Identifying Riemannian Singularities with Regular Non-Riemannian Geometry

CQUeST 2022 Workshop on Cosmology and Quantum Space Time

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Based on Phys.Rev.Lett. **128** (2022) 4 041602

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2022/06/29

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- In this talk, we will apply non-Riemannian geometries to the singularity problem in GR.

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   Changing the time coordinate to $v = t + r_*$ where:

   $r_* := r + 2M \ln\left|\frac{r}{2M} - 1\right|$ stands for the tortoise coordinate satisfying $\frac{dr_*}{dr} = \left(1 - \frac{2M}{r}\right)^{-1}$

   allows to reexpress the Schwarzschild metric in the ingoing Eddington–Finkelstein coordinates:

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   $\Rightarrow$ The region $r = 2M$ thus corresponds to a coordinate singularity.

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2 Although the Ricci scalar vanishes $(R = 0)$, the Kretschmann scalar

$$R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} = \frac{48M^2}{r^6} \quad \text{diverges at} \quad r \to 0.$$

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The determinant $\det g = -r^4 \sin^2\theta$ vanishes at $r = 0$, so that the metric is not invertible there.

$\Rightarrow$ There is a genuine curvature singularity at $r = 0$.

Contrarily to the case $r = 2M$, the latter is not an artifact of the coordinate system and hence cannot be removed by coordinate transformation.

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3. There are geodesics which have bounded proper time only, *i.e.* they reach the singularity at $r = 0$ after finite proper time.

   $\Rightarrow$ The Schwarzschild metric is geodesically incomplete.

Double Field Theory (DFT)

- The DFT action reads

$$S_{\mathsf{DFT}} = \int d^{2D}X \, e^{-2d} \, \mathcal{R}(\mathcal{H}, d)$$

  where the generalised Ricci scalar $\mathcal{R}(\mathcal{H}, d)$ is the unique $\mathbf{O}(D, D)$ scalar built in terms of second derivatives of the fundamental $\mathbf{O}(D, D)$ variables $\mathcal{H}$ and d.

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- Substituting into the $\mathbf{O}(D,D)$ variables the Riemannian parameterisation:

$$\begin{aligned} \mathbf{O}(D,D) \text{ dilaton} \quad e^{-2d} &= \sqrt{-g}e^{-2\phi} \\ \text{Generalized metric} \quad \mathcal{H}_{AB} &= \begin{pmatrix} g^{-1} & -g^{-1}B \\ Bg^{-1} & g - Bg^{-1}B \end{pmatrix} = \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ B & \mathbf{1} \end{pmatrix} \begin{pmatrix} g^{-1} & \mathbf{0} \\ \mathbf{0} & g \end{pmatrix} \begin{pmatrix} \mathbf{1} & -B \\ \mathbf{0} & \mathbf{1} \end{pmatrix} \end{aligned}$$

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  yields the universal spacetime low-energy action for the closed string massless (NS-NS) sector $(\phi, g_{\mu\nu}, B_{\mu\nu})$ ubiquitous in all string theories:

$$\int \mathrm{d}^D x\, \sqrt{-g}\, e^{-2\phi} \left(R_g + 4\partial_\mu\phi\partial^\mu\phi - \tfrac{1}{12}H_{\lambda\mu\nu}H^{\lambda\mu\nu}\right) \quad \text{where} \quad H = \mathrm{d}B$$

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- The action is invariant under the doubled diffeomorphisms:
  - (Undoubled) Diffeomorphisms: $\delta_\xi g_{\mu\nu} = \mathcal{L}_\xi g_{\mu\nu} \quad , \quad \delta_\xi B_{\mu\nu} = \mathcal{L}_\xi B_{\mu\nu} \quad , \quad \delta_\xi \phi = \mathcal{L}_\xi \phi$
  - B-gauge transformations: $\delta_\Lambda B_{\mu\nu} = \partial_\mu \Lambda_\nu - \partial_\nu \Lambda_\mu$

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  - $D=2$ Black hole solution Witten 1991
  - $D=4$ Spherical solution Burgess, Myers, Quevedo 1994
  - $D=10$ Black 5-brane Horowitz, Strominger 1991

Main ansatz

- We focus on the following supergravity ansatz, with $x^\mu = (t, y, z^i)$:

$$\begin{aligned}
\text{Metric} \quad \mathrm{d}s^2 &= \frac{1}{F(x)}\left(-\mathrm{d}t^2 + \mathrm{d}y^2\right) + G_{ij}(x)\mathrm{d}z^i\mathrm{d}z^j \\
\text{Kalb–Ramond field} \quad B_{(2)} &= \pm\frac{1}{F(x)}\,\mathrm{d}t \wedge \mathrm{d}y + \frac{1}{2}\,\beta_{\mu\nu}(x)\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \\
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- Generically, the metric features a curvature singularity

$$R \to \infty \quad , \quad R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} \to \infty \quad \text{whenever} \quad F \to 0.$$

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with Pauli matrices:

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- This suggests that the curvature singularities featured in the considered class of GR spacetimes are artifacts of Riemannian geometry, and have no counterparts in DFT geometry.

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- From the general behavior of particles and strings on non-Riemannian backgrounds, we expect that:
  - geodesics freeze on non-Riemannian points $F = 0$
  - strings become chiral at $F = 0$

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- The $D=2$ black hole solution from Witten 1991 reads:

$$\mathrm{d}s^2 = \frac{\mathrm{d}y^+ \mathrm{d}y^-}{F(y^+, y^-)} \qquad \text{and} \qquad H = 0 \qquad \text{with} \qquad F = -1 + \frac{y^+ y^-}{l^2} = \frac{F}{|F|} e^{-2\phi}.$$

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Example ($D = 10$ black 5-brane)

- We focus on the particular $D = 10$ black 5-brane geometry from Horowitz, Strominger 1991:

$$\mathrm{d}s^2 = \frac{-\mathrm{d}t^2 + \mathrm{d}r^2}{F(r)} + r^2 \mathrm{d}\Omega_3^2 + \mathrm{d}\vec{x}^{\,2} \qquad \text{and} \qquad H = 0 \qquad \text{where} \qquad F = 1 - (r_c/r)^2 = e^{-2\phi}\,.$$

