Scattering amplitudes and their classical limits

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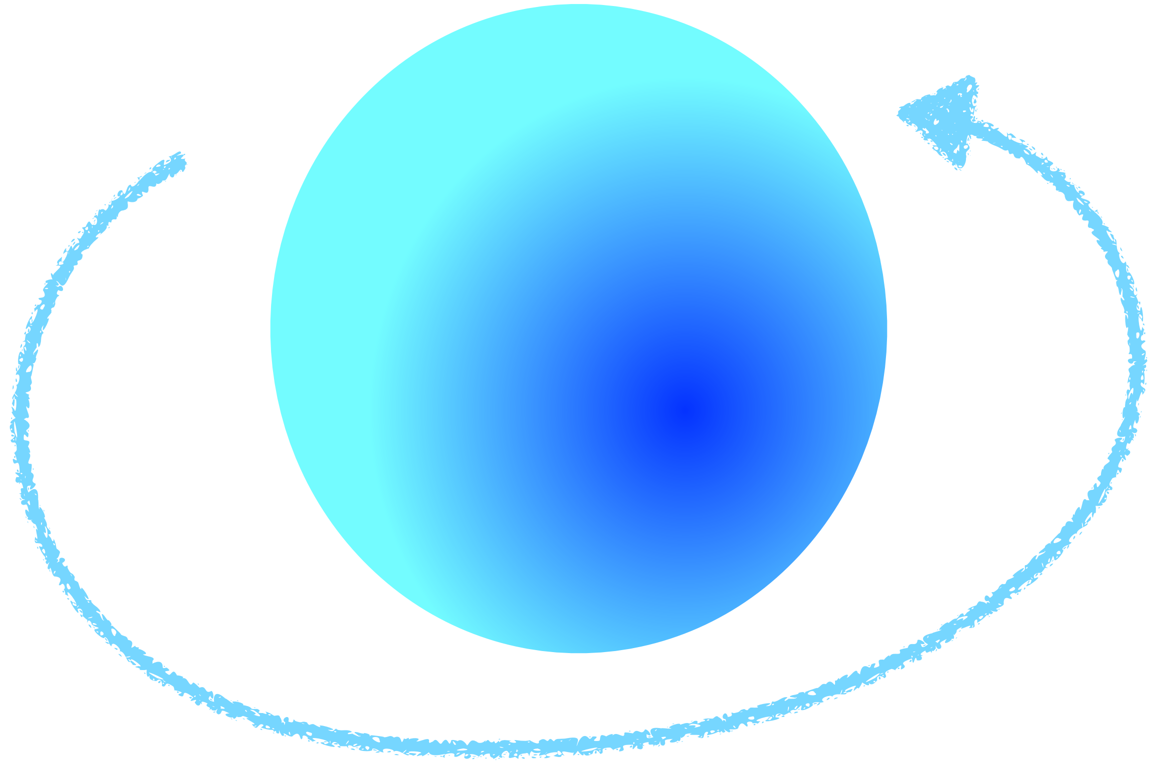
PART I

Spin-dependent



Scattering amplitudes
and their classical limits

Spinor-helicity variables

$$p_{a\dot{a}} = p_\mu (\sigma^\mu)_{a\dot{a}}$$

$$\det(p) = 0$$

Massless : $$p_{a\dot{a}} = -\lambda_a \bar{\lambda}_{\dot{a}} = -|p]_a \langle p|_{\dot{a}}$$

Massive : $$p_{a\dot{a}} = -\lambda_{aI} \bar{\lambda}^I{}_{\dot{a}} = -|p]_{aI} \langle p|^I{}_{\dot{a}}$$

$$\det(p) = m^2$$

$$\det(\lambda) = m$$

Spinor-helicity variables

"Should I learn how to use them?"

QFT course, 1st semester

$$\Psi(x) = \int \widetilde{dp} \left[e^{ipx} u_I(p) b^I + e^{-ipx} v^I(p) b_I^\dagger \right]$$

$$\{b^I(p), b_J^\dagger(q)\} = (2\pi)^3 (2\omega_{\vec{p}}) \delta^3(\vec{p} - \vec{q}) \, \delta^I{}_J$$

↑

Wigner's "little group" SU(2)
for massive particles

QFT course, 1st semester

[Peskin-Schoeder, chapter 3]

There are two linearly independent solutions for $u(p)$,

$$u^s(p) = \begin{pmatrix} \sqrt{p\cdot\sigma}\,\xi^s \\ \sqrt{p\cdot\bar\sigma}\,\xi^s \end{pmatrix}, \qquad s = 1, 2$$

which we normalize according to

$$\bar u^r(p) u^s(p) = 2m\delta^{rs} \qquad \text{or} \qquad u^{r\dagger}(p) u^s(p) = 2E_{\mathbf{p}}\delta^{rs}.$$

δ^{rs} makes no sense! $\delta^{\bar{r}s}$ or $\delta^{r}{}_{s}$ will do.

QFT course, 1st semester

Little-group covariant notation:

$$p_{a\dot{a}} = -\lambda_{aI}\bar{\lambda}^{I}{}_{\dot{a}}$$

$$u_I(p) = \begin{pmatrix} \lambda_{aI} \\ \bar{\lambda}^{\dot{a}}{}_I \end{pmatrix}, \quad v^I(p) = \begin{pmatrix} -\lambda_{aJ} \\ \bar{\lambda}^{\dot{a}}{}_J \end{pmatrix} \epsilon^{JI},$$

$$\bar{u}^I(p) = \begin{pmatrix} \lambda^{Ia} & \bar{\lambda}^{I}{}_{\dot{a}} \end{pmatrix}, \quad \bar{v}_I(p) = \epsilon_{IJ} \begin{pmatrix} \lambda^{Ja} & -\bar{\lambda}^{J}{}_{\dot{a}} \end{pmatrix}.$$

Fine. But, are they useful in practical computations?

Helicity basis

[Jacob-Wick 1959]

3d "helicity basis" $\hat{k} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$

$$\hat{k}\cdot\vec{\sigma} = |k_+)(k_+| - |k_-)(k_-| \,,$$

$$|k_+) = \begin{pmatrix} c_{\theta/2} \\ e^{+i\phi}s_{\theta/2} \end{pmatrix} \,, \quad |k_-) = \begin{pmatrix} -e^{-i\phi}s_{\theta/2} \\ c_{\theta/2} \end{pmatrix} \,.$$

