HOLOGRAPHIC RG FLOW TRIGGERED BY A CLASSICAL MARGINAL OPERATOR

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BASED ON :

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Contents

- Motivation
- RG flow in QFT
- Holographic description of RG flow
- Discussion

AdS/CFT correspondence (symmetry)

Classical SUGRA on AdS space-time	⟷ 1 to 1	Super-CFT at the AdS boundary (in a strong coupling regime)

For example

Isometry of AdS_5	←	$SO(2,4)$	→	Conformal symmetry on $R^{1,3}$
Isometry of S^5	←	$SO(6)$	→	R-symmetry of N=4 SUSY

$$Z_{gravity} \approx e^{-S_{on-shell}} \quad \longleftrightarrow \quad Z_{gauge} = \langle e^{-S_{CFT}} \rangle$$

one to one map

What is the role of the extra dimension of a bulk geometry?

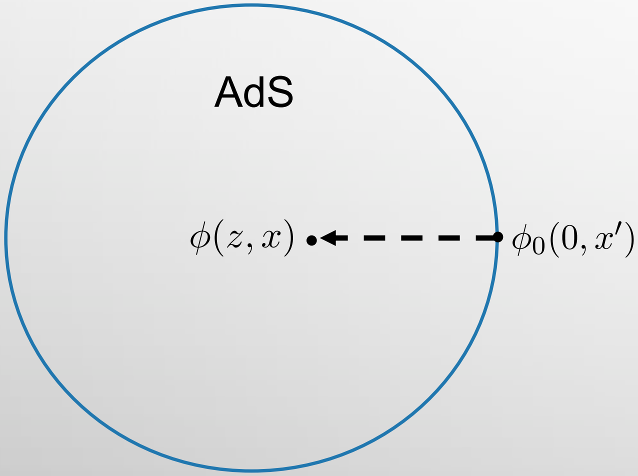
Two-point correlation function

The AdS metric becomes

$$ds^2 = \frac{dz^2 - dt^2 + d\vec{x}^2}{z^2}$$

A scalar field fluctuation in the AdS space (corresponding fluctuation in the dual CFT)

$$0 = \frac{1}{\sqrt{-g}}\partial_\mu \left(\sqrt{-g}g^{\mu\nu}\partial_\nu \phi(z,t,\vec{x})\right) - m^2\phi(z,t,\vec{x})$$



Solution

$$\phi(z,x) = \int d^4x' \mathcal{D}_\phi\left(z,x;0,x'\right)\phi_0(0,x')$$

with Bulk-to-boundary propagator

$$\mathcal{D}_\phi\left(z,x;0,x'\right) \sim \left(\frac{z}{z^2+(x-x')^2}\right)^\Delta$$

Near the boundary ($z \to 0$),

the bulk-to-boundary propagator reduces to

$$\mathcal{D}_\phi\left(z, x; 0, x'\right) \approx \frac{z^\Delta}{(x - x')^{2\Delta}}$$

which leads to the boundary-to-boundary propagator

$$\lim_{z \to 0} \phi(z, x) = z^\Delta \int d^4x' \frac{\phi_0(0, x')}{(x - x')^{2\Delta}}$$

The boundary (on-shell) action of matter becomes

$$S_{bd} \sim \int d^4x \int d^4x' \ \frac{\phi_0(0, x)\phi_0(0, x')}{(x - x')^{2\Delta}}$$

As a consequence, the two-point correlation function reads

$$\left\langle \mathcal{O}(x)\mathcal{O}(x') \right\rangle = \frac{\partial}{\partial\phi_0(0, x')} \frac{\partial}{\partial\phi_0(0, x)} Z = \frac{1}{(x - x')^{2\Delta}}$$

which is exactly the two-point function expected in the CFT.

Comparing the two-point correlation function with the bulk scalar field

$$\langle \mathcal{O}(x)\mathcal{O}(x')\rangle = \frac{\partial}{\partial\phi_0(0,x')}\frac{\partial}{\partial\phi_0(0,x)}Z = \frac{1}{(x-x')^{2\Delta}}$$

(in the asymptotic region)

$$\phi(z,x) \approx \phi_0(x)\ z^{4-\Delta}\,(1+\cdots) + \mathcal{O}(x)\ z^{\Delta}\,(1+\cdots)$$

the radial coordinate dependence is associated with the scaling dimension of an operator.