- The Ricci scalar diverges both at $r = 0$ and $r = r_c$ as $R = -\dfrac{4r_c^4}{r^4(r^2 - r_c^2)}$.
- Although the H-flux is trivial, we introduce a pure gauge B-field as $B_{(2)} = \pm \dfrac{1}{F(r)}\,\mathrm{d}t \wedge \mathrm{d}r$.
- The resulting generalised metric is non-Riemannian regular on the 3-sphere of the radius $r = r_c$ (but still singular at $r = 0$).
- In fact, one can show that the non-Riemannian sphere forms the boundary of a geodesically complete space $F > 0$ which excludes the dangerous point $r = 0$.
- More precisely, time-like and non-radial null geodesics cannot reach the non-Riemannian sphere. Only the radial null ones can, albeit taking infinite affine parameter with vanishing proper velocities.
- Moreover, the geodesic deviation is regular with vanishing norm, the only nontrivial values of $R^{\mu}{}_{\nu\rho\sigma}\dot{x}^{\nu}\dot{x}^{\rho}$ at $r = r_c$ being $\pm 2E^2/r_c^2$ for μ, σ being t or r.
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Summary

- Exploring the non-Riemannian sector of DFT allows to go beyond supergravity and to accommodate nonrelativistic physical theories (Gomis–Ooguri, Newton–Cartan, Carroll, *etc.*) as well as to shed new light on issues within GR.

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  1. coordinate singularity: The curvature singularity of Riemannian geometry appears as a coordinate singularity within DFT which can be removed by doubled diffeomorphisms.
  2. curvature singularity: All DFT curvature tensors are regular, as a consequence of the regularity of the generalised metric and dilaton field.

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    2. curvature singularity: All DFT curvature tensors are regular, as a consequence of the regularity of the generalised metric and dilaton field.
    3. geodesic incompleteness: Focusing on particular known supergravity solutions, it is shown that the non-Riemannian points $F = 0$ form an impenetrable sphere where particles freeze and strings become chiral. Computed in the string frame, geodesics outside the non-Riemannian sphere are complete with no singular deviation.

Summary

- Exploring the non-Riemannian sector of DFT allows to go beyond supergravity and to accommodate nonrelativistic physical theories (Gomis–Ooguri, Newton–Cartan, Carroll, *etc.*) as well as to shed new light on issues within GR.
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Thank you for your attention!

Gravitation and Geometry

- Following the paradigmatic example of General Relativity, one can distinguish between two facets of gravitation theories:
  - A kinematical content encoding the geometry of the theory (in turn prescribing the motion of test particules, *etc.*)
    - Differential geometry
    - Metric structure
    - Parallelism

Differential geometry	Metric structure	Parallelism
(Lie) $\mathcal{L}_X\,,\,[\cdot,\cdot]_{\text{Lie}}$	$g_{\mu\nu}$	$\Gamma^{\lambda}_{\mu\nu}$

  - A dynamical content encoding the dynamics of the geometry (as prescribed by the matter/energy content).

Einstein Field equations	
$G_{\mu\nu} = 8\pi G\, T_{\mu\nu}$	$\mu, \nu \in \{1, \dots, D\}$

- There are many options to modify GR *e.g.* :
  - Modify the (Riemannian) geometry of General Relativity by replacing the non-degenerate metric structure $g_{\mu\nu}$ with a degenerate (non-Riemannian) one *cf.* N. Obers talk on Wed and Fri talks
    - Newton–Cartan geometry (non-relativistic physics $c \to \infty$) $h^{\mu\nu}\psi_\nu = 0$
    - Carrollian geometry (ultra-relativistic physics $c \to 0$) $\gamma_{\mu\nu}\xi^\nu = 0$

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  - There are many options to modify GR *e.g.* :
    - Modify the underlying differential geometry to a stringy geometry: Double Field Theory

Double Field Theory (DFT)

Stringy geometry:

- Spacetime is doubled $x^A = (\tilde{x}_\mu, x^\mu)$ and $\partial_A = (\tilde{\partial}^\mu, \partial_\mu)$
  where $A \in \{1, \ldots, 2D\}$ and $\mu \in \{1, \ldots, D\}$
- Spacetime is endowed with a canonical $\mathbf{O}(D, D)$ metric $\mathcal{J}_{AB} = \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix}$.
- Section condition $\quad \partial^A \partial_A \sim 0$
- Generalized Lie derivative $\quad (\hat{\mathcal{L}}_X Y)^A = X^B \partial_B Y^A + (\partial^A X_C - \partial_C X^A) Y^C$
- C-bracket $\quad [X, Y]^A_{\mathsf{C}} := X^B \partial_B Y^A - Y^B \partial_B X^A + \frac{1}{2} Y^B \partial^A X_B - \frac{1}{2} X^B \partial^A Y_B$

$$\left[\hat{\mathcal{L}}_X, \hat{\mathcal{L}}_Y\right] \sim \hat{\mathcal{L}}_{[X,Y]_{\mathsf{C}}}$$ up to the section condition

- Fundamental objects of the theory are the DFT metric $\mathcal{H}_{AB}$ and the dilaton d.
- The Ricci calculus of General Relativity can be generalised to the semi-covariant calculus of DFT:
  - A semi-covariant connection Γ_{ABC}
  - A generalised Ricci tensor $\mathcal{R}_{AB}$
  - A generalised Ricci scalar $\mathcal{R}$

Stringy geometry	Metric structure	Parallelism
(Courant) $\mathcal{J}_{AB}, \hat{\mathcal{L}}_X, [\cdot,\cdot]_{\mathsf{C}}$	$\mathcal{H}_{AB}, d$	Γ_{ABC}

Double Field Theory (DFT)

Stringy geometry:

- The DFT action reads
$$S_{\text{DFT}} = \int d^{2D}X\, e^{-2d}\, \mathcal{R}(\mathcal{H}, d)$$
where the generalised Ricci scalar $\mathcal{R}(\mathcal{H}, d)$ is the unique $\mathbf{O}(D, D)$ scalar built in terms of second derivatives of the fundamental $\mathbf{O}(D, D)$ variables $\mathcal{H}$ and d.