Helicity basis

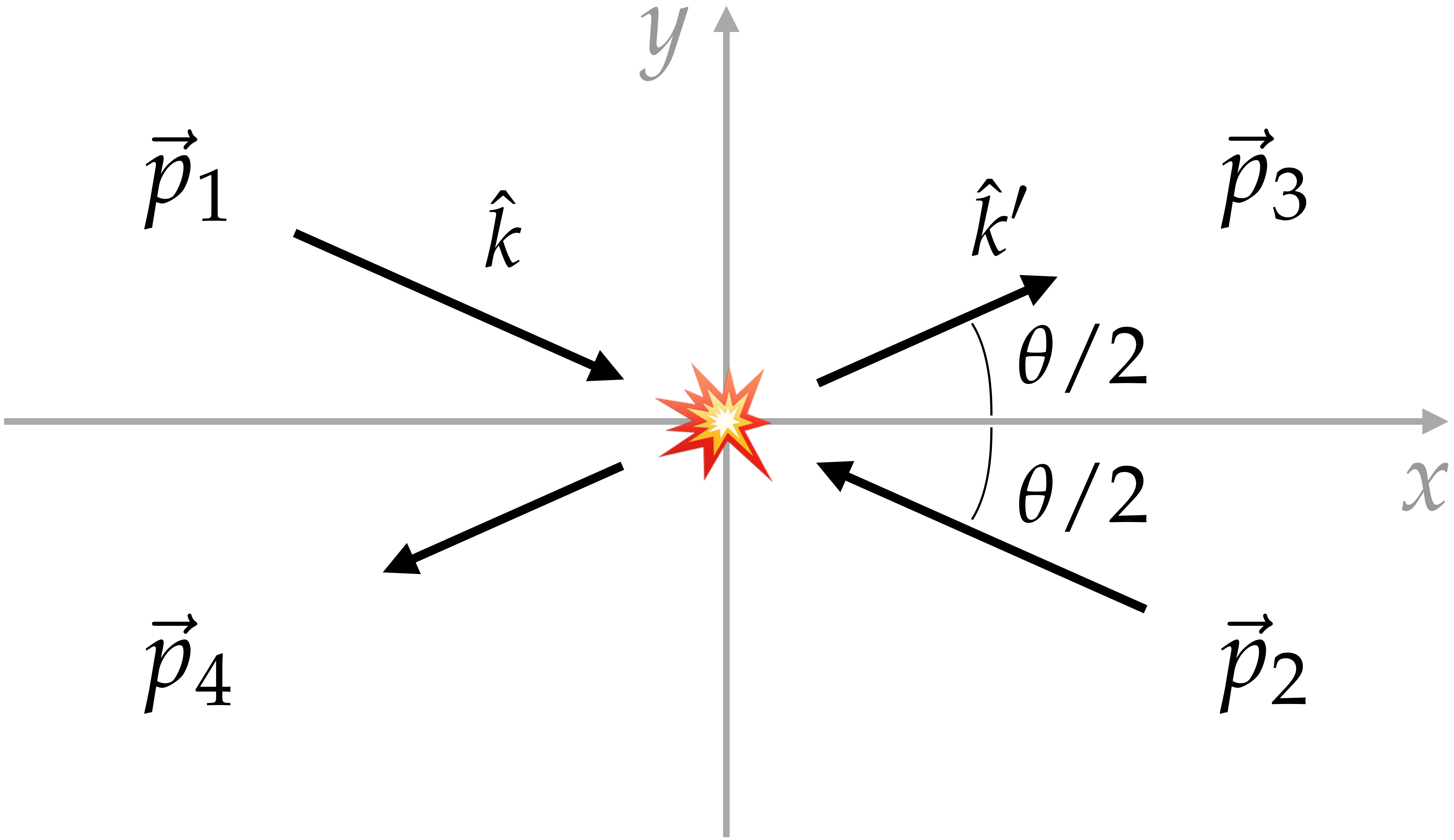
(1+3)d "helicity basis" $E = m\cosh\rho\,, \quad \vec{k} = m\sinh\rho\,\hat{k}\,.$

$$\lambda_{aI} = \sqrt{m}\left(e^{-\rho/2}|k_+) \quad e^{+\rho/2}|k_-)\right), \quad \bar{\lambda}^{I}{}_{\dot{a}} = \sqrt{m}\begin{pmatrix} e^{-\rho/2}(k_+| \\ e^{+\rho/2}(k_-| \end{pmatrix},$$

$$\bar{\lambda}^{\dot{a}}{}_{I} = \sqrt{m}\left(e^{+\rho/2}|k_+) \quad e^{-\rho/2}|k_-)\right), \quad \lambda^{Ia} = \sqrt{m}\begin{pmatrix} e^{+\rho/2}(k_+| \\ e^{-\rho/2}(k_-| \end{pmatrix}.$$

A QED process

$e^- e^+ \rightarrow \mu^+ \mu^-$



A QED process

$$e^- e^+ \to \mu^+ \mu^-$$

$$\mathcal{M}^{I_3 I_4}{}_{I_1 I_2} = \frac{e^2}{s} \left[\bar{u}^{I_3}(p_3) \gamma_\mu v^{I_4}(p_4) \right] \left[\bar{v}_{I_2}(p_2) \gamma^\mu u_{I_1}(p_1) \right]$$

↑

(4x4) matrix

↓

$$\mathcal{M} = -\frac{2e^2}{s} \left(\langle \mathbf{31} \rangle [\mathbf{42}] + \langle \mathbf{42} \rangle [\mathbf{31}] + \langle \mathbf{41} \rangle [\mathbf{32}] + \langle \mathbf{32} \rangle [\mathbf{41}] \right)$$

↑ ↑

(2x2) blocks

A QED process

$$e^- e^+ \to \mu^+ \mu^-$$

$$\widetilde{\mathcal{M}} = -\frac{1}{2m_e m_\mu}\left(\langle\mathbf{31}\rangle[\mathbf{42}] + \langle\mathbf{42}\rangle[\mathbf{31}] + \langle\mathbf{41}\rangle[\mathbf{32}] + \langle\mathbf{32}\rangle[\mathbf{41}]\right)$$

$$= \begin{pmatrix} c_\theta & -i\mathrm{ch}_e s_\theta & -i\mathrm{ch}_e s_\theta & c_\theta \\ -i\mathrm{ch}_\mu s_\theta & \mathrm{ch}_\mu \mathrm{ch}_e(c_\theta + 1) & \mathrm{ch}_\mu \mathrm{ch}_e(c_\theta - 1) & -i\mathrm{ch}_\mu s_\theta \\ -i\mathrm{ch}_\mu s_\theta & \mathrm{ch}_\mu \mathrm{ch}_e(c_\theta - 1) & \mathrm{ch}_\mu \mathrm{ch}_e(c_\theta + 1) & -i\mathrm{ch}_\mu s_\theta \\ c_\theta & -i\mathrm{ch}_e s_\theta & -i\mathrm{ch}_e s_\theta & c_\theta \end{pmatrix}$$