Therefore, we can identify the bulk quantities with the boundary quantities

ϕ_0 : coupling constant (or source) of the dual operator

$\mathcal{O}$: dual operator

Δ : conformal dimension of the dual operator

This is because of the scaling invariance

$$x \to \lambda x \qquad \text{and} \qquad z \to \lambda z$$

If we further take into account the gravitational backreaction of the massive scalar field, the inside geometry corresponding the IR region of the dual QFT is deviated from the AdS geometry. *The dual QFT is not conformal anymore and allows* the nontrivial RG flow of the dual QFT.

$$S_{on-shell} = S_{CFT} - \int d^4x \phi_0 \mathcal{O}$$

In general, the scaling dimension is well defined only in a scale invariant theory. Then, how can we connect the scale-dependent correlation functions to bulk solutions?

We expect that a coupling constant becomes scale-dependent for a nonconformal QFT. However, the coefficients of the bulk solution are given by two integral constants

$$\phi = \frac{\phi_0}{r^{4-\Delta}} \ (1 + \cdots) + \frac{\mathcal{O}}{r^{\Delta}} \ (1 + \cdots)$$

(ϕ_0 and $\mathcal{O}$ are two integral constants)

What are the scale-dependent coupling and vev of the operator?

Answering this question is important to understand IR physics along the RG flow.

For example)

It was known that QCD is asymptotically free. Therefore, QCD has a UV fixed point where a conformal symmetry is restored. The conformal symmetry requires

$$\langle T^{\mu}{}_{\mu} \rangle = 0$$

However, when QCD deforms by the gluon condensation, the trance anomaly appears at the one-loop level

$$\langle T^{\mu}{}_{\mu} \rangle = -\frac{N_c}{8\pi} \frac{\beta_\lambda}{\lambda^2} \langle G \rangle \quad \text{with} \quad G = -\mathrm{Tr}\, F^2$$

How can we describe such a deformation in the holographic setup?

RG flow in QFT

For CFT

- CFT has a vanishing beta-function $\beta_{CFT} = 0$ (due to the scale symmetry)

- CFT is a dual of an AdS geometry (isometry of AdS space = conformal symmetry)

Deformation of UV CFT by an operator with a conformal dimension Δ

$\bar{O}$

$$S_{QFT} = S_{CFT} + \int d^d x \sqrt{\gamma}\, \mu^{d-\Delta} \lambda\, \bar{O}$$

where μ and λ denote the RG scale and dimensionless coupling.

Under the RG (scale) transformation at the classical (tree) level, the coupling constant and operator scale by

$$\lambda \to \mu^{-(d-\Delta)} \lambda \quad \text{and} \quad O \to \mu^{-\Delta} O.$$

Then, a classical beta-function becomes

$$\beta_{cl} \equiv \frac{\partial \lambda}{\partial \log \mu} = -(d - \Delta)\lambda.$$

Then, we divide the operator into

- relevant $(\beta_q < 0)$ for $\Delta < d$
- marginal $(\beta_q = 0)$ for $\Delta = d$
- irrelevant $(\beta_q > 0)$ for $\Delta > d$

For the gluon condensation with $\Delta = d$,

the gluon condensation is classically marginal

$$\beta_{cl} = 0 \quad \text{and} \quad \langle T^{\mu}{}_{\mu} \rangle = 0$$

This is the story at the classical level. If we further consider quantum corrections, the scaling behavior of the coupling constant and operator change.

Near the UV fixed point, the beta-function is corrected due to the quantum corrections

$$\beta_\lambda = \beta_{cl} + \beta_q = -(d - \Delta)\lambda + \beta_q.$$

Even for a classically marginal operator with $\Delta = d,$ its beta-function becomes nontrivial

classically marginal operator → quantum corrections

- marginally relevant $(\beta_q < 0)$
- truly marginal $(\beta_q = 0)$
- marginally irrelevant $(\beta_q > 0)$