- The associated equations of motion can be unified into:

Einstein Double Field equations $G_{AB} = 8\pi G\, T_{AB}$	$A, B \in \{1, \dots, 2D\}$

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Stringy geometry:

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  where the generalised Ricci scalar $\mathcal{R}(\mathcal{H}, d)$ is the unique $\mathbf{O}(D, D)$ scalar built in terms of second derivatives of the fundamental $\mathbf{O}(D, D)$ variables $\mathcal{H}$ and d.

- Substituting into the $\mathbf{O}(D, D)$ variables the Riemannian parameterisation:

$$\text{$\mathbf{O}(D,D)$ dilaton} \quad e^{-2d} = \sqrt{-g}e^{-2\phi}$$

$$\text{Generalized metric} \quad \mathcal{H}_{AB} = \begin{pmatrix} g^{-1} & -g^{-1}B \\ Bg^{-1} & g - Bg^{-1}B \end{pmatrix} = \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ B & \mathbf{1} \end{pmatrix} \begin{pmatrix} g^{-1} & \mathbf{0} \\ \mathbf{0} & g \end{pmatrix} \begin{pmatrix} \mathbf{1} & -B \\ \mathbf{0} & \mathbf{1} \end{pmatrix}$$

  yields the universal spacetime low-energy action for the closed string massless (NS-NS) sector $(\phi, g_{\mu\nu}, B_{\mu\nu})$ ubiquitous in all string theories:

$$\int \mathrm{d}^D x\, \sqrt{-g}\, e^{-2\phi} \left(R_g + 4\partial_\mu \phi \partial^\mu \phi - \tfrac{1}{12} H_{\lambda\mu\nu} H^{\lambda\mu\nu}\right) \quad \text{where} \quad H = \mathrm{d}B$$

- The action is invariant under the doubled diffeomorphisms:
  - (Undoubled) Diffeomorphisms: $\delta_\xi g_{\mu\nu} = \mathcal{L}_\xi g_{\mu\nu} \quad , \quad \delta_\xi B_{\mu\nu} = \mathcal{L}_\xi B_{\mu\nu} \quad , \quad \delta_\xi \phi = \mathcal{L}_\xi \phi$
  - B-gauge transformations: $\delta_\Lambda B_{\mu\nu} = \partial_\mu \Lambda_\nu - \partial_\nu \Lambda_\mu$

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"Spacetime tells matter how to move. Matter tells spacetime how to curve." J.-A. Wheeler

Kinematical — Dynamical

- Wheeler's quote allows to distinguish between two facets of gravitation theories:
  - A kinematical content encoding the geometry of the theory (in turn prescribing the motion of test particules, *etc.*)
    Generically, the kinematical content of a gravitational theory can be sliced into three (hierarchized) layers, where each layer is supported by the ones above:
    - Differential geometry
    - Metric structure
    - Parallelism
  - A dynamical content encoding the dynamics of the geometry (as prescribed by the matter/energy content).

Light-speed review of GR

- Focusing on the kinematical side of GR, we distinguish between three layers of geometry:
  - Differential geometry

    Spacetime is a manifold $\mathscr{M}$ of dimension $D = d + 1$, carrying natural differential geometric notions such as: vector fields $\Gamma\left(T\mathscr{M}\right)$, differential forms $\Omega^{\bullet}\left(\mathscr{M}\right)$, Lie bracket $[\cdot,\cdot]_{\mathsf{Lie}}$, Lie derivative $\mathcal{L}_X$, Cartan calculus *etc.* (relying on the Lie algebroid structure on $T\mathscr{M}$).

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  - Metric structure

    The fundamental object of GR is a metric tensor $g_{\mu\nu}$ being:
    - Symmetric *i.e.* $g_{\mu\nu} = g_{\nu\mu}$
    - Non-degenerate *i.e.* invertible $g_{\mu\lambda}g^{\lambda\nu} = \delta_\mu{}^\nu$
    - Of Lorentzian signature $(-1, \underbrace{+1, \cdots, +1}_{D-1})$.

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    - Of Lorentzian signature $(-1, \underbrace{+1, \cdots, +1}_{D-1})$.
  - Parallelism

    A notion of parallelism is given in the guise of a connection on $\mathscr{M}$. Among all possible connections, the Levi–Civita connection provides a canonical choice being uniquely determined by:
    - Metric compatibility *i.e.* $\nabla_\lambda g_{\mu\nu} = 0$
    - Torsionfreeness *i.e.* $\Gamma^\lambda_{[\mu\nu]} = 0$.

$$\Rightarrow \Gamma^\lambda_{\mu\nu} = \frac{1}{2} g^{\lambda\rho}\left(\partial_\mu g_{\rho\nu} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu}\right)$$

Light-speed review of GR

- As for the dynamical side of GR:
  - Given our fundamental field, the next step consists in writing (dynamical) equations of motion. A convenient way to obtain equations of motion consists in deriving them from an action functional.
  - Since the square-root of the determinant of the metric tensor is a scalar density of weight 1:
$$\mathcal{L}_\xi\left(\sqrt{-g}\right) = \xi^\mu \partial_\mu\left(\sqrt{-g}\right) + \partial_\mu \xi^\mu \sqrt{-g} = \partial_\mu\left(\sqrt{-g}\,\xi^\mu\right)$$
a convenient way to parametrise the action is as:
$$S = \int_{\mathscr{M}} d^D x \sqrt{-g}\,\mathcal{L}(g)$$
where $\mathcal{L}(g)$ is a scalar tensor *i.e.* $(\delta_\xi - \mathcal{L}_\xi)\mathcal{L}(g) = 0$.
  - This last condition strongly constrains the possible terms that can enter the action.
  - The most general scalar quantity $\mathcal{L}(g)$ built solely in terms of the metric and linear in the second derivatives of g is the Ricci scalar $R(g)$ associated to g. (Vermeil's theorem, 1917).
  - This yields the Einstein–Hilbert action $\quad S = \int_{\mathscr{M}} d^D x \sqrt{-g}\, R(g)$
with associated Einstein equations of motion $G_{\mu\nu} = 8\pi G\, T_{\mu\nu}$ (in the presence of matter/energy).