$$m_e \cosh\rho_e = E = m_\mu \cosh\rho_\mu$$

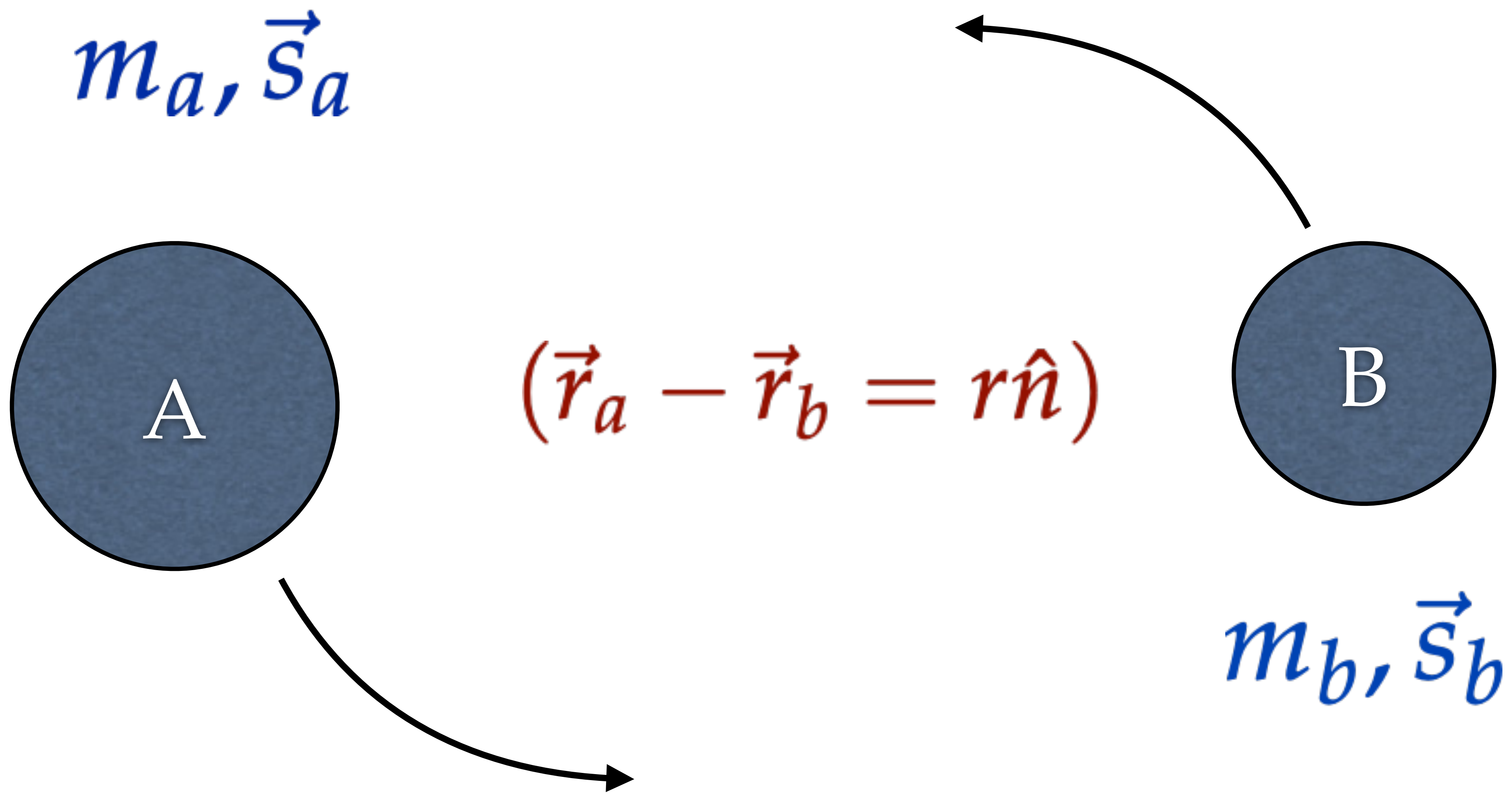
More info to be available in

[H. Lee, SL, S. Mazumdar]

in the 2nd half of 2022

PART II

Binary system with spin



Newton : $$H = \frac{1}{2m_a}\vec{p}_a^{\,2} + \frac{1}{2m_b}\vec{p}_b^{\,2} - \frac{G m_a m_b}{r}\,.$$

Einstein : $$\mathcal{S} = \frac{1}{16\pi G}\int d^4x\sqrt{-g}R \, + \, \mathcal{S}_{\text{matter}}\,.$$

Newton ~ leading approximation to Einstein

Between Newton and Einstein

Hamiltonian in the COM frame: $\vec{p}_a = \vec{p} = -\vec{p}_b$

Post-Newtonian (PN) :

$$H = \left(\frac{1}{2m_a} + \frac{1}{2m_b}\right)\vec{p}^2 - \frac{Gm_a m_b}{r} + (\text{corrections})$$

Post-Minkowskian (PM) : $E_{a,b} = \sqrt{m_{a,b}^2 + \vec{p}^2}$

$$H = E_a + E_b - \left(\frac{G}{r}\right)\left(\frac{m_a^2 m_b^2}{E_a E_b}\right)\cosh(2\theta) + (\text{corrections})$$

PN vs PM

(1686) (1938) (1974) (2000)

0PN 1PN 2PN 3PN

$$\left(\frac{Gm}{r}\right)\left(1 + v^2 + v^4 + v^6 + v^8 + \cdots\right) \quad \text{1PM}$$

$$\left(\frac{Gm}{r}\right)^2\left(1 + v^2 + v^4 + v^6 + \cdots\right) \quad \text{2PM}$$

$$\left(\frac{Gm}{r}\right)^3\left(1 + v^2 + v^4 + \cdots\right) \quad \text{3PM}$$

$$\left(\frac{Gm}{r}\right)^4\left(1 + v^2 + \cdots\right) \quad \text{4PM}$$

$$\vdots$$

QFT-inspired methods

[Donoghue 96]
[Goldberger, Rothstein 04]
[Hostein,Ross 08]
[Neill,Rothstein 13]

became main-stream around 2018

Small but not negligible

Not small : $|m_a/m_b| = \mathcal{O}(1)\,, \quad |\vec{s}_a/\vec{s}_b| = \mathcal{O}(1)$

Small : $\dfrac{Gm}{r}\,, \quad \left(\dfrac{\vec{a}}{r}\right)^2$ $\qquad \left(\vec{a} \equiv \dfrac{\vec{s}}{m} : \text{“spin-length”}\right)$

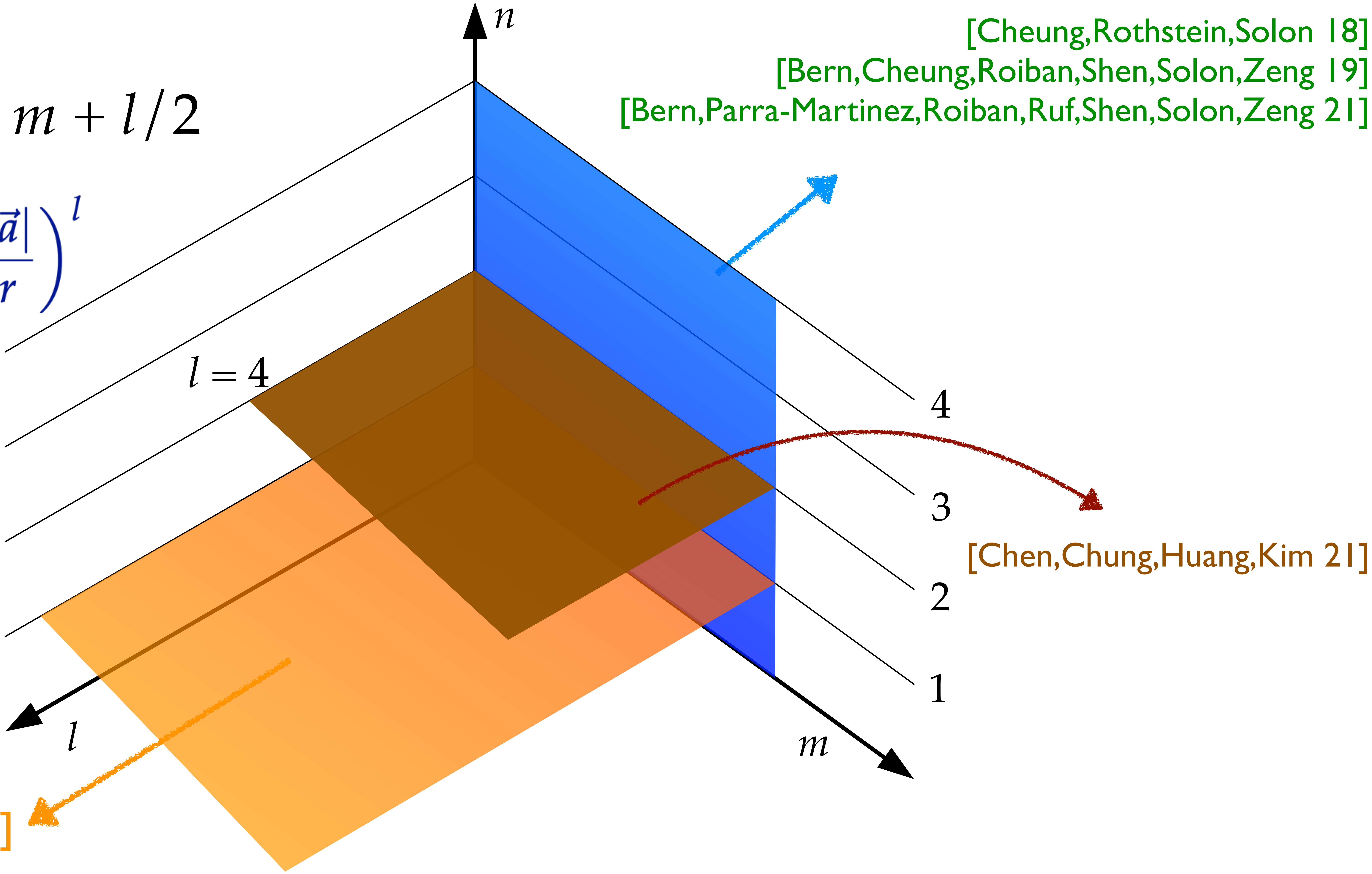
(not negligible)

