Toward Quantization of Inhomogeneous Field Theory

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CQUeST 2022 Workshop

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MOTIVATIONS

- Inhomogeneous Field Theory : **Position-dependent mass and/or couplings**

- **Position dependence of mass and couplings** breaks the Poincaré symmetry explicitly!

- **In IFT,** Poincaré invariant vacuum does not exist. ➤ **No perferred vacuum state**

- **Quantum field theory on curved spacetime** : Poincaré symmetry and Poincaré vacuum do not exist. ➤ **No a perferred vacuum state**

Preliminaries - Algebraic QFT on Curved ST

■ In this sense, one needs a formalism that establishes, a priori, a kind of 'democracy' between states, and that formulates the theory in terms of the elements that independent of any particular choices.

➤ Algebraic relations between the quantum fields holding in any states

Ex) it democratically treats the pure and the mixed states in the canonical quantization.

: Algebraic formulation of QFTCS.

Preliminaries - Algebraic QFT on Curved ST

- **Algebraic Formulation**

1. **Appropriate algebraic relations among quantum fields (local algebra)**
2. **Some appropriate properties or axioms: isotony, covariance, locality, and causality, existence of dynamics.**
3. **Algebraic states are introduced as normalized positive linear functionals on the field algebra.**
4. **Through the Gelfand-Naimark-Segal (GNS) construction, a relevant Hilbert space from the algebraic states can be constructed.**
5. **Especially, the free field case can be formulated in a rigorous way.**

Based on the classical equivalence,

$$\boxed{\text{IFT} \iff \text{FTCS}}$$

we propose one possible way to quantize IFT.

$$\boxed{\text{IQFT} \iff \text{QFTCS}}$$

We would like to emphasize that the classical equivalence does not automatically warrant its quantum version. As is well-known, the ordering ambiguity in the operator elevation of classical variables results in the trivial example of inequivalent quantum theories with the classical equivalence.

Therefore, our proposal should be taken as one possible way to quantize IFT and may be tested only by experiments.

Classical equivalence : Action

$$S_{\text{FTCS}} = \int d^2x \sqrt{-g}\Big[-\frac{1}{2}\nabla_\mu\phi\nabla^\mu\phi - \frac{1}{2}m_0^2\phi^2 - \frac{\xi}{2}\mathcal{R}\phi^2 \Big]$$

$$ds^2 = e^{2\omega(x)}(-dt^2 + dx^2)$$

$$S_{\text{IFT}} = \int d^2x \Big[-\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2(x)\,\phi^2 \Big]$$

$$m^2(x; m_0, \xi) = \sqrt{-g}(m_0^2 + \xi\mathcal{R}) = e^{2\omega(x)}m_0^2 - 2\omega''\xi$$

Classical equivalence : Equation of mothion

Klein-Gordon equation

of **the IFT with the mass function**

$$m(x) = m_0 e^{bx}$$

$$\left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} + m_0^2 e^{2bx} \right) \phi(t,x) = 0$$

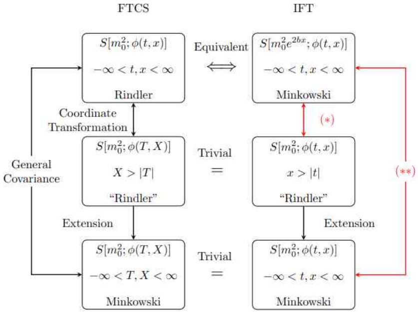
Klein-Gordon equation

of **the FTCS with constant mass**

on Rindler background

$$ds^2 = e^{2bx}(-dt^2 + dx^2)$$

$$\left(-\Box + m_0^2 + \xi \mathcal{R}\right) \phi = 0, \qquad \Box \equiv \frac{1}{\sqrt{-g}} \partial_\mu \left(\sqrt{-g} g^{\mu\nu} \partial_\nu\right)$$