Supergravity

- The universal spacetime low-energy action for the bosonic sector $(g_{\mu\nu}, B_{\mu\nu}, \phi)$ of oriented closed string theories reads:

$$\int \mathrm{d}^D x \, \sqrt{-g}\, e^{-2\phi} \, \left(R + 4\,\partial_\mu \phi\,\partial^\mu \phi - \tfrac{1}{12} H_{\lambda\mu\nu} H^{\lambda\mu\nu}\right) \quad \text{with} \quad H = \mathrm{d}B$$

where $D = 10$ or 26.

- The corresponding equations of motion take the form:

$$\begin{aligned} R_{\mu\nu} - \frac{1}{4} H_{\mu\rho\sigma} H_\nu{}^{\rho\sigma} + 2\,\nabla_\mu \nabla_\nu \phi &= 0 \text{ (Variation with respect to } g) \\ \frac{1}{2}\nabla^\rho H_{\rho\mu\nu} - H_{\mu\nu\rho}\nabla^\rho \phi &= 0 \text{ (Variation with respect to } B) \\ R + 4\,(\Box\phi - \partial^\mu \phi\,\partial_\mu \phi) - \frac{1}{12} H_{\lambda\mu\nu} H^{\lambda\mu\nu} &= 0 \text{ (Variation with respect to } \phi). \end{aligned}$$

- The action is invariant under the following doubled diffeomorphisms:
  - Diffeomorphisms:
    - $\delta_\xi g_{\mu\nu} = \mathcal{L}_\xi g_{\mu\nu} = \xi^\lambda \partial_\lambda g_{\mu\nu} + g_{\mu\lambda}\partial_\nu \xi^\lambda + g_{\lambda\nu}\partial_\mu \xi^\lambda$
    - $\delta_\xi B_{\mu\nu} = \mathcal{L}_\xi B_{\mu\nu} = \xi^\lambda \partial_\lambda B_{\mu\nu} + B_{\mu\lambda}\partial_\nu \xi^\lambda + B_{\lambda\nu}\partial_\mu \xi^\lambda$
    - $\delta_\xi \phi = \mathcal{L}_\xi \phi = \xi^\lambda \partial_\lambda \phi$
  - B-gauge transformations:
    - $\delta_\Lambda g_{\mu\nu} = 0$
    - $\delta_\Lambda B_{\mu\nu} = \partial_\mu \Lambda_\nu - \partial_\nu \Lambda_\mu$
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- Each term of the action is *separately* invariant under these symmetries hence the relative coefficients are left unfixed by diffeomorphisms and B-field symmetry.

- Athough non-manifest, the action enjoys an additional T-duality symmetry mixing (g, B, ϕ) in a non-trivial manner.

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where $D = 10$ or 26.

- Splitting the coordinates as $x^\mu = \left\{u, x^i\right\}$ and assuming that the coordinate u is an isometric direction (*i.e.* $\partial_u g_{\mu\nu} = \partial_u B_{\mu\nu} = \partial_u \phi = 0$), the following transformation preserves the form of the action:

$$\begin{aligned}
g_{ij} &\mapsto g_{ij} - \frac{g_{ui} g_{uj} - B_{ui} B_{uj}}{g_{uu}} \quad , & g^{ij} &\mapsto g^{ij} \ , \\
g_{ui} &\mapsto \frac{B_{ui}}{g_{uu}} \quad , & g^{ui} &\mapsto -B_{uj}\, g^{ji} \ , \\
g_{uu} &\mapsto \frac{1}{g_{uu}} \quad , & g^{uu} &\mapsto g_{uu} - B_{ui}\, g^{ij} B_{ju} \ , \\
B_{ij} &\mapsto B_{ij} - \frac{g_{ui} B_{uj} - B_{ui} g_{uj}}{g_{uu}} \quad , & B_{ui} &\mapsto \frac{g_{ui}}{g_{uu}} \ , \\
\det g &\mapsto \frac{\det g}{g_{uu}^2} \quad , & \phi &\mapsto \phi - \frac{1}{2} \ln g_{uu}
\end{aligned}$$

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- One can check that only the precise relative coefficients of the action allow for this transformation to be a symmetry. This non-linear and (highly) non-manifest discrete symmetry of the action is known as the Buscher transformation.

- As we will see, the DFT formalism will allow to make this symmetry both linear and manifest.

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GR	Differential geometry (Lie) $\mathcal{L}_X\,,\,[\cdot,\cdot]_{\text{Lie}}$	Metric structure $g_{\mu\nu}$	Parallelism $\Gamma^{\lambda}_{\mu\nu}$
DFT	Stringy geometry (Courant algebroid) $\mathcal{J}_{AB}\,,\,\hat{\mathcal{L}}_X\,,\,[\cdot,\cdot]_{\text{C}}$	Metric structure $\mathcal{H}_{AB}\,,\,d$	Parallelism Γ_{ABC}

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DFT	Einstein Double Field equations $G_{AB} = 8\pi G\, T_{AB}$	$A, B \in \{1, \ldots, 2D\}$

Non-Riemannian geometries

- Geometry is not the privilege of relativistic physics.
- As soon as 1923, E. Cartan proposed a geometrical reformulation of Newton's theory of gravitation.
- Modify the (Riemannian) geometry of General Relativity by replacing the non-degenerate metric structure $g_{\mu\nu}$ with a degenerate (non-Riemannian) one.
- Non-Riemannian geometries come in two (main) flavors:
  - Newton–Cartan geometry (non-relativistic physics $c \to \infty$) $h^{\mu\nu}\psi_\nu = 0$

    Condensed matter , Effective field theories , (fractional) Quantum Hall effect

    Hydrodynamics , Hořava-Lifshitz gravity , Galilean string , *etc.*

  - Carrollian geometry (ultra-relativistic physics $c \to 0$) $\gamma_{\mu\nu}\xi^\nu = 0$

    BMS group , Null Hydrodynamics , Flat holography , Carrollian string , *etc.*

- From the viewpoint of GR, any geometry featuring a degenerate metric is singular. Nevertheless, these are well-defined as non-Riemannian geometries.