Possibly small : $\left(\dfrac{\vec{p}}{m}\right)^2$

Amplitude to GR : the frontier

PN order $= n + m + l/2$

$$\left(\frac{Gm}{r}\right)^n \left(\frac{\vec{p}^2}{m^2}\right)^m \left(\frac{|\vec{a}|}{r}\right)^l$$



Our contribution

Ming-Zhi Chung, Yu-tin Huang, Jung-Wook Kim, SL
[1812.08752][1908.08463][2003.06600]

"Complete" 1PM Hamiltonian

Linear in $\frac{Gm}{r}$

"conservative dynamics"
(ignoring radiation)

Exact in $\left(\frac{\vec{p}}{m}\right)^2$ and $\left(\frac{\vec{a}}{r}\right)^2$

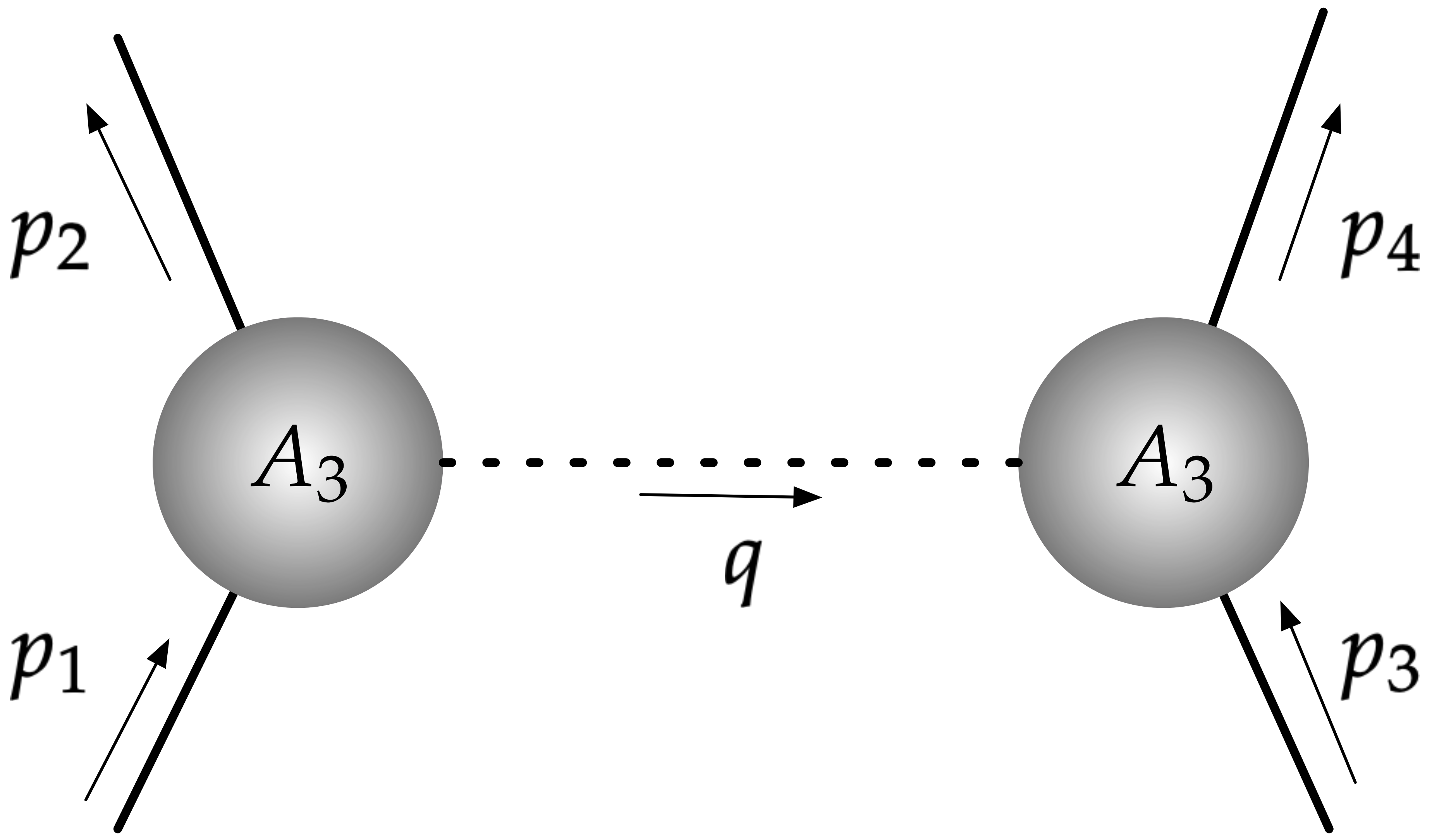
Applicable for general spinning body (beyond Kerr BH)

For further progress at 2PM see

[Kim, Levi, Yin 2112.01509]
[Chen, Chung, Huang, Kim 2111.13639]

1PM kinematics

$$p_2 = (E_a, \vec{p} - \vec{q}/2)$$
$$p_1 = (E_a, \vec{p} + \vec{q}/2)$$



$$p_4 = (E_b, -\vec{p} + \vec{q}/2)$$
$$p_3 = (E_b, -\vec{p} - \vec{q}/2)$$

$$E_{a,b} = \sqrt{m_{a,b}^2 + \vec{p}^2}\,, \quad E = E_a + E_b\,, \quad \cosh\theta = u_a \cdot u_b = \frac{p_a \cdot p_b}{m_a m_b}$$

Spin-less 1PM potential:

[Cheung,Rothstein,Solon 18]
[Bern,Cheung,Roiban,Shen,Solon,Zeng 19]

$$V_{\mathrm{1PM}}^{(0)} = -\left(\frac{G}{r}\right)\frac{m_a^2 m_b^2}{E_a E_b}\cosh(2\theta)$$

1PM potential with spin

"spin kernel"

$$V_{1\text{PM}} = -\frac{m_a^2 m_b^2}{E_a E_b} \int \frac{d^3\vec{q}}{(2\pi)^3} \left(\frac{4\pi}{\vec{q}^2}\right) K(\vec{q}, \vec{p}, \vec{a}_{a,b}.m_{a,b})$$

$K = A \times R$ "(amplitude) x (Thomas precession)"

Ex) Kerr BH

$W(x) = e^x$

$$A = \frac{1}{2}\left[e^{2\theta} W_a(\tau_a) W_b(\tau_b) + e^{-2\theta} W_a(-\tau_a) W_b(-\tau_b)\right]$$

"spin multipole factors"

$$R = e^{\tilde{\tau}_a} e^{\tilde{\tau}_b}$$

$$\tau_a = \frac{i}{\sinh\theta} \varepsilon(q, u_a, u_b, a_a)\,, \quad \tilde{\tau}_a = -i\left(\frac{m_b}{r_a E}\right) \varepsilon(q, u_a, u_b, a_a)\,, \quad r_a \equiv 1 + \frac{E_a}{m_a}$$

From Amplitude to Potential

Extract classical contributions from GR amplitudes

[Holstein,Ross 08]
[Neill,Rothstein 13]

Full theory vs PM-EFT

Cheung, Rothstein, Solon [1808.02489]

Spin-multipole moments

Levi, Steinhoff [1501.04956]
Arkani-Hamed, Huang, Huang [1709.04891]

Spin-multipole moments : C and c

Amplitude :

$$\mathcal{A}_3^{+h} = \kappa_h m x^h \left(\frac{\langle \mathbf{12} \rangle^{2S}}{m^{2S}} + c_1 x \frac{\langle \mathbf{12} \rangle^{2S-1} \langle \mathbf{1}3 \rangle \langle 3\mathbf{2} \rangle}{m^{2S+1}} + \cdots \right) .$$

Potential :

$$L_{\rm SI} = \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n)!} \frac{C_{2n}}{m^{2n-1}} D_{\mu_{2n}} \cdots D_{\mu_3} \frac{E_{\mu_1 \mu_2}}{\sqrt{u^2}} S^{\mu_1} \ldots S^{\mu_{2n}} + \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n+1)!} \frac{C_{2n+1}}{m^{2n}} D_{\mu_{2n+1}} \cdots D_{\mu_3} \frac{B_{\mu_1 \mu_2}}{\sqrt{u^2}} S^{\mu_1} \ldots S^{\mu_{2n+1}} .$$

In the large S limit, the generating functions satisfy

$$A(x) \equiv \sum_{n=0}^{\infty} c_n (x/S)^n , \quad W(x) \equiv \sum_{n=0}^{\infty} \frac{C_n}{n!} x^n \quad \Longrightarrow \quad W(x) = e^x A(x) .$$

BH = minimally coupled particle !

