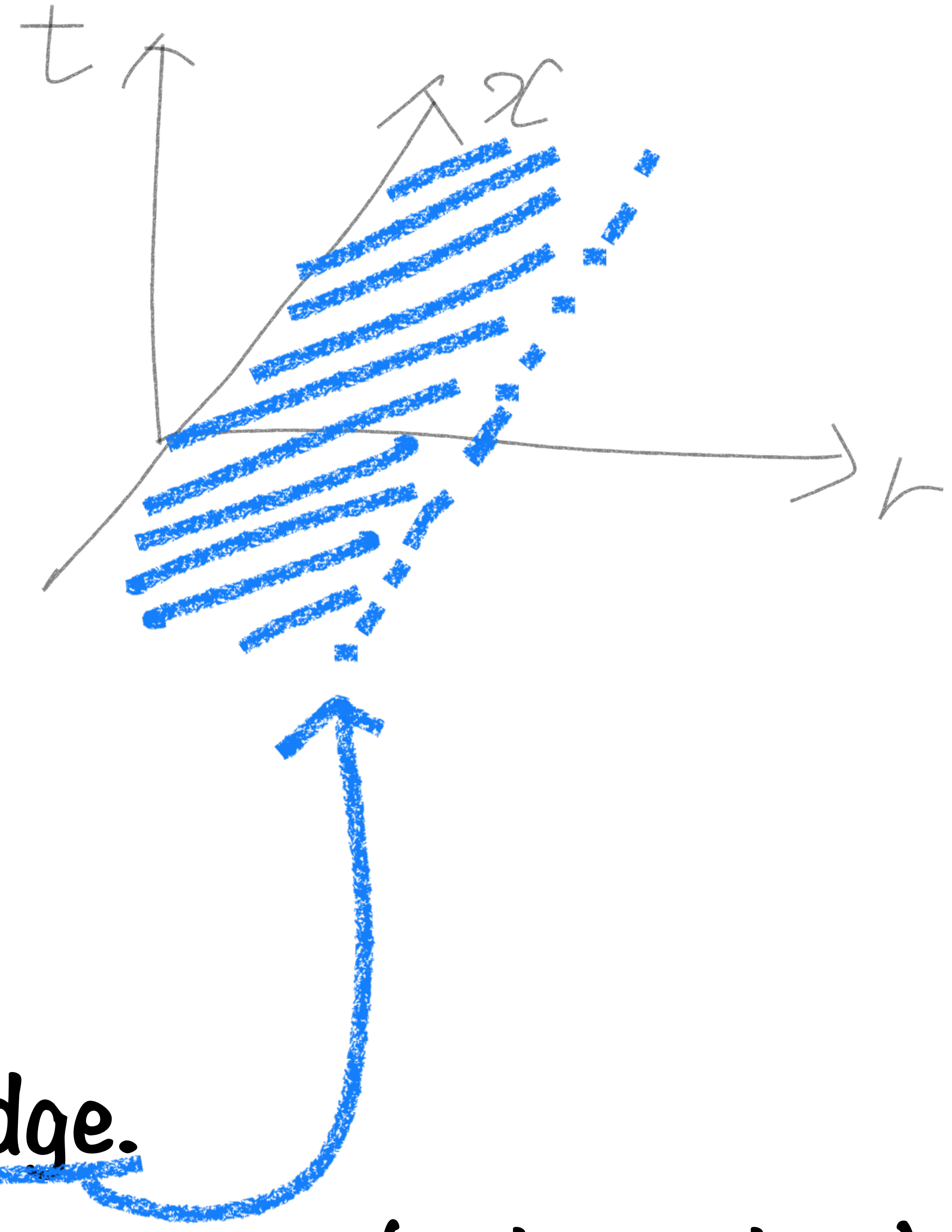
Motivation

In particular, in three dimensions, it has been known that a spherically symmetric black hole (BTZ black hole) can be formed through homogeneous and inhomogeneous dust clouds.

[Ross, Mann (1993)] [Gutti (2005)]

How about a cylindrically symmetric dust cloud in three dimensions?

Can cylindrical black holes (black strings) be formed from collapsing?



[Banados,Teitelboim,Zanelli (1992)] [Horowitz and Welch (1993)]

Non-rotating BTZ black hole =(dual)=> uncharged black string

[Hyun, Jeong,Kim,Oh (2007)]

This subject was investigated from the numerical argument on the dust edge.

Interestingly, it was claimed that the dust cloud should be spatially inhomogeneous (r-dependent).