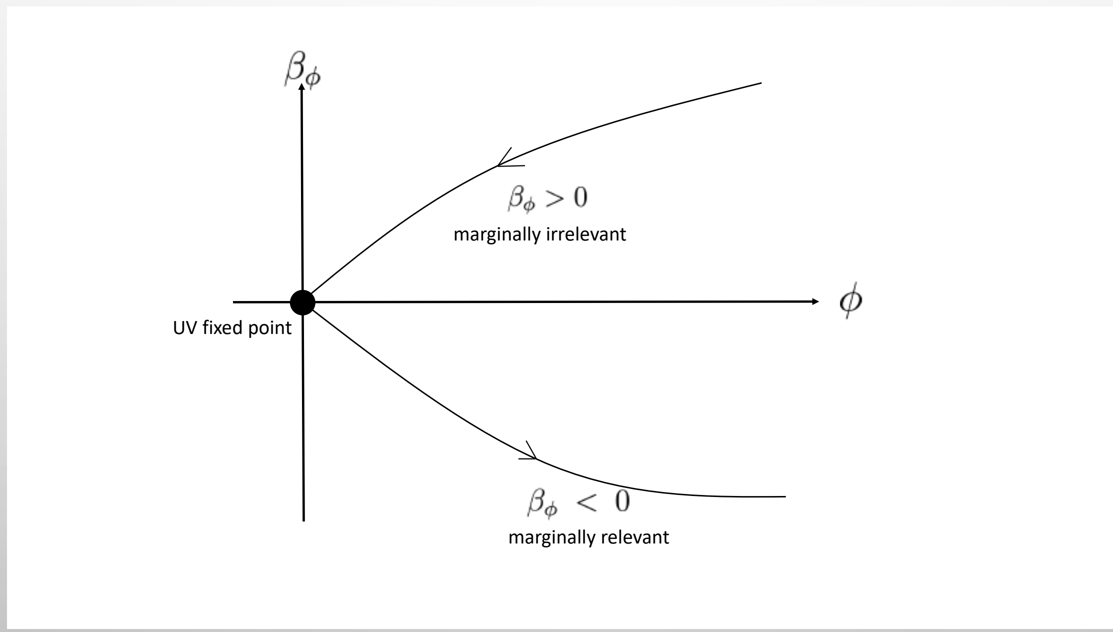


Figure 1. The RG flows caused by marginally relevant and irrelevant operators.

On the QFT side, after an appropriate renormalization procedure, the renormalized partition function is given by a functional of the coupling constants

$$\mathcal{Z} = \int \mathcal{D}\phi \, e^{-(S_{QFT}+S_{ct})} = e^{-\Gamma[\gamma_{\mu\nu}(\mu),\lambda(\mu);\mu]},$$

Here, we took into account the metric as a coupling (the method was also used to explain the conformal anomaly in CFT).

Since the partition function must be independent of the cutoff scale, it should satisfy

$$0 = \frac{d\Gamma}{d\mu}$$

which leads to the RG equation

$$0 = \frac{\mu}{\sqrt{\gamma}}\frac{\partial\Gamma}{\partial\mu} + \gamma^{\mu\nu}\langle T_{\mu\nu}\rangle + \beta_\lambda \langle O \rangle$$

where

$$\beta_\lambda = \frac{d\lambda}{d\log\mu},$$

$$\langle T_{\mu\nu}\rangle = -\frac{2}{\sqrt{\gamma}}\frac{\partial\Gamma}{\partial\gamma^{\mu\nu}},$$

$$\langle O \rangle = \frac{1}{\sqrt{\gamma}}\frac{\partial\Gamma}{\partial\lambda}.$$

(1) When the metric is scale invariant, the traditional RG equation occurs

$$0 = \mu\frac{\partial\Gamma}{\partial\mu} + \beta_\lambda \langle O \rangle$$

(2) The vev of operators are derived quantities

(3) For $\partial\Gamma/\partial\mu = 0$, we obtain the trace anomaly caused by the deformation, like the previous gluon condensation,

$$\langle T^\mu{}_\mu \rangle \sim -\beta_\lambda \langle O \rangle$$

Holographic dual of a classically marginal operator

We take into account a 5-dimensional Einstein-scalar gravity theory

$$S = -\frac{1}{2\kappa^2}\int d^5X\sqrt{g}\left(\mathcal{R} - 2\Lambda - \frac{1}{2}g^{MN}\partial_M\phi\partial_N\phi - \frac{1}{2}\frac{m^2}{R^2}\phi^2\right) + \frac{1}{\kappa^2}\int_{\partial\mathcal{M}} d^4x\sqrt{\gamma}\,K,$$

where the bulk scalar field is the dual of a deformation operator.