[Hansen 74]
[Steinhoff,Vines 16]
[Vines, 1709.06016]

Kerr BH solution to all orders in spin

$$\bar{h}^{\mu\nu} = \left(u^\mu u^\nu \cos(a\cdot\partial) + u^{(\mu}\epsilon^{\nu)}{}_{\rho\sigma\gamma} u^\rho a^\sigma \partial^\gamma \frac{\sin(a\cdot\partial)}{(a\cdot\partial)} \right) \frac{4GM}{r}$$

Kerr BH : $C_{2n} = C_{2n+1} = 1$

$(a^\mu = S^\mu / m)$

In terms of on-shell 3-point coupling,

$A(x) = 1\,,\quad W(x) = e^x$

Kerr BH : $c_0 = 1\,,\;\; c_n = 0\;\; (n \geq 1)$

1PM parts of PN computations

	LO	NLO	NNLO	N^3LO
S^1	✓	✓	✓	
S^2	✓	✓	✓	—
S^3	✓	⋆	—	—
S^4	✓	—	—	—

★ [Levi, McLeod, von Hippel 2003]

★ [Levi, McLeod, von Hippel 2003]

[Levi, Mougiakakos, Vieira 1912]

★ [Levi, Teng 2008]

A canonical transformation is needed
to compare results in different coordinate choices.

We work in the "isotropic gauge".

No dependence on $(\vec{p}\cdot\hat{n})$ or $(\vec{p}\cdot\vec{a})$

Factorization of the spin-dependence.

Linear in spin

$$V_{S_a^1 S_b^0} = \left(\frac{G}{r^2}\right) m_b [\hat{n} \cdot (\vec{p} \times \vec{a}_a)] X_{(1,0)} \,.$$

$$X^{\mathrm{LO}}_{(1,0)} = \frac{4m_a + 3m_b}{2m_b} \,.$$

[Tulczyjew 59]
[Damour 82]

$$X^{\mathrm{NLO}}_{(1,0)} = \left(\frac{18m_a^2 + 8m_a m_b - 5m_b^2}{8m_a^2 m_b^2}\right) \vec{p}^{\,2} \,.$$

[Tagoshi,Ohashi,Owen 00]
[Faye,Blanchet,Buonanno 06]
[Damour,Jaranowski,Schäfer 07]

$$X^{\mathrm{N^2LO}}_{(1,0)} = \left(\frac{-15m_a^4 - 15m_a^2 m_b^2 - 12m_a m_b^3 + 7m_b^4}{16m_a^4 m_b^4}\right) \vec{p}^{\,4} \,.$$

[Hartung,Steinhoff 11][Levi 11]
[Marsat,Bohe,Faye,Blanchet 12]
[Hartung,Steinhoff,Schäfer 13]
[Levi,Steinhoff 15]

$$X^{\mathrm{N^3LO}}_{(1,0)} = \left(\frac{84m_a^6 + 50m_a^4 m_b^2 + 84m_a^2 m_b^4 + 80m_a m_b^5 - 45m_b^6}{128m_a^6 m_b^6}\right) \vec{p}^{\,6} \,.$$

[NEW]

Quadratic in spin

$$V_{S_a^2 S_b^0} = -\left(\frac{G}{r^3}\right) m_a m_b \left[\vec{a}_a^2 - 3(\vec{a}_a \cdot \hat{n})^2\right] X_{(2,0)} \,.$$

$$X^{\mathrm{LO}}_{(2,0)} = \frac{1}{2} C_2^{(a)} \,.$$

[D'Eath 75]
[Thorne,Hartle 85]

$$X^{\mathrm{NLO}}_{(2,0)} = \frac{C_2^{(a)}(6m_a^2 + 16 m_a m_b + 6 m_b^2) - (8m_a + 7m_b)m_b}{8 m_a^2 m_b^2} \vec{p}^2 \,.$$

[Porto,Rothstein 08][Steinhoff,Hergt,Schäfer 08]

$$X^{\mathrm{N^2LO}}_{(2,0)} = \frac{C_2^{(a)}(-5m_a^4 + 18 m_a^2 m_b^2 - 5 m_b^4) - 3(7m_a^2 + 4 m_a m_b - 2 m_b^2) m_b^2}{16 m_a^4 m_b^4} \vec{p}^4 \,.$$

[Levi 11][Levi,Steinhoff 15-16]

Cubic in spin

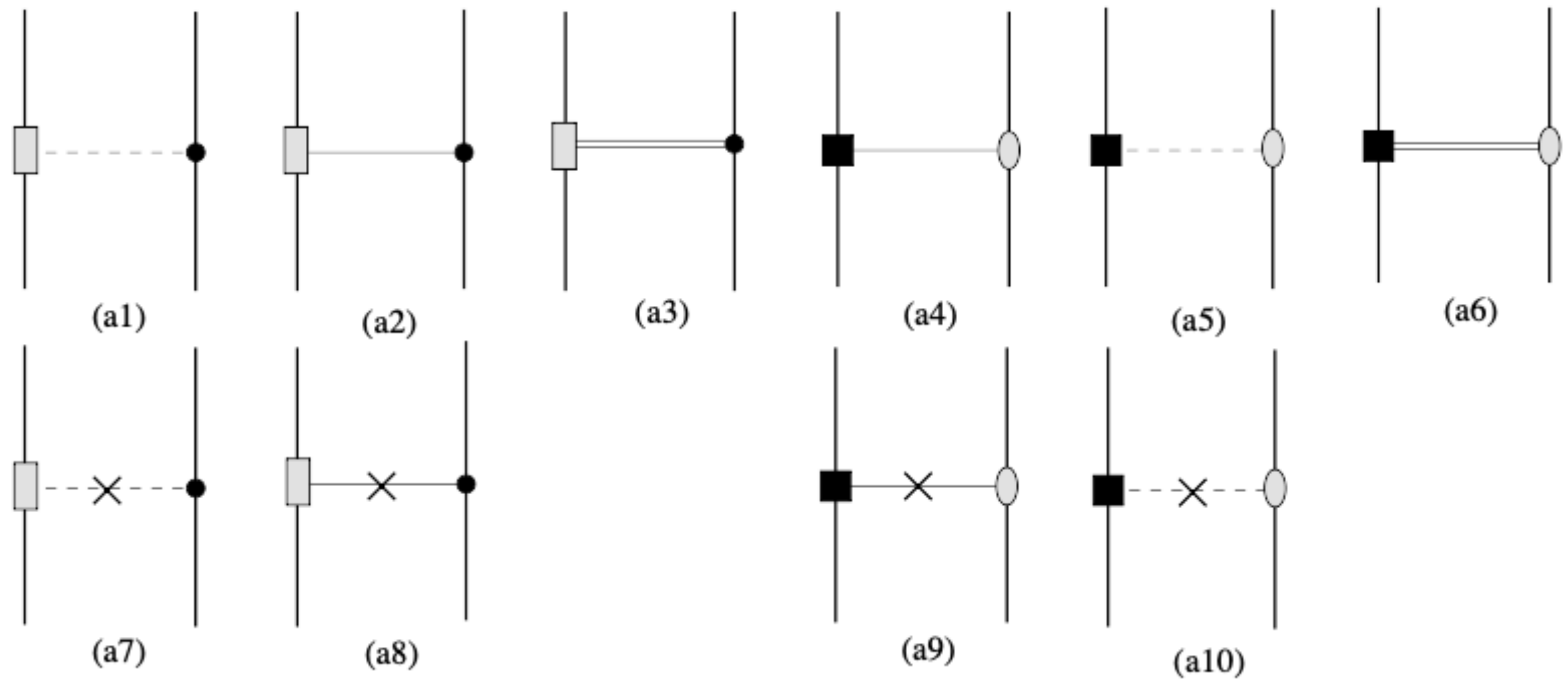
$$V^{\rm LO}_{S_a^3 S_b^0} = \left(\frac{G}{r^4}\right) m_b \left[\hat{n}\cdot(\vec{p}\times\vec{a}_a)\right]\left[\vec{a}_a^2 - 5(\vec{a}_a\cdot\hat{n})^2\right] X_{(3,0)}\,.$$

$$X^{\rm LO}_{(3,0)} = \left[C_3^{(a)}\left(\frac{m_a+m_b}{m_b}\right) - \frac{3}{4}C_2^{(a)}\right].$$