Stringy geometry

- Spacetime is doubled

$$X^A = (\tilde{x}_\mu, x^\mu) \text{ and } \partial_A = (\tilde{\partial}^\mu, \partial_\mu)$$

  where $A \in \{1, \ldots, 2D\}$ and $\mu \in \{1, \ldots, D\}$.

- The double spacetime is naturally endowed with a canonical $\mathbf{O}(D, D)$ metric $\mathcal{J}_{AB} = \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix}$.

- A further condition is imposed so that the fields of the theory only depend on half of the coordinates.

- The so-called section condition or strong constraint reads

$$\partial^A \partial_A (\cdot) := \mathcal{J}^{AB} \partial_A \partial_B (\cdot) = 0$$

  where $(\cdot)$ is a place holder for fields and gauge parameters of the theory *as well as any products* between those. This implies that $\partial^A \partial_A \Phi = 0$ and $\partial^A \Phi \, \partial_A \Psi = 0$ for any field and gauge parameter of the theory.

- In terms of D coordinates, the section condition reads $\partial_\mu \tilde{\partial}^\mu (\cdot) = 0$.

- The solution $\tilde{\partial}^\mu (\cdot) = 0$ (*i.e.* all the fields are independent of the tilde coordinates $\tilde{x}_\mu$) is called the supergravity frame.

Generalised Lie derivative

Similarly to differential geometry, one can define a stringy geometry avatar of the notion of Lie derivative.

- The generalised Lie derivative on the tensorial density V^A of weight $\omega(V)$ along the vector field ξ^A reads:
$$\hat{\mathcal{L}}_\xi V^A := \xi^C \partial_C V^A + (\partial^A \xi_C - \partial_C \xi^A) V^C + \omega(V) \partial_C \xi^C V^A .$$
where doubled indices are raised and lowered with the $\mathbf{O}(D, D)$ metric $\mathcal{J}$.

- The generalised Lie derivative annihilates the $\mathbf{O}(D, D)$ metric $\mathcal{J}$ as well as the Kronecker delta *i.e.* :
$$\hat{\mathcal{L}}_\xi \mathcal{J}^{AB} = 0 \quad , \quad \hat{\mathcal{L}}_\xi \mathcal{J}_{AB} = 0 \quad , \quad \hat{\mathcal{L}}_\xi \delta^A_B = 0 .$$

- The generalised Lie derivative has a non-trivial kernel. Namely, any vector field of the form $\xi = \mathcal{J}^{-1} \partial \chi$ (*i.e.* $\xi^A := \partial^A \chi$) is annihiliated as:
$$\hat{\mathcal{L}}_{\mathcal{J}^{-1}\partial\chi} V \sim 0 \quad \text{for any } \chi, V.$$

- As was the case in differential geometry, the generalised Lie derivative can be used to formulate an operative definition of "$\mathbf{O}(D, D)$ tensoriality":

  A quantity T is $\mathbf{O}(D, D)$ tensorial if and only if $(\delta_\xi - \hat{\mathcal{L}}_\xi) T = 0$.

- Example:
  Let V^A be a $\mathbf{O}(D, D)$ vector field so that $(\delta_\xi - \hat{\mathcal{L}}_\xi) V^A = 0$.
  Now, consider the partial derivative $\partial_A V^B$.
  We compute $(\delta_\xi - \hat{\mathcal{L}}_\xi)(\partial_A V^B) \sim V^C (\partial_A \partial^B \xi_C - \partial_{AC} \xi^B) \neq 0$.

C-bracket

- The generalised Lie derivative satisfies the closure property

$$\hat{\mathcal{L}}_X \circ \hat{\mathcal{L}}_Y T - \hat{\mathcal{L}}_Y \circ \hat{\mathcal{L}}_X T \sim \hat{\mathcal{L}}_{[X,Y]_{\mathsf{C}}} T$$

where $[\cdot,\cdot]_{\mathsf{C}}$ stands for the C-bracket of vector fields and is defined explicitly as:

$$[X,Y]^A_{\mathsf{C}} := X^B \partial_B Y^A - Y^B \partial_B X^A + \frac{1}{2} Y^B \partial^A X_B - \frac{1}{2} X^B \partial^A Y_B .$$

- The C-bracket is skewsymmetric and satisfies a deformed Jacobi identity:

$$\Big[X,[Y,Z]_{\mathsf{C}}\Big]^A_{\mathsf{C}} + \Big[Y,[Z,X]_{\mathsf{C}}\Big]^A_{\mathsf{C}} + \Big[Z,[X,Y]_{\mathsf{C}}\Big]^A_{\mathsf{C}} \sim \partial^A N(X,Y,Z)$$

with $N(X,Y,Z) = \frac{1}{6}\Big(\Big\langle [X,Y]_{\mathsf{C}} \, , Z \Big\rangle + \text{cycl.}\Big)$ the Nijenhuis tensor.

- The fact that the obstruction $\partial^A N(X,Y,Z)$ is of the form $\mathcal{J}^{-1}\partial\chi$ (hence in the kernel of $\hat{\mathcal{L}}$) ensures that the closure condition for generalised Lie derivatives is well-defined.

Classification of generalised metrics

- Solutions to the defining equations of the DFT generalised metric:

$$\underbrace{\mathcal{H}_{AB}=\mathcal{H}_{BA}}_{\text{Symmetric}}\quad,\quad\underbrace{\mathcal{H}_{A}{}^{C}\mathcal{H}_{B}{}^{D}\mathcal{J}_{CD}=\mathcal{J}_{AB}}_{\mathbf{O(D,D)}}$$

are classified by two non-negative integers $(n,\bar{n})$ such that $0\leq n+\bar{n}\leq D$.