FTCS
IFT
$S[m_0^2;\phi(t,x)]$
$-\infty < t,x < \infty$
Rindler
Equivalent
$\Longleftrightarrow$
$S[m_0^2 e^{2bx};\phi(t,x)]$
$-\infty < t,x < \infty$
Minkowski
Coordinate
Transformation
(*)
General
Covariance
$S[m_0^2;\phi(T,X)]$
$X > |T|$
"Rindler"
Trivial
$=$
$S[m_0^2;\phi(t,x)]$
$x > |t|$
"Rindler"
(**)
Extension
Extension
$S[m_0^2;\phi(T,X)]$
$-\infty < T,X < \infty$
Minkowski
Trivial
$=$
$S[m_0^2;\phi(t,x)]$
$-\infty < t,x < \infty$
Minkowski

Based on the classical equivalence,

$$\boxed{\text{IFT} \iff \text{FTCS}}$$

we propose one possible way to quantize IFT.

$$\boxed{\text{IQFT} \iff \text{QFTCS}}$$

Canonical Quantization

From our equivalence $\boxed{\mathrm{IQFT} \iff \mathrm{QFTCS}}$**, we can apply all the results in the Rindler spacetime to the IFT with the exponential mass function.**

$$\left(\frac{\partial^2}{\partial t^2}-\frac{\partial^2}{\partial x^2}+m_0^2 e^{2bx}\right)\phi(t,x)=0$$

$$u_\Omega(t,x)=\frac{1}{\sqrt{2\Omega}}\theta(\rho)h_\Omega(\rho)e^{-i\Omega\eta}\,,$$
$$\eta\equiv bt\,,\quad \rho\equiv b^{-1}e^{bx}\,,$$

where θ is the step function and h is given by the modified Bessel function of second kind

$$h_\Omega(\rho)=R^*\sqrt{\frac{2}{\pi}}\frac{K_{i\Omega}(m_0\rho)}{|\Gamma(i\Omega)|}\,,\qquad R^*\equiv\left(\frac{m_0}{2b}\right)^{2i\Omega}\left(\frac{\Gamma(-i\Omega)}{|\Gamma(i\Omega)|}\right)^2$$

Equal time commutator

$$[\phi(t,x),\pi(t,y)]=i\delta(x-y)\,,$$
$$[\phi(t,x),\phi(t,y)]=[\pi(t,x),\pi(t,y)]=0$$

Commutation relation

$$[b_\Omega,b^\dagger_{\Omega'}]=\delta(\Omega-\Omega')$$

Annihilation/Creation op's

$$b_\Omega/b^\dagger_\Omega$$

Rindler Vacuum $b_\Omega|0\rangle_{\rm R}=0$ ➤ **the vacuum of the IQFT with an exp. mass function** $|\underline{0}\rangle_{\rm IFT}$

Algebraic Method of Quantization in QFTCS

■ Yet, it has been known that the canonical quantization of FTCS is insufficient to provide a comprehensive general framework, at least conceptually. A more adequate and conceptually superior framework is based on an algebraic construction, which is called an algebraic formulation of QFTCS.

Algebraic Method of Quantization in QFTCS

In the algebraic formulation, a physically important class of quantum states are given by

Gaussian Hadamard states not by a Preferred Vacuum State.

■ Contrary to the ordinary vacuum state, the Hadamard state is not unique for a given background spacetime but forms a class in general.

■ In the case of a Gaussian Hadamard state,

- one can obtain a Fock space representation of the algebra of quantum fields.
- one can identify the Gaussian Hadamard state with the Fock space vacuum.

■ In this way, one can see that some well-known vacuums belong to the Hadamard class.

■ The Hadamard method encompasses the usual Fock space canonical quantization and implements appropriately relevant requirements such as general covariance of stress tensor, while it connects unitarily inequivalent representations of the algebras of observables.

Algebraic Method of Quantization in QFTCS

Hadamard state is defined as an algebraic state satisfying the Hadamard conditions.