[Hergt,Schäfer 07]
[Levi,Steinhoff 14]

$$X^{\rm NLO}_{(3,0)} = \frac{24C_3^{(a)} m_a(m_a+m_b) - 3C_2^{(a)}(6m_a^2+16m_a m_b+5m_b^2) + (12m_a+11m_b)m_b}{16 m_a^2 m_b^2}\vec{p}^2\,.$$

[Levi,Mougiakakos,Vieira 19]*



[Levi,Mougiakakos,Vieira 19]

$$
\begin{aligned}
\text{Fig. 1(a1)} = & -\frac{1}{2}C_{1(BS^3)}\frac{G}{r^4}\frac{m_2}{m_1^2}\Big[\vec{S}_1\cdot\vec{v}_1\times\vec{v}_2\big(2S_1^2\,\vec{v}_1\cdot\vec{n}+3\vec{S}_1\cdot\vec{v}_1\ \vec{S}_1\cdot\vec{n}-10\big(\vec{S}_1\cdot\vec{n}\big)^2\vec{v}_1\cdot\vec{n}\big) \\
& +2\vec{S}_1\cdot\vec{v}_1\times\vec{n}\big(S_1^2-5\big(\vec{S}_1\cdot\vec{n}\big)^2\big)\vec{v}_1\cdot\vec{v}_2+\vec{S}_1\cdot\vec{v}_2\times\vec{n}\Big(S_1^2\big(v_1^2+v_2^2\big) \\
& -\vec{S}_1\cdot\vec{v}_1\big(\vec{S}_1\cdot\vec{v}_1-5\vec{S}_1\cdot\vec{n}\ \vec{v}_1\cdot\vec{n}\big)-5\big(\vec{S}_1\cdot\vec{n}\big)^2\big(v_1^2+v_2^2\big)\Big) \\
& -\vec{v}_1\cdot\vec{v}_2\times\vec{n}\ \vec{S}_1\cdot\vec{v}_1\big(S_1^2-5\big(\vec{S}_1\cdot\vec{n}\big)^2\big)\Big] \\
& +\frac{1}{3}C_{1(BS^3)}\frac{G}{r^3}\frac{m_2}{m_1^2}\ \Big[2\vec{S}_1\cdot\vec{v}_1\times\vec{v}_2\big(\dot{\vec{S}}_1\cdot\vec{S}_1-3\dot{\vec{S}}_1\cdot\vec{n}\ \vec{S}_1\cdot\vec{n}\big) \\
& +\vec{S}_1\cdot\vec{a}_1\times\vec{v}_2\big(S_1^2-3\big(\vec{S}_1\cdot\vec{n}\big)^2\big)+\dot{\vec{S}}_1\cdot\vec{v}_1\times\vec{v}_2\big(S_1^2-3\big(\vec{S}_1\cdot\vec{n}\big)^2\big) \\
& -3\vec{S}_1\cdot\vec{v}_1\times\vec{n}\ \big(\vec{S}_1\cdot\vec{v}_2\ \dot{\vec{S}}_1\cdot\vec{n}+\dot{\vec{S}}_1\cdot\vec{v}_2\ \vec{S}_1\cdot\vec{n}\big)-3\vec{S}_1\cdot\vec{a}_1\times\vec{n}\ \vec{S}_1\cdot\vec{v}_2\ \vec{S}_1\cdot\vec{n} \\
& -3\dot{\vec{S}}_1\cdot\vec{v}_1\times\vec{n}\ \vec{S}_1\cdot\vec{v}_2\ \vec{S}_1\cdot\vec{n}\Big],
\end{aligned}
\tag{3.1}
$$

$$
\begin{aligned}
\text{Fig. 1(a2)} = & \frac{1}{2}C_{1(BS^3)}\frac{G}{r^4}\frac{m_2}{m_1^2}\ \vec{S}_1\cdot\vec{v}_1\times\vec{n}\Big[S_1^2\big(v_1^2+3v_2^2\big)-\vec{S}_1\cdot\vec{v}_1\big(\vec{S}_1\cdot\vec{v}_1-5\ \vec{S}_1\cdot\vec{n}\ \vec{v}_1\cdot\vec{n}\big) \\
& -5(\vec{S}_1\cdot\vec{n})^2\big(v_1^2+3v_2^2\big)\Big],
\end{aligned}
\tag{3.2}
$$

$$
\begin{aligned}
\text{Fig. 1(a3)} = & -C_{1(BS^3)}\frac{G}{r^4}\frac{m_2}{m_1^2}\Big[\big(\vec{S}_1\cdot\vec{v}_1\times\vec{n}\ v_2^2-\vec{S}_1\cdot\vec{v}_2\times\vec{n}\ \vec{v}_1\cdot\vec{v}_2\big)\big(S_1^2-5(\vec{S}_1\cdot\vec{n})^2\big)\Big] \\
& +C_{1(BS^3)}\frac{G}{r^3}\frac{m_2}{m_1^2}\Big[\vec{S}_1\cdot\vec{v}_2\times\vec{n}\big(\vec{S}_1\cdot\vec{v}_2\dot{\vec{S}}_1\cdot\vec{n}+\dot{\vec{S}}_1\cdot\vec{v}_2\vec{S}_1\cdot\vec{n}\big) \\
& +\dot{\vec{S}}_1\cdot\vec{v}_2\times\vec{n}\ \vec{S}_1\cdot\vec{v}_2\ \vec{S}_1\cdot\vec{n}\Big],
\end{aligned}
\tag{3.3}
$$