Let us start with the action given as

$$S = \frac{1}{2\kappa^2} \int d^3x \sqrt{-g} e^{-2\phi} \left(g^{\mu\nu} R_{\mu\nu} + 4 \left(\nabla \phi \right)^2 + \frac{4}{\ell^2} \right) + S_M,$$

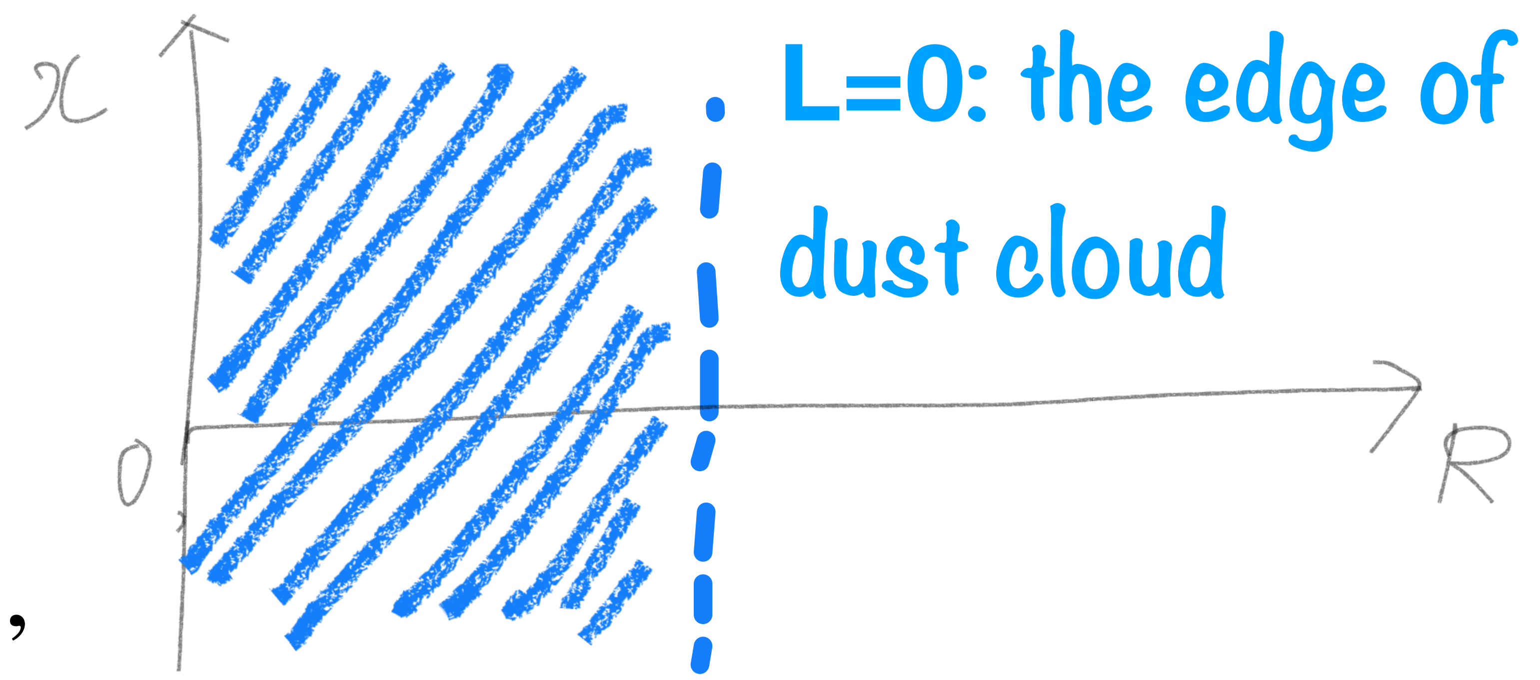
where κ^2 is a parameter length scale, ϕ is a dilation field, ℓ is the anti-de sitter radius and S_M is a mass action.

The equations of motions with respect to $g_{\mu\nu}$ and ϕ are obtained as

$$e^{-2\phi} \left(R_{\mu\nu} + 2 \nabla_\mu \nabla_\nu \phi \right) = T^M_{\mu\nu},$$

$$g^{\mu\nu} R_{\mu\nu} - 4 \left(\nabla \phi \right)^2 + 4 \Box \phi + \frac{4}{\ell^2} = 0,$$

where $T^M_{\mu\nu} = \left(-2/\sqrt{-g} \right) \left(\delta S_M / \delta g^{\mu\nu} \right)$.



The combined metric tensor and the dilation field are

$g_{\mu\nu} = \Theta(L) g_{\mu\nu}^{(\text{out})} + \Theta(-L) g_{\mu\nu}^{(\text{in})}$ and $\phi = \Theta(L)\phi^{(\text{out})} + \Theta(-L)\phi^{(\text{in})}$,

where (out) and (in) in the superscripts denote the outer and inner space, respectively.