For a constant ϕ, the geometric solution becomes an AdS space

$$ds^2 = \frac{R^2}{z^2}\left(dz^2 + \delta_{ij}dx^i dx^j\right)$$

which corresponds to the undeformed CFT.

If we further consider the gravitational backreaction of the scalar field, the CFT deforms by the dual operator.

In the asymptotic region ($z = 0$), the bulk scalar field has the following expansion

$$\phi = c_1 z^{4-\Delta}\left(1 + \cdots\right) + c_2 z^{\Delta}\left(1 + \cdots\right)$$

with

$$\Delta = 2 + \sqrt{4 + \frac{m^2}{R^2}}.$$

From $$\phi = c_1 z^{4-\Delta}\left(1+\cdots\right) + c_2 z^{\Delta}\left(1+\cdots\right)$$

$$\Delta = 2 + \sqrt{4 + \frac{m^2}{R^2}}.$$

- for $m^2 < 0$, the deformation is relevant $(\Delta < 4)$
- for $m^2 = 0$, the deformation is marginal $(\Delta = 4)$
- for $m^2 > 0$, the deformation is marginal $(\Delta > 4)$

Noting that c_1 and c_2 are two integral constants,

- when we naively identify c_1 with a coupling constant, the beta-function always vanishes

$$\beta = 0$$

- the vev of the operator is not determined from the partition function.

We need to improve the holographic definition of the coupling constant and operator's vev in order to describe the RG flow correctly.

We begin with the following gravity theory

$$S = -\frac{1}{2\kappa^2}\int d^5X\sqrt{g}\left(\mathcal{R} - 2\Lambda - \frac{1}{2}g^{MN}\partial_M\phi\partial_N\phi\right) + \frac{1}{\kappa^2}\int_{\partial\mathcal{M}} d^4x\sqrt{\gamma}\ K,$$

Then, the dual theory is a CFT deformed by a marginal operator.

We can investigate the gravity theory in two different ways.

- (1) Einstein equation (2nd-order differential equation)

using the following metric ansatz in the normal coordinate system

$$ds^2 = e^{2A(y)}\delta_{\mu\nu}dx^\mu dx^\nu + dy^2$$

Einstein equations are

$$0 = 24\dot{A}^2 - \dot{\phi}^2 + 4\Lambda,$$

$$0 = 12\ddot{A} + 24\dot{A}^2 + \dot{\phi}^2 + 4\Lambda,$$

$$0 = \ddot{\phi} + 4\dot{A}\dot{\phi},$$

and the solution is given by

$$\phi = \phi_0 + \eta\sqrt{\frac{3}{2}}\log\left(\frac{4\sqrt{6} - \phi_1 z^4/R^4}{4\sqrt{6} + \phi_1 z^4/R^4}\right),$$

$$e^{2A(y)} = \frac{R^2}{z^2}\sqrt{1 - \frac{\eta^2\phi_1^2}{96}\frac{z^8}{R^8}}, \qquad \text{with} \quad z = Re^{-y/R}$$

These are the second-order differential equations.

However, we need to the first-order differential equations to describe the RG flow.

- (2) Hamilton-Jacobi formalism (1st-order differential equation)

Since the RG equation is given by the first-order differential equation, the Hamilton-Jacobi formulation is useful to describe the RG flow of the dual QFT.

After the ADM decomposition

$$ds^2 = N^2 dy^2 + \gamma_{\mu\nu}(x,y) dx^\mu dx^\nu \qquad \text{with} \qquad \gamma_{\mu\nu} = e^{2A(y)} \delta_{\mu\nu}.$$

the Einstein-scalar theory can be rewritten as

$$S = \int d^4x dy \sqrt{g}\, \mathcal{L},$$

with

$$\mathcal{L} = \frac{1}{2\kappa^2} \left[N \left(-\mathcal{R}^{(4)} + K_{\mu\nu} K^{\mu\nu} - K^2 + 2\Lambda \right) + \frac{1}{2N} \dot{\phi}^2 \right],$$

where the extrinsic curvature is given by

$$K_{\mu\nu} = \frac{1}{2N} \frac{\partial \gamma_{\mu\nu}}{\partial y}$$

and the intrinsic curvature of the boundary vanishes for a flat boundary

$$\mathcal{R}^{(4)} = 0$$

The canonical momenta of the boundary metric and scalar field are defined

$$\pi_{\mu\nu} \equiv \frac{\partial S}{\partial \dot{\gamma}^{\mu\nu}} = -\frac{1}{2\kappa^2}\left(K_{\mu\nu} - \gamma_{\mu\nu}K\right),$$
$$\pi_\phi \equiv \frac{\partial S}{\partial \dot{\phi}} = \frac{1}{2\kappa^2}\dot{\phi}.$$