- **The short distance singularity structure of the n-point functions of the Hadamard state on curved spacetime should be given by that of the n-point functions of the vacuum state in the Minkowski spacetime.**
- **The ultra-high energy mode of quantum fields resides essentially in the ground state.**
- **The singular structure of the n-point functions should be of positive frequency type.**

Algebraic Method of Quantization in QFTCS

Hadamard renormalization

The Gaussian Hadamard state, ω_H is defined by the renormalized two point function of scalar field φ as

$$\omega_{\mathrm{H}}\big(\phi(\mathbf{x})\phi(\mathbf{x}')\big) = F(\mathbf{x},\mathbf{x}') - H(\mathbf{x},\mathbf{x}')$$

- **Hadamard function, $F(\mathbf{x},\mathbf{x}')$ is an unrenormalized two point function.**
- **Hadamard parametrix, $H(\mathbf{x},\mathbf{x}')$ is a local covariant function of the half of squared geodesic length, σ(x, x′) between two points x and x′, written in terms of the metric and the curvature and takes the same form for any Hadamard states.**

Algebraic Method of Quantization in QFTCS

Hadamard renormalization

$$\omega_{\rm H}\big(\phi(\mathbf{x})\phi(\mathbf{x}')\big) = F(\mathbf{x},\mathbf{x}') - H(\mathbf{x},\mathbf{x}')$$

$$H(\mathbf{x},\mathbf{x}') = \alpha_D \frac{U(\mathbf{x},\mathbf{x}')}{\sigma^{\frac{D}{2}-1}(\mathbf{x},\mathbf{x}')} + \beta_D V(\mathbf{x},\mathbf{x}') \ln \mu^2 \sigma(\mathbf{x},\mathbf{x}') \quad \text{for even } D\,,$$

$$H(\mathbf{x},\mathbf{x}') = \alpha_D \frac{U(\mathbf{x},\mathbf{x}')}{\sigma^{\frac{D}{2}-1}(\mathbf{x},\mathbf{x}')} \quad \text{for odd } D\,,$$

where α_D, β_D are numerical constants depending on the dimension D and μ is a certain mass scale introduced from the dimensional reason.

Symmetric bi-scalars U(x, x′) and V(x, x′), which are regular for x′ → x, are universal geometrical objects independent of any Hadamard states.

Algebraic Method of Quantization in QFTCS

$$U(\mathbf{x},\mathbf{x}') = \sum_{n=0}^{D/2-2} U_n(\mathbf{x},\mathbf{x}')\sigma^n(\mathbf{x},\mathbf{x}'), \qquad V(\mathbf{x},\mathbf{x}') = \sum_{n=0}^{+\infty} V_n(\mathbf{x},\mathbf{x}')\sigma^n(\mathbf{x},\mathbf{x}')$$

- **In two dimensions,** $\alpha_2 = 0$ **and** $H(\mathbf{x},\mathbf{x}') = \frac{V(\mathbf{x},\mathbf{x}')}{2\pi} \ln \mu^2 \sigma(\mathbf{x},\mathbf{x}')$

$$V(\mathbf{x},\mathbf{x}') = -1 - \frac{1}{24}\mathcal{R} g_{\mu\nu}\nabla^\mu\sigma\nabla^\nu\sigma - \frac{1}{2}\left(m_0^2 + \xi - \frac{1}{6}\right)\mathcal{R}\sigma + \mathcal{O}(\sigma^{3/2})$$

One can obtain the renormalized stress tensor by acting an appropriate differential bi-vector operator, $\mathcal{T}_{\mu\nu'}$ **on the renormalized 2-point function as ('point-splitting method')**

$$\langle T_{\mu\nu}(\mathbf{x})\rangle_{\mathrm{H}} = \lim_{\mathbf{x}'\to\mathbf{x}} \mathcal{T}_{\mu\nu'}\langle\phi(\mathbf{x})\phi(\mathbf{x}')\rangle_{\mathrm{H}}$$

$$\begin{aligned}\mathcal{T}_{\mu\nu'} =& (1-2\xi)\partial_\mu\partial_{\nu'} + \left(2\xi - \frac{1}{2}\right) g_{\mu\nu'} g^{\alpha\beta'}\partial_\alpha\partial_{\beta'} - \frac{1}{2} g_{\mu\nu'} m_0^2 \\ & - 2\xi\delta_\mu^{\mu'}\partial_{\mu'}\partial_{\nu'} + 2\xi g_{\mu\nu'} g^{\alpha\beta}\nabla_\alpha\nabla_\beta + \xi\left(\mathcal{R}_{\mu\nu'} - \frac{1}{2} g_{\mu\nu'}\mathcal{R}\right)\end{aligned}$$