[Israel (1996)]

Then the EOM for the metric tensor are divided into 3 parts:

$$\Theta(L)e^{-2\phi^{(\text{out})}}\left(R_{\mu\nu}^{(\text{out})} + 2\nabla_\mu^{(\text{out})}\nabla_\nu^{(\text{out})}\phi^{(\text{out})}\right) + \Theta(-L)e^{-2\phi^{(\text{in})}}\left(R_{\mu\nu}^{(\text{in})} + 2\nabla_\mu^{(\text{in})}\nabla_\nu^{(\text{in})}\phi^{(\text{in})}\right)$$

$$+\delta(L)e^{-2\phi}\left\{\frac{1}{2}\left(n_\mu\kappa_{\nu\alpha}n^\alpha + \kappa_{\mu\alpha}n^\alpha n_\nu - \kappa_{\mu\nu} - \kappa^\alpha_\alpha n_\mu n_\nu\right) + 2n_\mu\left[\partial_\nu\phi\right]\right\} = T^M_{\mu\nu}$$

The combined metric tensor and the dilation field are

$g_{\mu\nu} = \Theta(L) g_{\mu\nu}^{\text{(out)}} + \Theta(-L) g_{\mu\nu}^{\text{(in)}}$ and $\phi = \Theta(L)\phi^{\text{(out)}} + \Theta(-L)\phi^{\text{(in)}}$,

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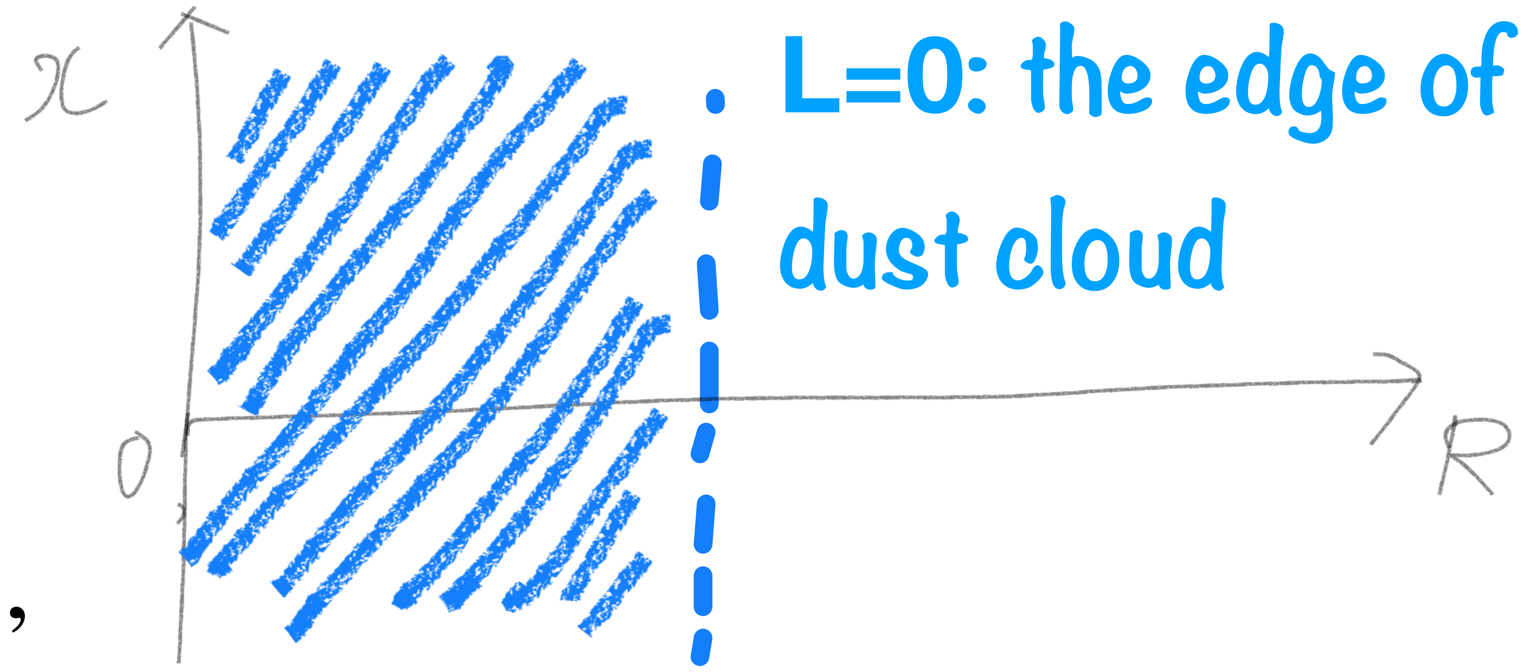


[Israel (1996)]

Then the EOM for the metric tensor is divided into 3 parts:

The first junction condition is assumed. (will be explained in the next slide)

$$\Theta(L) e^{-2\phi^{\text{(out)}}} \left(R_{\mu\nu}^{\text{(out)}} + 2\nabla_\mu^{\text{(out)}} \nabla_\nu^{\text{(out)}} \phi^{\text{(out)}} \right) + \Theta(-L) e^{-2\phi^{\text{(in)}}} \left(R_{\mu\nu}^{\text{(in)}} + 2\nabla_\mu^{\text{(in)}} \nabla_\nu^{\text{(in)}} \phi^{\text{(in)}} \right)$$

the outer space — the inner space

$$+\delta(L) e^{-2\phi} \left\{ \frac{1}{2} \left(n_\mu \kappa_{\nu\alpha} n^\alpha + \kappa_{\mu\alpha} n^\alpha n_\nu - \kappa_{\mu\nu} - \kappa^\alpha_\alpha n_\mu n_\nu \right) + 2 n_\mu [\partial_\nu \phi] \right\} = T^M_{\mu\nu}$$

on the edge

defined only on the edge

$[A] = A\big|_{L\to 0+} - A\big|_{L\to 0-}$

$[\partial_\gamma g_{\mu\nu}] = n_\gamma \kappa_{\mu\nu}$

n_μ: a unit vector normal to $L = 0$

Likewise, the EOM for the dilaton field is

$$\Theta(L)\left(g^{(\text{out})\mu\nu}R^{(\text{out})}_{\mu\nu}-4\left(\nabla^{(\text{out})}\phi^{(\text{out})}\right)^2+4\,\Box^{(\text{out})}\phi^{(\text{out})}+\frac{4}{\ell^2}\right)$$
$$+\Theta(-L)\left(g^{(\text{in})\mu\nu}R^{(\text{in})}_{\mu\nu}-4\left(\nabla^{(\text{in})}\phi^{(\text{in})}\right)^2+4\,\Box^{(\text{in})}\phi^{(\text{in})}+\frac{4}{\ell^2}\right)+\delta(L)\left(n^\alpha\kappa_{\alpha\beta}n^\beta-\kappa^\alpha_\alpha+n^\alpha\left[\partial_\alpha\phi\right]\right)=0.$$

So the stress tensor can also be expressed as

$$T^M_{\mu\nu}=\Theta(L)T^{M(\text{out})}_{\mu\nu}+\Theta(-L)T^{M(\text{in})}_{\mu\nu}+\delta(L)T^{M(\text{edge})}_{\mu\nu}.$$

Dust cloud $\Rightarrow$ $T^{M(\text{out})}_{\mu\nu}=T^{M(\text{edge})}_{\mu\nu}=0,\qquad T^{M(\text{in})}_{\mu\nu}=\rho u_\mu u_\nu$

Now we are in a position to introduce junction conditions on the edge.

The first junction condition is given as

$$[g_{\mu\nu}] = 0, \quad [\phi] = 0,$$

which should be satisfied.

Also, the absence of matter on the edge ($T_{\mu\nu}^{M(\text{edge})} = 0$) gives the second junction condition.

$$[K_{\mu\nu}] = 0, \quad n^{\alpha}[\partial_{\alpha}\phi] = 0,$$

where the extrinsic curvature of the hypersurface of $L = 0$ is $K_{\mu\nu} = h_{\mu}^{\gamma}\nabla_{\gamma}n_{\nu}$ with $h_{\mu\nu} = g_{\mu\nu} - n_{\mu}n_{\nu}$.

We considered the following line elements:

$$ds^2_{\text{(out)}} = -\left(1-\frac{M}{R}\right)dT^2 + \frac{\ell^2}{4R^2}\left(1-\frac{M}{R}\right)^{-1}dR^2 + d\sigma^2, \quad \phi^{\text{(out)}}(R) = -\frac{1}{2}\ln\left(R\ell\right),$$

(M: mass parameter of the black string, R: radial coordinate)

$$ds^2_{\text{(in)}} = -dt^2 + a^2(t,r)dr^2 + d\sigma^2, \qquad \phi^{\text{(in)}} = \phi^{\text{(in)}}(t,r),$$

((t, r, σ) :comoving coordinates, $a(t,r)$: a scale factor, r_0 : constant)

$$L = 0 \\ \Leftrightarrow R = \mathscr{R}(t) \\ \Leftrightarrow r = r_0$$

$$[g_{\mu\nu}] = 0, \quad [\phi] = 0,$$

1st junction condition

$$[K_{\mu\nu}] = 0, \quad n^{\alpha}[\partial_\alpha \phi] = 0,$$

(1) → 2nd junction condition

$$\dot{T} = \frac{1}{F(\mathscr{R})}\sqrt{F(\mathscr{R}) + \frac{\ell^2\dot{\mathscr{R}}^2}{4\mathscr{R}^2}}$$