Then, the bulk action reexpresses as

$$S = \int d^4x dy \sqrt{g}\ \left(\pi_{\mu\nu}\dot{\gamma}^{\mu\nu} + \pi_\phi \dot{\phi} - N\mathcal{H}\right)$$

with the following Hamiltonian constraint

$$\mathcal{H} = 2\kappa^2 \left(\gamma^{\mu\rho}\gamma^{\nu\sigma}\pi_{\mu\nu}\pi_{\rho\sigma} - \frac{1}{3}\pi^2 + \frac{1}{2}\pi_\phi^2\right) - \frac{\Lambda}{\kappa^2}. = 0$$

Here, the Hamiltonian corresponds to a generator of the translation in the y-direction. All solutions connected by this transformation are gauge-equivalent.

$\longrightarrow$ This corresponds to the RG transformation of the dual QFT.

The variation of the on-shell bulk action reduces to the variation of the boundary action

$$\delta S_B = \int_{\partial\mathcal{M}} d^4x \sqrt{\gamma}\left(\pi_{\mu\nu}\delta\gamma^{\mu\nu} + \pi_\phi \delta\phi\right),$$

which satisfies

$$\pi_{\mu\nu} = \frac{1}{\sqrt{\gamma}}\frac{\delta S_B}{\delta\gamma^{\mu\nu}} \quad \text{and} \quad \pi_\phi = \frac{1}{\sqrt{\gamma}}\frac{\delta S_B}{\delta\phi}.$$

These two relations and the previous Hamiltonian constraint are equivalent to the Einstein equations

According to the AdS/CFT correspondence, the above boundary action corresponds to the generating functional of the dual QFT.

$$\mathcal{Z} \approx e^{-S_B}$$

Since the above boundary action suffers from the UV divergence,

we need to renormalize it by adding appropriate counterterms (holographic renormalization).

The marginal deformation does not generate additional UV divergence, so that only the counter term renormalize the AdS geometry is required

$$S_{ct} = -\frac{1}{2\kappa^2}\int d^4x\sqrt{\gamma}\,\mathcal{L}_{ct}. \qquad \text{with} \qquad \mathcal{L}_{ct} = \frac{6}{R}.$$

As a consequence, the renormalized boundary action (generating functional) is given by

$$\Gamma[\gamma_{\mu\nu}, \phi; \bar{y}] = S_B - S_{ct}.$$

Since the generating function must be independent of the UV cutoff,
it has to satisfy the following RG equation

$$0 = \frac{\mu}{\sqrt{\gamma}}\frac{\partial\Gamma}{\partial\mu} + \gamma^{\mu\nu}\langle T_{\mu\nu}\rangle + \beta_\phi\langle O\rangle$$

with

$$\beta_\phi \equiv \frac{\partial\phi}{\partial\log\mu},$$

$$\langle T_{\mu\nu}\rangle \equiv -\frac{2}{\sqrt{\gamma}}\frac{\partial\Gamma}{\partial\gamma^{\mu\nu}} = -\left(2\pi_{\mu\nu} - \frac{1}{2\kappa^2}\gamma_{\mu\nu}\mathcal{L}_{ct}\right),$$

$$\langle O\rangle \equiv \frac{1}{\sqrt{\gamma}}\frac{\delta\Gamma}{\delta\phi} = \pi_\phi + \frac{1}{2\kappa^2}\frac{\partial\mathcal{L}_{ct}}{\partial\phi}.$$

Two prescriptions for the holographic RG flow

(1) In the normal coordinate system,

the scaling of the dual QFT is related to translation in the radial direction of the dual gravity.