- The explicit form of the most general DFT metric is given by:

$$\mathcal{H}_{AB}=\begin{pmatrix}H^{\mu\nu} & -H^{\mu\sigma}B_{\sigma\lambda}+Y_{i}^{\mu}X_{\lambda}^{i}-\bar{Y}_{\bar{\imath}}^{\mu}\bar{X}_{\lambda}^{\bar{\imath}}\\ B_{\kappa\rho}H^{\rho\nu}+X_{\kappa}^{i}Y_{i}^{\nu}-\bar{X}_{\kappa}^{\bar{\imath}}\bar{Y}_{\bar{\imath}}^{\nu} & K_{\kappa\lambda}-B_{\kappa\rho}H^{\rho\sigma}B_{\sigma\lambda}+2X_{(\kappa}^{i}B_{\lambda)\rho}Y_{i}^{\rho}-2\bar{X}_{(\kappa}^{\bar{\imath}}B_{\lambda)\rho}\bar{Y}_{\bar{\imath}}^{\rho}\end{pmatrix}$$

- In matrix form:

$$\mathcal{H}_{AB}=\begin{pmatrix}\mathbf{1} & \mathbf{0}\\ B & \mathbf{1}\end{pmatrix}\begin{pmatrix}H & Y_{i}(X^{i})^{T}-\bar{Y}_{\bar{\imath}}(\bar{X}^{\bar{\imath}})^{T}\\ X^{i}(Y_{i})^{T}-\bar{X}^{\bar{\imath}}(\bar{Y}_{\bar{\imath}})^{T} & K\end{pmatrix}\begin{pmatrix}\mathbf{1} & -B\\ \mathbf{0} & \mathbf{1}\end{pmatrix}$$

where the metrics $H^{\mu\nu}$ and $K_{\mu\nu}$ are of rank $D-(n+\bar{n})$.

Classification of generalised metrics

- The explicit form of the most general DFT metric is given by:

$$\mathcal{H}_{AB} = \begin{pmatrix} H^{\mu\nu} & -H^{\mu\sigma}B_{\sigma\lambda} + Y_i^\mu X_\lambda^i - \bar{Y}_{\bar{\imath}}^\mu \bar{X}_\lambda^{\bar{\imath}} \\ B_{\kappa\rho}H^{\rho\nu} + X_\kappa^i Y_i^\nu - \bar{X}_\kappa^{\bar{\imath}} \bar{Y}_{\bar{\imath}}^\nu & K_{\kappa\lambda} - B_{\kappa\rho}H^{\rho\sigma}B_{\sigma\lambda} + 2X^i_{(\kappa}B_{\lambda)\rho}Y_i^\rho - 2\bar{X}^{\bar{\imath}}_{(\kappa}B_{\lambda)\rho}\bar{Y}_{\bar{\imath}}^\rho \end{pmatrix}$$

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  where the metrics $H^{\mu\nu}$ and $K_{\mu\nu}$ are of rank $D - (n + \bar{n})$.

- Symmetries: $H^{\mu\nu} = H^{\nu\mu},\ K_{\mu\nu} = K_{\nu\mu},\ B_{\mu\nu} = -B_{\nu\mu}$
- Kernels: Both H and K admit two kinds of zero eigenvectors, with $i = 1, \ldots, n$ and $\bar{\imath} = 1, \ldots, \bar{n}$:

$$H^{\mu\nu}X_\mu^i = 0 \quad , \quad H^{\mu\nu}\bar{X}_\mu^{\bar{\imath}} = 0 \quad , \quad K_{\mu\nu}Y_i^\mu = 0 \quad , \quad K_{\mu\nu}\bar{Y}_{\bar{\imath}}^\mu = 0$$

- Completeness: $H^{\mu\rho}K_{\rho\nu} + Y_i^\mu X_\nu^i + \bar{Y}_{\bar{\imath}}^\mu \bar{X}_\nu^{\bar{\imath}} = \delta^\mu{}_\nu$
- Invariance: $\mathcal{H}$ is preserved by boosts $Y_i^\mu \mapsto H^{\mu\nu}V_{\nu i} \quad , \quad \bar{Y}_{\bar{\imath}}^\mu \mapsto H^{\mu\nu}\bar{V}_{\nu\bar{\imath}}$
  together with corresponding transformations of K and B.

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  where the metrics $H^{\mu\nu}$ and $K_{\mu\nu}$ are of rank $D-(n+\bar{n})$.

- Different choices of $(n,\bar{n})$ yield different parameterisations of the DFT metric.
- The $(0,0)$ case reproduces the Riemannian ansatz yielding supergravity.
- For all other cases (*i.e.* $n+\bar{n}>0$), the induced metric structure is necessarily degenerate.
- The DFT framework thus allows to go beyond supergravity by including non-Riemannian geometries.

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- Upon a generic $(n,\bar{n})$ background:
  - Free particles freeze along the $n+\bar{n}$ non-Riemannian directions: $X_{\mu}^{i}\,\dot{x}^{\mu}=0 \quad , \quad \bar{X}_{\mu}^{\bar{\imath}}\,\dot{x}^{\mu}=0$
  - Strings become
    - chiral along n directions: $X_{\mu}^{i}\,\partial_{+}x^{\mu}(\tau,\sigma)=0$
    - anti-chiral along $\bar{n}$ directions: $\bar{X}_{\mu}^{\bar{\imath}}\,\partial_{-}x^{\mu}(\tau,\sigma)=0$

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where the metrics $H^{\mu\nu}$ and $K_{\mu\nu}$ are of rank $D-(n+\bar{n})$.

- The classification is point-wise, so that the $(n,\bar{n})$-type of a given generalised metric can vary according to the spacetime point.
- As shown in Cho, Park 19' the $(n,\bar{n})$-type is not dynamically protected. However, the trace $\mathcal{H}_A{}^A = 2\,(n-\bar{n})$ remains invariant.
- The minimal non-Riemannian deviation from $(0,0)$ is therefore $(1,1)$ which have been found to accommodate nonrelativistic theories such as Gomis–Ooguri, Newton–Cartan, Carroll *etc.*

From Lie to Courant

- The previous relations differ from the usual differential geometry case and hence do not define a Lie algebroid but rather constitute the defining equations of a Courant algebroid (introduced by Liu, Weinstein and Xu 97').

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DFT

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DFT Courant Algebroids K. Morand (21')

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DFT — Courant Algebroids — K. Morand (21')

- As in the Lie algebroid case, the Courant structure underlying the stringy geometry of DFT allows to define notions of connection, torsion, curvature, etc.