Time relation (1)

$$\phi^{\text{(out)}}(\mathscr{R}(t)) = \phi^{\text{(in)}}(t, r_0)$$

(2)

$$-\frac{2\mathscr{R}}{\ell\dot{\mathscr{R}}}\frac{d}{dt}\sqrt{F(\mathscr{R}) + \frac{\ell^2\dot{\mathscr{R}}^2}{4\mathscr{R}^2}} = 0$$

(3)

$$\sqrt{\frac{F(\mathscr{R})}{\ell^2} + \frac{\dot{\mathscr{R}}^2}{4\mathscr{R}^2}} + \frac{\phi^{\text{(in)}}{}'(t, r_0)}{a(t, r_0)} = 0$$

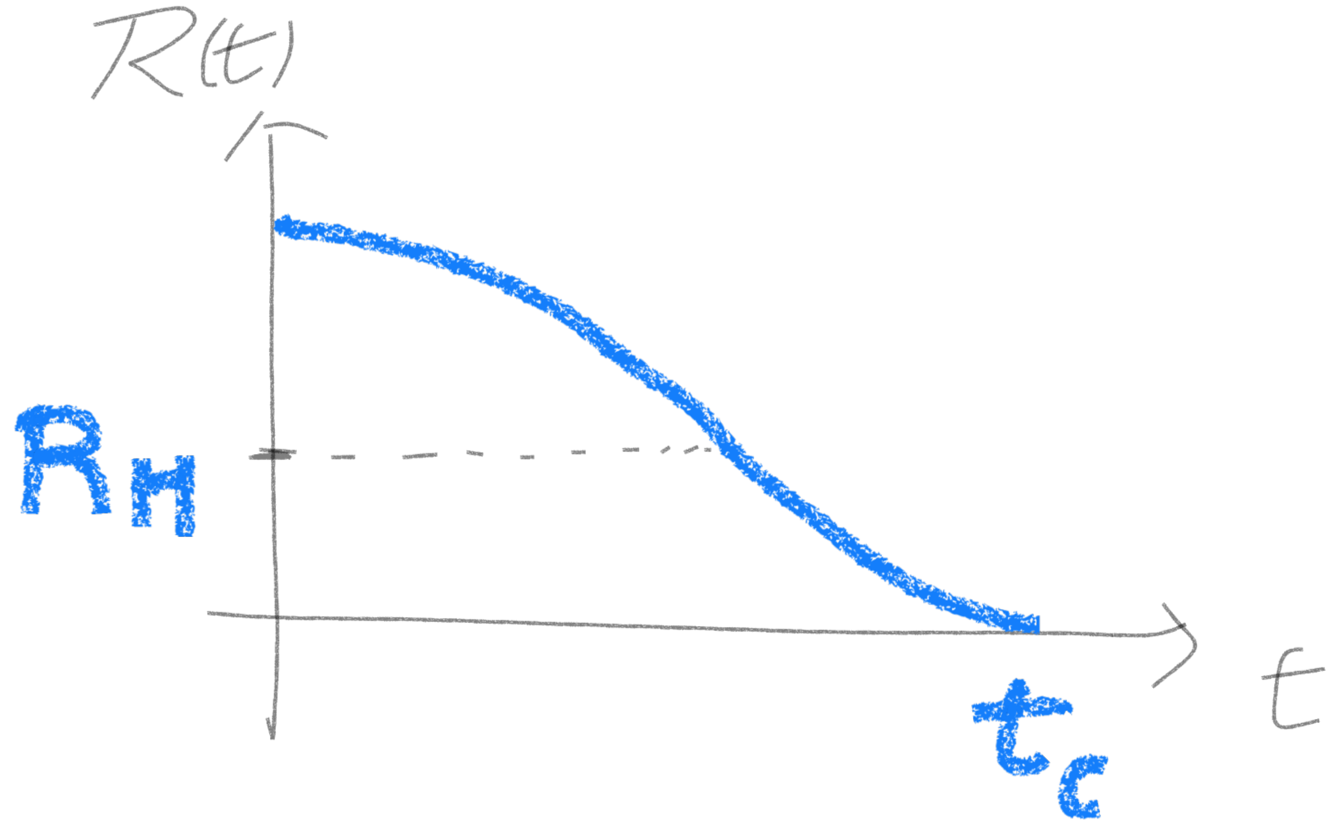
(4)

From ③, we get $\mathscr{R}(t)$..

the inner time coordinate

the outer radial coordinate

$$\mathscr{R}(t) = \frac{M}{1-\eta^2}\sin^2\left(\frac{\sqrt{1-\eta^2}}{\ell}(t-t_c)\right),$$



where a collapse time t_c is defined by $\mathscr{R}(t_c) = 0$ which gives

$$t_c = \frac{\ell}{\sqrt{1-\eta^2}}\sin^{-1}\sqrt{\frac{(1-\eta^2)\mathscr{R}_0}{M}}.$$

So we should check that the outer and inner solutions should satisfy ② and ④ on the edge.