On the QFT side, the scaling behavior of the coordinate and momentum is

$$x \to e^{-\sigma} x \text{ or } \mu \to e^{\sigma} \mu$$

At the boundary of the bulk geometry

$$ds^2 = \gamma_{\mu\nu} dx^\mu dx^\nu \qquad \text{with} \qquad \gamma_{\mu\nu} = e^{2A(y)} \delta_{\mu\nu}.$$

the bulk coordinate must transform

$$e^{A(\bar{y})} \to e^{\sigma} e^{A(\bar{y})}$$

Therefore, we have to identify the radial coordinate with the RG scale of the dual QFT

$$\mu = \frac{e^{A(\bar{y})}}{R}$$

(2) When the CFT deforms with a nontrivial beta-function,

the coupling constant becomes a function of the RG scale and the vev of the operator must be derived from the generating functional.

To describe the scale dependence of the coupling constant,

we identify the value of the bulk field at the boundary with the strength of the coupling constant

For example,

$$\phi = \underbrace{c_1 z^{4-\Delta}}_{\text{classical}} (1 + \cdots) + \underbrace{c_2 z^{\Delta} (1 + \cdots)}_{\text{quantum corrections}}$$

at the leading order, we obtain

$$\langle O \rangle \equiv \frac{1}{\sqrt{\gamma}} \frac{\delta \Gamma}{\delta \phi} \sim c_2 + \cdots$$

This is consistent with the identification $\langle \mathcal{O} \rangle = c_2$ at the UV fixed point.

From the bulk solution of the Einstein-scalar gravity

$$\phi = \phi_0 + \eta\sqrt{\frac{3}{2}}\log\left(\frac{4\sqrt{6}-\phi_1 z^4/R^4}{4\sqrt{6}+\phi_1 z^4/R^4}\right),$$

$$e^{2A(y)} = \frac{R^2}{z^2}\sqrt{1-\frac{\eta^2\phi_1^2}{96}\frac{z^8}{R^8}},$$

with $z = Re^{-y/R}$

1) For $\phi = 0$, a pure AdS is a solution which corresponds to an undeformed CFT.

2) The massless bulk scalar field at the boundary is matched to the coupling constant of a classically marginal operator.

3) The beta-function of a marginal operator with quantum corrections

for $\phi \sim \lambda$

$$\beta_\phi \equiv \frac{\partial \phi}{\partial \log \mu}$$

$\beta_\phi < 0$ - marginally relevant $(\eta < 0)$

$\beta_\phi = 0$ - truly marginal $(\eta = 0)$

$\beta_\phi > 0$ - marginally irrelevant $(\eta > 0)$

Gluon condensation in QCD

In a 4-dimensional space, QCD is asymptotically free (conformal at the UV fixed point)

For QCD, the condensations are usually associated with the spontaneous symmetry breaking and responsible for the mass of hadrons

If there is a non-vainshing gluon condensation

$$\langle G \rangle \neq 0 \quad \text{with} \quad G = -\mathrm{Tr}\, F^2$$

QCD deforms by the condensation which gives rise to a new ground state.

The quantum correction at the one-loop level leads to the following trace anomaly

$$\langle T^{\mu}{}_{\mu} \rangle = -\frac{N_c}{8\pi} \frac{\beta_\lambda}{\lambda^2} \langle G \rangle$$

where $\lambda = N_c\, g_{YM}^2$ is the ‘t Hooft coupling.

Holographic dual of the gluon condensation

From the kinetic term of the Yang-Mills theory

$$S_{YM} = -\frac{1}{4g_{YM}^2}\int d^4x\sqrt{\gamma}\;\; \mathrm{Tr}\, F^2$$

we identify the bulk scalar field with the inverse of the Yang-Mills coupling or ‘t Hoot coupling

$$\phi = \frac{N_c}{4\lambda}.$$

Then, the beta-function of ϕ is related to that of the ‘t Hooft coupling

$$\beta_\phi \equiv \frac{\partial\phi}{\partial\log\mu} = -\frac{N_c}{4}\frac{\beta_\lambda}{\lambda^2}$$