Field content of DFT

- Recall that the universal gravitational massless sector ubiquitous in string theory (bosonic, heterotic, Type II, closed sector of Type I) takes the form:
  - $g_{\mu\nu}$: D-dimensional metric
  - $B_{\mu\nu}$: two-form field
  - ϕ: scalar dilaton field
- These are not $\mathbf{O}(D, D)$-tensors.
- Since we are looking for an $\mathbf{O}(D, D)$ invariant theory, the fundamental fields are required to be $\mathbf{O}(D, D)$ tensors with $2D$ dimensional indices $A, B, \ldots$
  - $g_{\mu\nu}$ and $B_{\mu\nu}$ are unified in the generalised metric $\mathcal{H}_{AB}$.
  - ϕ is combined with $\det g$ to yield a $\mathbf{O}(D, D)$ singlet dilaton density d.

Generalised metric

- The generalised metric $\mathcal{H}$ is defined as a symmetric $\mathbf{O}(D,D)$ element *i.e.* satisfies the following relations:

$$\underbrace{\mathcal{H}_{AB} = \mathcal{H}_{BA}}_{\text{Symmetric}} \quad , \quad \underbrace{\mathcal{H}_A{}^C \mathcal{H}_B{}^D \mathcal{J}_{CD} = \mathcal{J}_{AB}}_{\mathbf{O(D,D)}} .$$

- Under the decomposition $X^A = (\tilde{x}_\mu, x^\mu)$, the generalised metric splits into:

$$\mathcal{H}_{AB} = \begin{pmatrix} \mathcal{H}^{\mu\nu} & \mathcal{H}^\mu{}_\lambda \\ \mathcal{H}_\kappa{}^\nu & \mathcal{H}_{\kappa\lambda} \end{pmatrix}$$

so that the defining conditions decompose as:

$$\mathcal{H}^{\mu\nu} = \mathcal{H}^{\nu\mu} \quad , \qquad \mathcal{H}_{\mu\nu} = \mathcal{H}_{\nu\mu} \quad , \qquad \mathcal{H}_\mu{}^\nu = \mathcal{H}^\nu{}_\mu \, ,$$
$$\mathcal{H}^{(\mu}{}_\rho \mathcal{H}^{\nu)\rho} = 0 \quad , \qquad \mathcal{H}_{\rho(\mu} \mathcal{H}^\rho{}_{\nu)} = 0 \quad , \qquad \mathcal{H}^\mu{}_\rho \mathcal{H}^\rho{}_\nu + \mathcal{H}^{\mu\rho} \mathcal{H}_{\rho\nu} = \delta^\mu{}_\nu \, .$$

- Assuming that the upper left block $\mathcal{H}^{\mu\nu}$ is non-degenerate, we may identify it as the inverse of a Riemannian metric *i.e.* $\mathcal{H}^{\mu\nu} = g^{\mu\nu}$.

- The remaining constraints are all solved by a skew-symmetric B-field ($B_{\mu\nu} = -B_{\nu\mu}$) such that the most general DFT-metric in this case takes the well-known form:

$$\mathcal{H}_{AB} = \begin{pmatrix} g^{\mu\nu} & -g^{\mu\sigma} B_{\sigma\lambda} \\ B_{\kappa\rho} g^{\rho\nu} & g_{\kappa\lambda} - B_{\kappa\rho} g^{\rho\sigma} B_{\sigma\lambda} \end{pmatrix} .$$

Generalised metric

- The generalised Lie derivative of the DFT-metric takes the form:

$$\hat{\mathcal{L}}_\xi \mathcal{H}_{AB} = \xi^C \partial_C \mathcal{H}_{AB} + (\partial_A \xi^C - \partial^C \xi_A)\mathcal{H}_{CB} + (\partial_B \xi^C - \partial^C \xi_B)\mathcal{H}_{AC}\,.$$

- Under the decomposition $X^A = (\tilde{x}_\mu, x^\mu)$, defining $\xi^A = (\tilde{\xi}_\mu, \xi^\mu)$ and assuming $\tilde{\partial}^\mu = 0$, the latter yields:
  - $(\hat{\mathcal{L}}_\xi \mathcal{H})^{\mu\nu} = \mathcal{L}_\xi \mathcal{H}^{\mu\nu}$
  - $(\hat{\mathcal{L}}_\xi \mathcal{H})^{\mu}{}_{\nu} = \mathcal{L}_\xi \mathcal{H}^{\mu}{}_{\nu} + 2\,\mathcal{H}^{\mu\lambda}\partial_{[\nu}\tilde{\xi}_{\lambda]}$
  - $(\hat{\mathcal{L}}_\xi \mathcal{H})_{\mu}{}^{\nu} = \mathcal{L}_\xi \mathcal{H}_{\mu}{}^{\nu} + 2\,\mathcal{H}^{\lambda\nu}\partial_{[\mu}\tilde{\xi}_{\lambda]}$
  - $(\hat{\mathcal{L}}_\xi \mathcal{H})_{\mu\nu} = \mathcal{L}_\xi \mathcal{H}_{\mu\nu} + 2\,\mathcal{H}_{\mu}{}^{\lambda}\partial_{[\nu}\tilde{\xi}_{\lambda]} + 2\,\mathcal{H}^{\lambda}{}_{\nu}\partial_{[\mu}\tilde{\xi}_{\lambda]}$
- In the Riemannian parameterisation, this is equivalent to:
  - $\delta g_{\mu\nu} = \mathcal{L}_\xi g_{\mu\nu}$
  - $\delta B_{\mu\nu} = \mathcal{L}_\xi B_{\mu\nu} + 2\,\partial_{[\mu}\tilde{\xi}_{\nu]}$.
- The generalised metric $\mathcal{H}$ thus allows to combine (g, B) in an $\mathbf{O}(D, D)$ invariant manner while the generalised Lie derivative recovers the symmetries of the supergravity action as doubled diffeomorphisms.

DFT action

- The DFT action reads $$S_{\text{DFT}} = \int d^{2D}X\, e^{-2d}\, \mathcal{R}(\mathcal{H}, d)$$

  where $\mathcal{R}(\mathcal{H}, d)$ is called the generalised Ricci scalar and is the unique $\mathbf{O}(D, D)$ scalar built in terms of second derivatives of the fundamental $\mathbf{O}(D, D)$ variables $\mathcal{H}$ and d.