Note that ④ indicates that the dilation solution is spatially inhomogeneous on the edge.

$$④: \sqrt{\frac{F(\mathscr{R})}{\ell^2} + \frac{\dot{\mathscr{R}}^2}{4\mathscr{R}^2}} + \frac{\phi^{(\mathrm{in})\prime}(t, r_0)}{a(t, r_0)} = 0.$$

nonzero in general

Next, let as find out the inner solution satisfying the EOMs for $g_{\mu\nu}^{(\mathrm{in})}$ and $\phi^{(\mathrm{in})}$ given as

$$e^{-2\phi^{(\mathrm{in})}}\left(-\frac{\ddot{a}}{a}+2\ddot{\phi}^{(\mathrm{in})}\right)=\rho, \qquad \dot{\phi}^{(\mathrm{in})\prime}-\frac{\dot{a}}{a}\phi^{(\mathrm{in})\prime}=0,$$

$$\frac{\ddot{a}}{a}+\frac{2}{a^2}\left(\phi^{(\mathrm{in})\prime\prime}-a\dot{a}\dot{\phi}^{(\mathrm{in})}-\frac{a'}{a}\phi^{(\mathrm{in})\prime}\right)=0, \qquad \left(\dot{\phi}^{(\mathrm{in})}\right)^2-\left(\frac{\phi^{(\mathrm{in})\prime}}{a}\right)^2-\ddot{\phi}^{(\mathrm{in})}+\frac{1}{\ell^2}=0.$$

By solving these equations we get two types of solutions.

	(i) $c_1(r)\neq 0$	(ii) $c_1(r)=0$
The existence of homogeneous limit	✗	○

$$\left(\frac{\phi^{(\mathrm{in})\prime}(t,r)}{a(t,r)}=\frac{c_1(r)}{\ell}\right)$$

$c_1(r)$: some aribitrary function indicating inhomogeneous property of the dilaton field

The explicit forms of the solutions are...

$$\phi^{(\mathrm{in})}(t,r) = -\frac{1}{2}\ln\left[c_3(r)\frac{\sin^2\left(\sqrt{1-c_1^2(r)}\left(\frac{t}{\ell}+c_2(r)\right)\right)}{1-c_1^2(r)}\right]$$ for the both cases of (i)&(ii),

(i) $c_1(r) \neq 0$

$$a(t,r) = -\frac{\ell}{c_1(r)}\left\{\frac{c_1(r)c_1'(r)}{1-c_1^2(r)} + \frac{c_3'(r)}{2c_3(r)} + \left(-\frac{c_1(r)c_1'(r)}{\sqrt{1-c_1^2(r)}}\left(\frac{t}{\ell}+c_2(r)\right) + \sqrt{1-c_1^2(r)}c_2'(r)\right)\cot\left(\sqrt{1-c_1^2(r)}\left(\frac{t}{\ell}+c_2(r)\right)\right)\right\}$$

(ii) $c_1(r) = 0$

$$a(t,r) = c_4(r)\cot\left(\frac{t}{\ell}+c_2(r)\right) + c_5(r)$$

This is actually identical with the homogeneous cosmological solutions in two-dimensional dilaton gravity.

[Mann, Ross (1993)]

As we expected, it is found that the outer spacetime and the inner spacetime for the case (ii) cannot be smoothly matched.

On the other hand, the case (i) is possible if $g_{\mu\nu}^{\text{(in)}}$ and $\phi^{\text{(in)}}$ satisfy the conditions on the edge.