Algebraic Method of Quantization in QFTCS

Understanding Unruh Effect

■ The quantum expectation value of the stress tensor should be local, covariant, covariantly-conserved, and etc. The Hadamard renormalization is well suited to this purpose.

$$\langle T_{\mu\nu}(\mathbf{x})\rangle_{\mathrm{H}} = \frac{\partial y^\alpha}{\partial x^\mu}\frac{\partial y^\beta}{\partial x^\nu}\,\langle T_{\alpha\beta}(\mathbf{y})\rangle_{\mathrm{H}}$$

The vacuum states in each coordinate x and y do not need to be realized on the same Fock space vacuum in general.

in terms of the algebraic state ω_M

$$\omega_{\mathrm{M}}(T^{\mathrm{M}}_{\mu\nu}(\mathbf{x})) = \frac{\partial y^\alpha}{\partial x^\mu}\frac{\partial y^\beta}{\partial x^\nu}\,\omega_{\mathrm{M}}(T^{\mathrm{R}}_{\alpha\beta}(\mathbf{y}))$$

$T^{\mathrm{M}}_{\mu\nu}(\mathbf{x})$ stress tensor on the Minkowski spacetime

$T^{\mathrm{R}}_{\alpha\beta}(\mathbf{y})$ stress tensor on the Rindler spacetime

Algebraic Method of Quantization in QFTCS

Understanding Unruh Effect

$$\omega_{\mathrm{M}}(T^{\mathrm{M}}_{\mu\nu}(\mathbf{x})) = \frac{\partial y^{\alpha}}{\partial x^{\mu}}\frac{\partial y^{\beta}}{\partial x^{\nu}}\,\omega_{\mathrm{M}}(T^{\mathrm{R}}_{\alpha\beta}(\mathbf{y}))$$

In the Fock space representation

$$\omega_{\mathrm{M}}(T^{\mathrm{M}}_{\mu\nu}(\mathbf{x})) = {}_{\mathrm{M}}\langle 0|T^{\mathrm{M}}_{\mu\nu}(\mathbf{x})|0\rangle_{\mathrm{M}}$$

- **We take the Minkowski vacuum energy to be zero** $\omega_{\mathrm{M}}(T^{\mathrm{M}}_{\mu\nu}) = 0$ ➤ $\omega_{\mathrm{M}}(T^{\mathrm{R}}_{\mu\nu}) = 0$

- **Gaussian Hadamard state on the right Rindler wedge, $\underline{\omega}_{\mathrm{R}}$, which is not a global Hadamard state on the Minkowski spacetime, since it diverges on the Rindler horizon.**

$$\underline{\omega}_{\mathrm{R}}(T^{\mathrm{R}}_{\mu\nu}(\mathbf{x})) = \langle T^{\mathrm{R}}_{\mu\nu}(\mathbf{x})\rangle^{\mathrm{R}}_{\mathrm{H}} = {}_{\mathrm{R}}\langle 0|T^{\mathrm{R}}_{\mu\nu}(\mathbf{x})|0\rangle_{\mathrm{R}} \neq 0$$

Algebraic Method of Quantization in QFTCS

Understanding Unruh Effect in normal ordering

$$: T^{\mathrm{M}}_{\mu\nu}(\mathbf{x}) : \equiv T^{\mathrm{M}}_{\mu\nu}(\mathbf{x}) - {}_{\mathrm{M}}\langle 0|T^{\mathrm{M}}_{\mu\nu}(\mathbf{x})|0\rangle_{\mathrm{M}}\ \mathbf{1}$$

Our choice of $\omega_{\mathrm{M}}(T^{\mathrm{M}}_{\mu\nu}) = 0$ means $: T^{\mathrm{M}}_{\mu\nu}(\mathbf{x}) :=\ T^{\mathrm{M}}_{\mu\nu}(\mathbf{x})$