- Explicitly, the generalised Ricci scalar takes the form:

$$\begin{aligned} \mathcal{R}(\mathcal{H}, d) \;\; &:= \;\; \mathcal{H}^{AB}\left(\frac{1}{8}\,\partial_A \mathcal{H}_{CD}\,\partial_B \mathcal{H}^{CD} + \frac{1}{2}\,\partial^C \mathcal{H}_{AD}\,\partial^D \mathcal{H}_{BC} - 4\,\partial_A d\,\partial_B d + 4\,\partial_A \partial_B d\right) \\ &\quad -\partial_A \partial_B \mathcal{H}^{AB} + 4\,\partial_A \mathcal{H}^{AB}\,\partial_B d. \end{aligned}$$

- The variation of the DFT action with respect to the generalised metric $\mathcal{H}_{AB}$ and dilaton d yields:

  DFT vacuum equations $$\mathcal{R}_{AB} = 0 \quad , \quad \mathcal{R} = 0.$$

- Upon solving the section condition as $\tilde{\partial}^\mu = 0$ (supergravity frame) and plugging the Riemann ansatz

$$\mathcal{H}_{AB} = \begin{pmatrix} g^{\mu\nu} & -g^{\mu\sigma} B_{\sigma\lambda} \\ B_{\kappa\rho} g^{\rho\nu} & g_{\kappa\lambda} - B_{\kappa\rho} g^{\rho\sigma} B_{\sigma\lambda} \end{pmatrix} \text{ and } e^{-2d} = \sqrt{-g}\, e^{-2\phi}$$

  we recover the supergravity action:

$$S_{\text{DFT}} = \int \mathrm{d}^D x\, \sqrt{-g}\, e^{-2\phi} \left(R + 4\,\partial_\mu \phi\,\partial^\mu \phi - \tfrac{1}{12} H_{\lambda\mu\nu} H^{\lambda\mu\nu}\right) \quad \text{with} \quad H = \mathrm{d}B.$$

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- The Ricci calculus of General Relativity can be generalised to the semi-covariant calculus of DFT:
  - A semi-covariant connection Γ_{ABC}
  - A semi-covariant Riemann curvature S_{ABCD}
  - A generalised Ricci tensor $\mathcal{R}_{AB}$
  - A generalised Ricci scalar $\mathcal{R}$
  - A generalised Einstein tensor G_{AB}:
    - $\nabla^A G_{AB} = 0$ (divergenceless)
    - $G_{[AB]} \neq 0$ (not symmetric)
    - $G_{AB} = 0$ unifies both equations of motion

      $\mathcal{R}_{AB} = 0 \quad , \quad \mathcal{R} = 0.$

Symmetries of the DFT action

- On top of the generalised diffeomorphism invariance, the action enjoys a global $\mathbf{O}(D,D)$ invariance:

$$h \in \mathbf{O}(D,D) \quad , \quad X^A \mapsto h_B{}^A X^B \quad , \quad \mathcal{H}_{AB}(X) \mapsto h_A{}^C h_B{}^D \mathcal{H}_{CD}(hX) \quad , \quad d(X) \mapsto d(hX).$$

- Choosing h as a factorized T-duality transformation $h_A{}^B(t) = \begin{pmatrix} \delta^i{}_j - t^i{}_j & t^{ik} \\ t_{lj} & \delta_l{}^k - t_l{}^k \end{pmatrix}$ where t is a $D \times D$ matrix of the form $t = \text{diag}(\underbrace{0 \cdots 0}_{n-1}\, 1\, 0 \cdots 0)$.

- We will denote u the n^{th}-direction. The factorized T-duality transformation exchanges u and $\tilde{u}$ *i.e.* :

$$X^A = (\tilde{u}, \tilde{x}_i, u, x^i) \Rightarrow h_B{}^A X^B = (u, \tilde{x}_i, \tilde{u}, x^i).$$

- Under the following assumptions:

   1. Supergravity frame: $\tilde{\partial}^\mu = 0$ (*i.e.* $\tilde{\partial}^u = 0$, $\tilde{\partial}^i = 0$)
   2. The direction u is an isometry (*i.e.* $\partial_u = 0$)

   the factorized T-duality transformation allows to recast the *non-linear* Buscher rules on (g, B, ϕ) as a *linear* transformation on $\mathcal{H}$ and d.

- The isometry condition is necessary for the transformed fields to satisfy $\tilde{\partial}^u = 0$ and thus to land inside the supergravity frame.

Summary

- The stringy geometry of DFT possesses a firm algebraic underpinning provided by the underlying $\mathbf{O}(D,D)$ Courant algebroid structure $(\mathcal{J}, \hat{\mathcal{L}}_X, [\cdot,\cdot]_{\mathrm{C}})$.
- The fundamental fields $(\mathcal{H}_{AB}, d)$ of DFT are $\mathbf{O}(D,D)$ tensors. In the Riemannian parameterisation, these unify the low-energy spectrum of closed string theories $(g_{\mu\nu}, B_{\mu\nu}, \phi)$ as well as the corresponding symmetries.
- The symmetries of DFT allow to uniquely fix the dynamics of the fundamental $\mathbf{O}(D,D)$ fields $(\mathcal{H}_{AB}, d)$. In the Riemannian parameterisation, the corresponding action reproduces the universal low-energy effective action of closed string theories, while making manifest the underlying T-duality symmetry.
- The Ricci calculus of General Relativity can be generalised to the semi-covariant calculus of DFT. The latter allows to construct genuine tensorial quantities, including the generalised Einstein tensor G_{AB} forming the left (geometric) side of Einstein Double Field Equations (*cf.* Stephen's talk).
- Exploring the non-Riemannian sector of DFT allows to go beyond supergravity and to accommodate nonrelativistic physical theories (Newton–Cartan, Carroll, Gomis–Ooguri, *etc.*) as well as to shed new light on well-known GR problems (*cf.* Miok's talk).

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Thank you for your attention!