junction conditions

We found a simple model satisfying the conditions given as

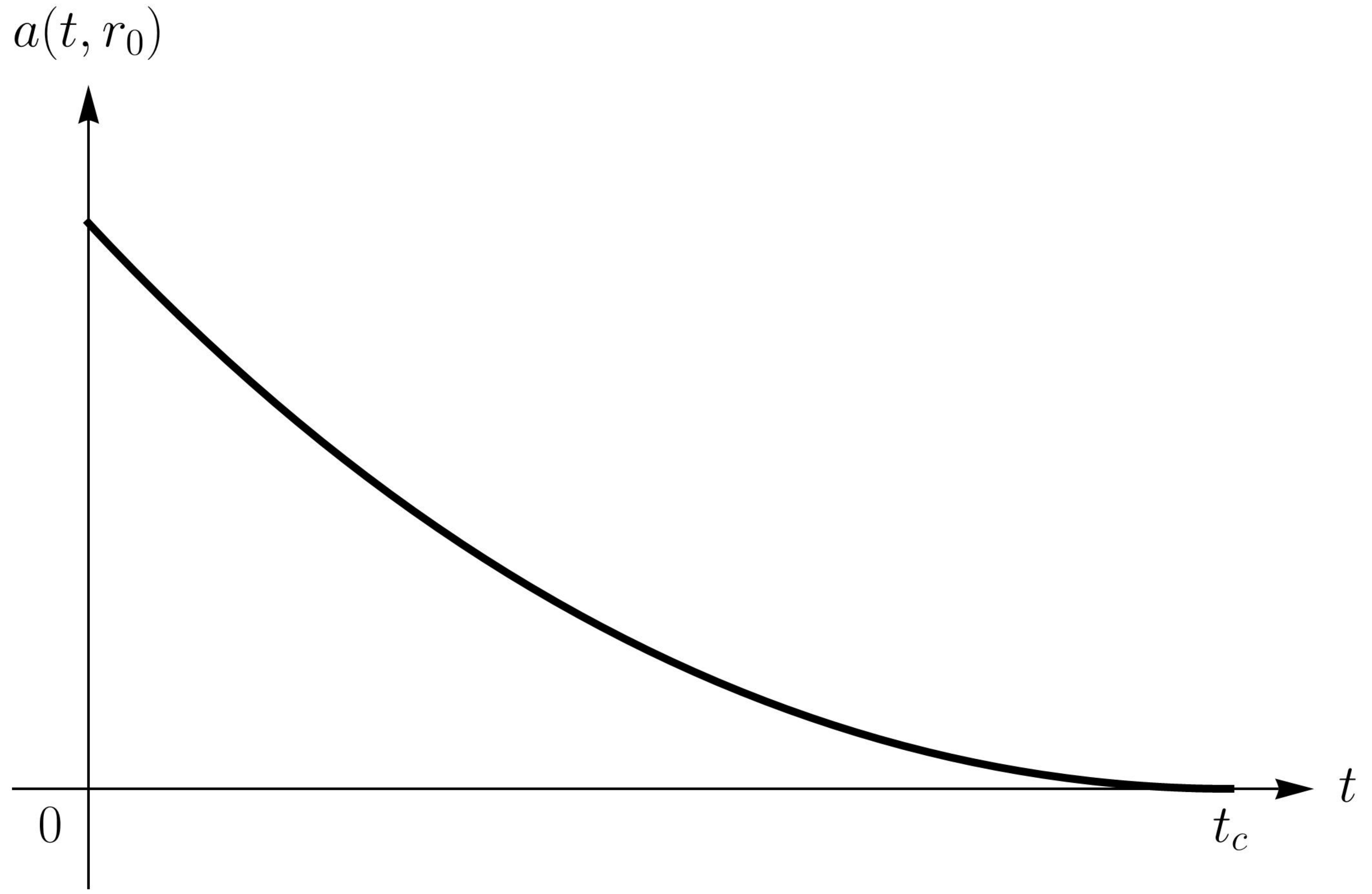
$$c_1(r) = -|\eta|\left(1+\frac{\Delta}{2}\right), \quad c_2(r) = -\frac{t_c}{\ell}, \quad c_3(r) = M\ell - M\ell\,|\eta|\,\Delta^2\left(1+\frac{\Delta}{3}\right),$$