From the previous gravity solution,

$\eta = 1$: marginally relevant $\langle T^\mu{}_\mu\rangle \neq 0$

$\eta = 0$: truly marginal $\langle T^\mu{}_\mu\rangle = 0$

$\eta = -1$: marginally irrelevant $\langle T^\mu{}_\mu\rangle = 0$

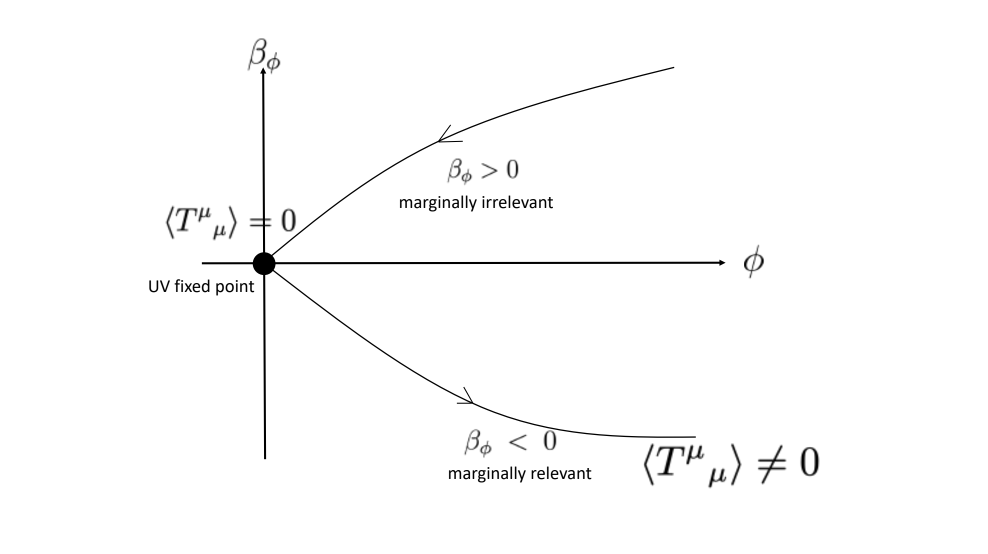


Figure 1. The RG flows caused by marginally relevant and irrelevant operators.

For $\eta = 1$, the asymptotic free theory at the UV fixed point flow into a new IR theory which has a non-vanishing gluon condensation.

The holographic calculation allows the following beta-function and gluon condensate

$$\beta_\phi = \frac{\phi_1}{R^4}\frac{1}{\mu^4} - \frac{\phi_1^3}{48R^{12}}\frac{1}{\mu^{12}} + \mathcal{O}\left(\mu^{-20}\right),$$
$$\langle G \rangle = -\frac{\phi_1}{2\kappa^2 R^5}\frac{1}{\mu^4} + \mathcal{O}\left(\mu^{-28}\right).$$

which rely on the RG scale.

Rewriting these result in terms of the ‘t Hooft coupling, we obtain

$$\beta_\lambda \sim -\lambda \text{ and } \langle G \rangle \sim -1/\lambda \text{ in the UV region.}$$

Since the ‘t Hooft coupling is dimensionless, its beta-function is at the tree level

$$\beta_\lambda = 0$$

Therefore, $\beta_\lambda \sim -\lambda$ comes from the quantum correction

Varying the holographic generating functional with respect to the metric, we obtain the following trace anomaly

$$\langle T^{\mu}{}_{\mu} \rangle = -\frac{\phi_1^2}{4\kappa^2 R^9}\frac{1}{\mu^8} + \frac{\phi_1^4}{384\kappa^2 R^{17}}\frac{1}{\mu^{16}} + \mathcal{O}\left(\mu^{-24}\right) .$$

Comparing the obtained holographic results, we finally find the following relation

$$\langle T^{\mu}{}_{\mu} \rangle = \underbrace{-\frac{N_c}{8}\frac{\beta_\lambda}{\lambda^2}}_{\text{one-loop}} \langle G \rangle + \underbrace{\mathcal{O}\left(\lambda^{-4}\right)}_{\text{higher-loop}} .$$

- The one-loop result is the expected trace anomaly caused by the gluon condensation.
- The nonvanishing higher order correction can modify the one-loop trace anomaly.

Conclusion

- We discuss how to realize the RG flow in the holographic setup.

- By applying the holographic RG flow,
we reproduced the expected trace anomaly caused by the gluon condensation.

- Future works,
  - Higher loop corrections
  - RG flow caused by relevant deformations
  - Nonperturbative IR physics after the RG flow

THANK YOU !