$$
\begin{aligned}
\text{Fig. 1(a4)} = & \frac{3}{2}C_{1(ES^2)}\frac{G}{r^4}\frac{1}{m_1}\Big[2\vec{S}_1\cdot\vec{S}_2\times\vec{v}_2\Big(\vec{S}_1\cdot\vec{v}_1\ \vec{v}_1\cdot\vec{n}-\vec{S}_1\cdot\vec{n}\big(3v_1^2+v_2^2\big)\ \Big) \\
& +2\vec{S}_2\cdot\vec{v}_1\times\vec{v}_2\Big(2S_1^2\ \vec{v}_1\cdot\vec{n}-\vec{S}_1\cdot\vec{v}_1\ \vec{S}_1\cdot\vec{n}\Big) \\
& -\vec{S}_2\cdot\vec{v}_2\times\vec{n}\Big(S_1^2\big(5v_1^2+v_2^2-10\big(\vec{v}_1\cdot\vec{n}\big)^2\big)-2\vec{S}_1\cdot\vec{v}_1\big(\vec{S}_1\cdot\vec{v}_1-5\vec{S}_1\cdot\vec{n}\ \vec{v}_1\cdot\vec{n}\big) \\
& -5\big(\vec{S}_1\cdot\vec{n}\big)^2\big(3v_1^2+v_2^2\big)\Big)\Big] \\
& +C_{1(ES^2)}\frac{G}{r^3}\frac{1}{m_1}\Big[2\ \vec{S}_1\cdot\vec{S}_2\times\vec{v}_2\ \big(\vec{S}_1\cdot\vec{a}_1+\dot{\vec{S}}_1\cdot\vec{v}_1\big)+2\dot{\vec{S}}_1\cdot\vec{S}_2\times\vec{v}_2\ \vec{S}_1\cdot\vec{v}_1 \\
& +4\vec{S}_2\cdot\vec{v}_1\times\vec{v}_2\ \dot{\vec{S}}_1\cdot\vec{S}_1\ +2\vec{S}_2\cdot\vec{v}_2\times\vec{a}_1\ S_1^2\ +\vec{S}_2\cdot\vec{v}_2\times\vec{a}_2\big(S_1^2-3\big(\vec{S}_1\cdot\vec{n}\big)^2\big) \\
& -6\vec{S}_2\cdot\vec{v}_2\times\vec{n}\Big(S_1^2\ \vec{a}_1\cdot\vec{n}-2\dot{\vec{S}}_1\cdot\vec{S}_1\ \vec{v}_1\cdot\vec{n}+\vec{S}_1\cdot\vec{v}_1\ \dot{\vec{S}}_1\cdot\vec{n}+\dot{\vec{S}}_1\cdot\vec{v}_1\ \vec{S}_1\cdot\vec{n} \\
& +\vec{S}_1\cdot\vec{a}_1\ \vec{S}_1\cdot\vec{n}\Big)\Big] \\
& -4C_{1(ES^2)}\frac{G}{r^2}\frac{1}{m_1}\vec{S}_2\cdot\vec{v}_2\times\vec{n}\Big(\dot{S}_1^2+\ddot{\vec{S}}_1\cdot\vec{S}_1\Big),
\end{aligned}
\tag{3.4}
$$

$$
\begin{aligned}
\text{Fig. 1(a5)} = & -\frac{3}{2}C_{1(ES^2)}\frac{G}{r^4}\frac{1}{m_1}\Big[2\vec{S}_1\cdot\vec{S}_2\times\vec{v}_1\big(\vec{S}_1\cdot\vec{v}_1\ \vec{v}_1\cdot\vec{n}-\vec{S}_1\cdot\vec{n}\,v_1^2\big) \\
& -6\vec{S}_1\cdot\vec{S}_2\times\vec{v}_2\ \vec{S}_1\cdot\vec{n}\ \vec{v}_1\cdot\vec{v}_2+\vec{S}_2\cdot\vec{v}_1\times\vec{v}_2\Big(S_1^2\vec{v}_2\cdot\vec{n}+2\vec{S}_1\cdot\vec{v}_2\ \vec{S}_1\cdot\vec{n} \\
& -5\big(\vec{S}_1\cdot\vec{n}\big)^2\vec{v}_2\cdot\vec{n}\Big)-\vec{S}_2\cdot\vec{v}_1\times\vec{n}\Big(S_1^2\ \big(3v_1^2-10\big(\vec{v}_1\cdot\vec{n}\big)^2\big)-2\big(\vec{S}_1\cdot\vec{v}_1\big)^2 \\
& +10\vec{S}_1\cdot\vec{v}_1\ \vec{S}_1\cdot\vec{n}\ \vec{v}_1\cdot\vec{n}-5\big(\vec{S}_1\cdot\vec{n}\big)^2\ \vec{v}_1^2\Big) \\
& -3\vec{S}_2\cdot\vec{v}_2\times\vec{n}\ \Big(S_1^2-5\big(\vec{S}_1\cdot\vec{n}\big)^2\Big)\vec{v}_1\cdot\vec{v}_2\ \Big] \\
& -3C_{1(ES^2)}\frac{G}{r^3}\frac{1}{m_1}\ \Big[\vec{S}_1\cdot\vec{S}_2\times\vec{n}\,\dot{\vec{S}}_1\cdot\vec{n}+\dot{\vec{S}}_1\cdot\vec{S}_2\times\vec{n}\,\vec{S}_1\cdot\vec{n}\Big] \\
& -\frac{1}{2}C_{1(ES^2)}\frac{G}{r^3}\frac{1}{m_1}\Big[\vec{S}_1\cdot\vec{S}_2\times\vec{v}_2\big(4\ \dot{\vec{S}}_1\cdot\vec{v}_2-3\dot{\vec{S}}_1\cdot\vec{n}\ \vec{v}_2\cdot\vec{n}\big) \\
& +\dot{\vec{S}}_1\cdot\vec{S}_2\times\vec{v}_2\big(4\ \vec{S}_1\cdot\vec{v}_2-3\vec{S}_1\cdot\vec{n}\ \vec{v}_2\cdot\vec{n}\big)-3\vec{S}_1\cdot\vec{S}_2\times\vec{n}\Big(\vec{S}_1\cdot\vec{v}_1\ \vec{a}_1\cdot\vec{n} \\
& +\dot{\vec{S}}_1\cdot\vec{v}_1\ \vec{v}_1\cdot\vec{n}+\vec{S}_1\cdot\vec{a}_1\ \vec{v}_1\cdot\vec{n}-2\vec{S}_1\cdot\vec{n}\ \vec{v}_1\cdot\vec{a}_1-\ \dot{\vec{S}}_1\cdot\vec{n}\,v_1^2\Big) \\
& -3\dot{\vec{S}}_1\cdot\vec{S}_2\times\vec{n}\big(\vec{S}_1\cdot\vec{v}_1\ \vec{v}_1\cdot\vec{n}-\vec{S}_1\cdot\vec{n}\,v_1^2\big) \\
& +4\big(\vec{S}_2\cdot\vec{v}_1\times\vec{a}_2+\dot{\vec{S}}_2\cdot\vec{v}_1\times\vec{v}_2\big)\Big(S_1^2-3\big(\vec{S}_1\cdot\vec{n}\big)^2\Big) \\
& -3\vec{S}_2\cdot\vec{v}_1\times\vec{n}\Big(2S_1^2\ \vec{a}_1\cdot\vec{n}+4\dot{\vec{S}}_1\cdot\vec{S}_1\ \vec{v}_1\cdot\vec{n}-\vec{S}_1\cdot\vec{v}_1\ \dot{\vec{S}}_1\cdot\vec{n}-\dot{\vec{S}}_1\cdot\vec{v}_1\ \vec{S}_1\cdot\vec{n} \\
& -\vec{S}_1\cdot\vec{a}_1\ \vec{S}_1\cdot\vec{n}\Big)-3\vec{S}_2\cdot\vec{a}_1\times\vec{n}\big(2S_1^2\ \vec{v}_1\cdot\vec{n}-\vec{S}_1\cdot\vec{v}_1\ \vec{S}_1\cdot\vec{n}\big) \\
& +3\vec{S}_2\cdot\vec{v}_2\times\vec{n}\big(8\dot{\vec{S}}_1\cdot\vec{S}_1\ \vec{v}_2\cdot\vec{n}-3\vec{S}_1\cdot\vec{v}_2\dot{\vec{S}}_1\cdot\vec{n}-3\dot{\vec{S}}_1\cdot\vec{v}_2\ \vec{S}_1\cdot\vec{n}\big)\Big] \\
& +2C_{1(ES^2)}\frac{G}{r^2}\frac{1}{m_1}\Big[\big(\vec{S}_1\cdot\vec{S}_2\times\vec{a}_2+\vec{S}_1\cdot\dot{\vec{S}}_2\times\vec{v}_2\big)\dot{\vec{S}}_1\cdot\vec{n} \\
& +\big(\dot{\vec{S}}_1\cdot\dot{\vec{S}}_2\times\vec{v}_2+\dot{\vec{S}}_1\cdot\vec{S}_2\times\vec{a}_2\big)\vec{S}_1\cdot\vec{n}-2\big(\dot{\vec{S}}_2\cdot\vec{v}_2\times\vec{n}+\vec{S}_2\cdot\vec{a}_2\times\vec{n}\big)\dot{\vec{S}}_1\cdot\vec{S}_1\Big],
\end{aligned}
\tag{3.5}
$$