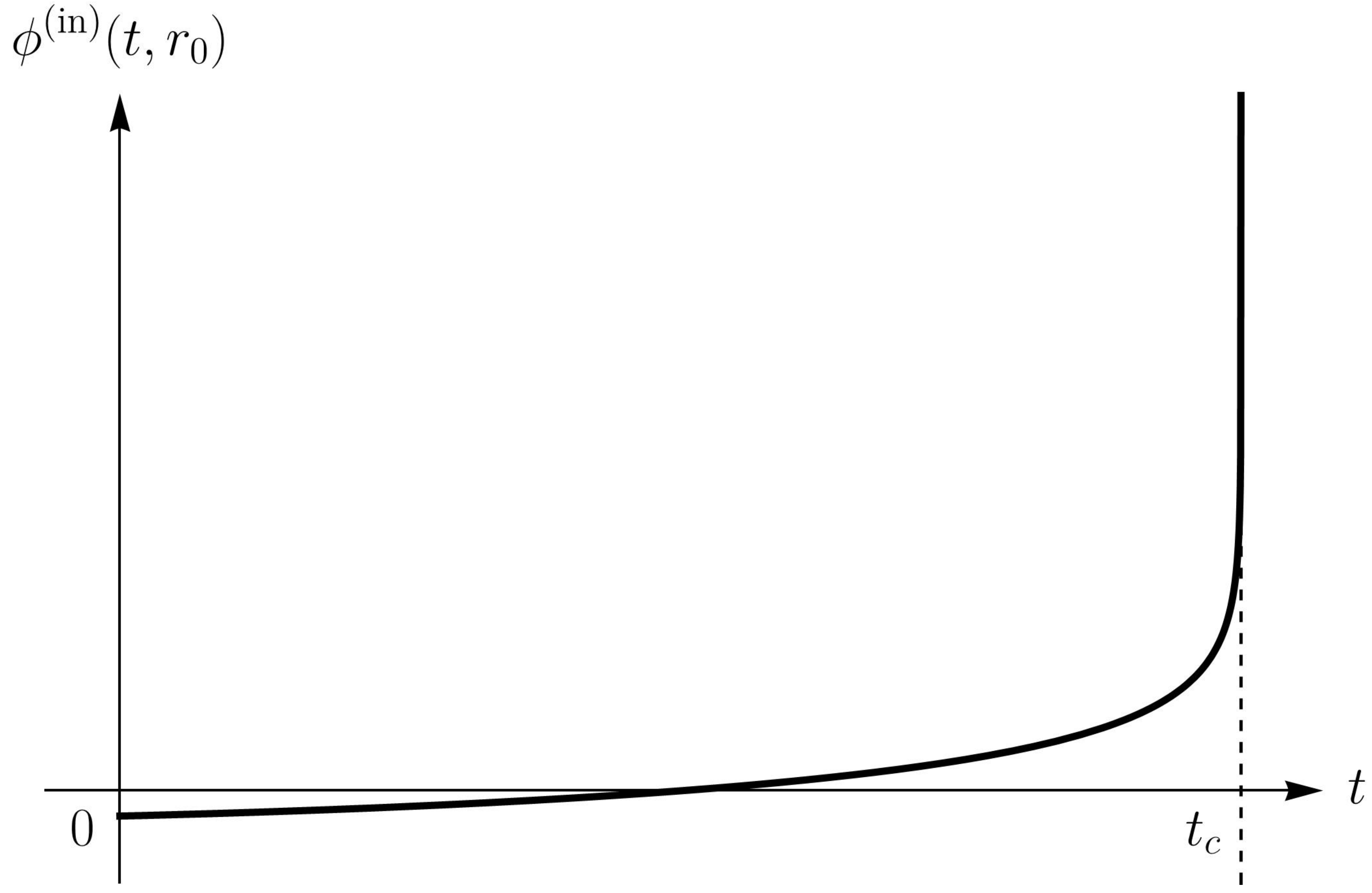
where $\Delta(r) = (r/r_0) - 1$.

In this simple model, the total mass of the dust cloud corresponds to M.

$$\int_0^{r_0} \rho(t,r)a(t,r)dr = M$$

The profiles of the scale factor and dilaton field on the edge





The initial finite scale factor eventually vanishes when the comoving time approaches the collapse time and the dilaton field diverges at the collapse time($t \to t_c$).

Also the curvature scalar on the edge diverges as $t \to t_c$.

$$R_{\text{curvature}}(t, r_0) = \frac{4(1-\eta^2)}{\ell^2 \sin^2\left(\sqrt{1-\eta^2}\left(\frac{t-t_c}{\ell}\right)\right)}$$

Conclusions & Discussions

- Motivated by the formation of the BTZ black hole which is dual to the black string in three dimensions, we studied the formation of the black string from the dust cloud.

- The equations of motions for the inner spacetime describing the dust cloud were exactly solved for arbitrary inhomogeneous distribution of the dust.

- In order to smoothly match the inner spacetime and the outer spacetime of the black string, the Israel junction conditions was imposed on the dust edge, which tells us that the inhomogeneous dust distribution is unavoidable.

- The curvature singularity occurs at the end of collapsing on the edge, but it is cloaked inside the horizon of the black string.

Conclusions & Discussions

◆ We proposed one simple model and it is not unique. There would exist infinitely many models satisfying the conditions on the edge. For example, a slightly general model is given as

$$c_3(r) = M\ell + |\eta|\,\ell \sum_{n=1}^{N} (-1)^n \alpha_n \left\{ \Delta^{n+1} + \frac{(n+1)\Delta^{n+2}}{2(n+2)} \right\}, \text{ where } \sum_{n=1}^{N} \alpha_n = M\,.$$

◆ As the previous work for the two-dimensional dilaton gravity with a homogeneous dust, one may match the outer spacetime of the black string solution and the inner spacetime of the homogeneous dust solution by introducing additional junction matter, i.e., $T^{M(\text{edge})}_{\mu\nu} \neq 0$. However this model with the junction matter may not ensure analytically soluble equations of motion, which deserves further study.

Thank you for listening!