By taking the same definition of a normal ordering in the Rindler spacetime,

$$: T^{\mathrm{R}}_{\mu\nu}(\mathbf{y}) : = T^{\mathrm{R}}_{\mu\nu}(\mathbf{y}) - {}_{\mathrm{R}}\langle 0|T^{\mathrm{R}}_{\mu\nu}(\mathbf{y})|0\rangle_{\mathrm{R}}\ \mathbf{1}\,,$$

$$\omega_{\mathrm{M}}(: T^{\mathrm{R}}_{\mu\nu}(\mathbf{y}) :) = \omega_{\mathrm{M}}(T^{\mathrm{R}}_{\mu\nu}(\mathbf{y})) - {}_{\mathrm{R}}\langle 0|T^{\mathrm{R}}_{\mu\nu}(\mathbf{y})|0\rangle_{\mathrm{R}} = -{}_{\mathrm{R}}\langle 0|T^{\mathrm{R}}_{\mu\nu}(\mathbf{y})|0\rangle_{\mathrm{R}}\ >\ 0$$

Unruh Effect in IQFT

QFT on the Rindler spacetime background ➤ IQFT with the exponential mass function

$$\langle T^{\mathrm{IFT}}_{\mu\nu}(\mathbf{x})\rangle^{\mathrm{IFT}}_{\mathrm{H}} \equiv \lim_{\mathbf{x}'\to\mathbf{x}} \mathcal{T}^{\mathrm{IFT}}_{\mu\nu'}\langle\phi(\mathbf{x})\phi(\mathbf{x}')\rangle^{\mathrm{IFT}}_{\mathrm{H}} \qquad \langle T_{\mu\nu}\rangle_{\mathrm{H}} = \lim_{\mathbf{x}'\to\mathbf{x}} \mathcal{T}_{\mu\nu'}\left[F(\mathbf{x},\mathbf{x}') + \frac{1}{4\pi}\ln\mu^2\sigma(\mathbf{x},\mathbf{x}')\right]$$

$$F_{\mathrm{IFT}}(\mathbf{x},\mathbf{x}') + \frac{1}{4\pi}\ln 2\sigma = \frac{b^2}{48\pi}\left[(x-x')^2 + (t-t')^2\right] + \cdots \qquad \mathcal{T}^{\mathrm{IFT}}_{\mu\nu'} = \partial_\mu\partial_{\nu'} - \frac{1}{2}\eta_{\mu\nu'}\eta^{\alpha\beta'}\partial_\alpha\partial_{\beta'} - \frac{1}{2}\eta_{\mu\nu'}m^2(x)$$

$$\langle T^{\mathrm{IFT}}_{tt}\rangle^{\mathrm{IFT}}_{\mathrm{H}} = \langle T^{\mathrm{IFT}}_{xx}\rangle^{\mathrm{IFT}}_{\mathrm{H}} = -\frac{b^2}{24\pi}, \quad \langle T_{\mathrm{IFT}}{}^{\mu}{}_{\mu}\rangle^{\mathrm{IFT}}_{\mathrm{H}} = 0$$

$$\langle T^{\mathrm{R}}_{\eta\eta}\rangle^{\mathrm{R}}_{\mathrm{H}} = -\frac{1}{24\pi}, \qquad \langle T^{\mathrm{R}}_{\rho\rho}\rangle^{\mathrm{R}}_{\mathrm{H}} = -\frac{1}{24\pi}\frac{1}{\rho^2}, \qquad \langle T_{\mathrm{R}}{}^{\mu}{}_{\mu}\rangle^{\mathrm{R}}_{\mathrm{H}} = 0$$

SUMMARY

Based on the classical equivalence, $\boxed{\text{IFT} \iff \text{FTCS}}$

we propose one possible way to quantize IFT. $\boxed{\text{IQFT} \iff \text{QFTCS}}$

The energy of the vacuum in the IQFT with the exponential mass function is negative and it can be interpreted as a Unruh-like effect in IQFT corresponding to the Unruh effect in QFTCS.

감사합니다.

Thank You