$$
\begin{aligned}
\text{Fig. 1(a6)} = & -3C_{1(ES^2)}\frac{G}{r^4}\frac{1}{m_1}\Big[2\vec{S}_1\cdot\vec{S}_2\times\vec{v}_1\ \vec{S}_1\cdot\vec{n}\ \vec{v}_1\cdot\vec{v}_2-2\vec{S}_1\cdot\vec{S}_2\times\vec{v}_2\ \vec{S}_1\cdot\vec{n}\,v_1^2 \\
& +\Big(\vec{S}_2\cdot\vec{v}_1\times\vec{n}\ \vec{v}_1\cdot\vec{v}_2-\vec{S}_2\cdot\vec{v}_2\times\vec{n}\ v_1^2\Big)\Big(S_1^2-5\big(\vec{S}_1\cdot\vec{n}\big)^2\Big)\Big] \\
& +C_{1(ES^2)}\frac{G}{m_1r^3}\Big[\vec{S}_1\cdot\vec{S}_2\times\vec{v}_1\dot{\vec{S}}_1\cdot\vec{v}_2+\dot{\vec{S}}_1\cdot\vec{S}_2\times\vec{v}_1\vec{S}_1\cdot\vec{v}_2+\vec{S}_1\cdot\vec{S}_2\times\vec{a}_1\vec{S}_1\cdot\vec{v}_2 \\
& -2\vec{S}_1\cdot\vec{S}_2\times\vec{v}_2\big(\dot{\vec{S}}_1\cdot\vec{v}_1+\vec{S}_1\cdot\vec{a}_1\big)-2\dot{\vec{S}}_1\cdot\vec{S}_2\times\vec{v}_2\ \vec{S}_1\cdot\vec{v}_1 \\
& +3\vec{S}_1\cdot\vec{S}_2\times\vec{n}\Big(\vec{S}_1\cdot\vec{n}\ \vec{a}_1\cdot\vec{v}_2+\dot{\vec{S}}_1\cdot\vec{n}\ \vec{v}_1\cdot\vec{v}_2\Big)+3\dot{\vec{S}}_1\cdot\vec{S}_2\times\vec{n}\ \vec{S}_1\cdot\vec{n}\ \vec{v}_1\cdot\vec{v}_2 \\
& -6\vec{S}_2\cdot\vec{v}_1\times\vec{v}_2\dot{\vec{S}}_1\cdot\vec{S}_1+3\vec{S}_2\cdot\vec{v}_2\times\vec{a}_1S_1^2+3\vec{S}_2\cdot\vec{v}_1\times\vec{n}\Big(2\dot{\vec{S}}_1\cdot\vec{S}_1\ \vec{v}_2\cdot\vec{n} \\
& -\vec{S}_1\cdot\vec{v}_2\ \dot{\vec{S}}_1\cdot\vec{n}-\dot{\vec{S}}_1\cdot\vec{v}_2\ \vec{S}_1\cdot\vec{n}\Big)-3\vec{S}_2\cdot\vec{v}_2\times\vec{n}\Big(2S_1^2\ \vec{a}_1\cdot\vec{n}+4\dot{\vec{S}}_1\cdot\vec{S}_1\ \vec{v}_1\cdot\vec{n} \\
& -2\vec{S}_1\cdot\vec{v}_1\ \dot{\vec{S}}_1\cdot\vec{n}-2\dot{\vec{S}}_1\cdot\vec{v}_1\ \vec{S}_1\cdot\vec{n}-2\vec{S}_1\cdot\vec{n}\ \vec{S}_1\cdot\vec{a}_1\Big) \\
& +3\vec{S}_2\cdot\vec{a}_1\times\vec{n}\big(S_1^2\vec{v}_2\cdot\vec{n}-\vec{S}_1\cdot\vec{v}_2\ \vec{S}_1\cdot\vec{n}\big)\Big] \\
& -C_{1(ES^2)}\frac{G}{r^2}\frac{1}{m_1}\Big[\vec{S}_1\cdot\vec{S}_2\times\vec{n}\ \ddot{\vec{S}}_1\cdot\vec{v}_2+2\dot{\vec{S}}_1\cdot\vec{S}_2\times\vec{n}\ \dot{\vec{S}}_1\cdot\vec{v}_2+\ddot{\vec{S}}_1\cdot\vec{S}_2\times\vec{n}\ \vec{S}_1\cdot\vec{v}_2 \\
& -2\vec{S}_2\cdot\vec{v}_2\times\vec{n}\ \big(\dot{S}_1^2+\ddot{\vec{S}}_1\cdot\vec{S}_1\big)\Big],
\end{aligned}
\tag{3.6}
$$

$$
\begin{aligned}
\text{Fig. 1(a7)} = & \frac{1}{2}C_{1(BS^3)}\frac{G}{r^4}\frac{m_2}{m_1^2}\Big[\vec{S}_1\cdot\vec{v}_1\times\vec{v}_2\Big(S_1^2\vec{v}_2\cdot\vec{n}+2\vec{S}_1\cdot\vec{v}_2\ \vec{S}_1\cdot\vec{n}-5\big(\vec{S}_1\cdot\vec{n}\big)^2\vec{v}_2\cdot\vec{n}\Big) \\
& -\vec{S}_1\cdot\vec{v}_2\times\vec{n}\Big(S_1^2\big(\vec{v}_1\cdot\vec{v}_2-5\vec{v}_1\cdot\vec{n}\ \vec{v}_2\cdot\vec{n}\big)+2\vec{S}_1\cdot\vec{v}_1\ \vec{S}_1\cdot\vec{v}_2 \\
& -10\vec{S}_1\cdot\vec{v}_1\ \vec{S}_1\cdot\vec{n}\ \vec{v}_2\cdot\vec{n}-10\vec{S}_1\cdot\vec{v}_2\ \vec{S}_1\cdot\vec{n}\ \vec{v}_1\cdot\vec{n} \\
& -5\big(\vec{S}_1\cdot\vec{n}\big)^2\big(\vec{v}_1\cdot\vec{v}_2-7\vec{v}_1\cdot\vec{n}\ \vec{v}_2\cdot\vec{n}\big)\Big)\Big] \\
& +\frac{1}{6}C_{1(BS^3)}\frac{G}{r^3}\frac{m_2}{m_1^2}\Big[\vec{S}_1\cdot\vec{v}_1\times\vec{a}_2\big(S_1^2-3\big(\vec{S}_1\cdot\vec{n}\big)^2\big) \\
& +6\vec{S}_1\cdot\vec{v}_2\times\vec{n}\big(\dot{\vec{S}}_1\cdot\vec{S}_1\vec{v}_2\cdot\vec{n}+\dot{\vec{S}}_1\cdot\vec{v}_2\vec{S}_1\cdot\vec{n}+\vec{S}_1\cdot\vec{v}_2\dot{\vec{S}}_1\cdot\vec{n}-5\dot{\vec{S}}_1\cdot\vec{n}\ \vec{S}_1\cdot\vec{n}\ \vec{v}_2\cdot\vec{n}\big) \\
& +3\vec{S}_1\cdot\vec{a}_2\times\vec{n}\big(S_1^2\vec{v}_1\cdot\vec{n}+2\vec{S}_1\cdot\vec{v}_1\ \vec{S}_1\cdot\vec{n}-5\big(\vec{S}_1\cdot\vec{n}\big)^2\vec{v}_1\cdot\vec{n}\big) \\
& +3\dot{\vec{S}}_1\cdot\vec{v}_2\times\vec{n}\big(S_1^2\vec{v}_2\cdot\vec{n}+2\vec{S}_1\cdot\vec{v}_2\ \vec{S}_1\cdot\vec{n}-5\big(\vec{S}_1\cdot\vec{n}\big)^2\vec{v}_2\cdot\vec{n}\big)\Big] \\
& -\frac{1}{6}C_{1(BS^3)}\frac{G}{r^2}\frac{m_2}{m_1^2}\Big[2\vec{S}_1\cdot\vec{a}_2\times\vec{n}\big(\dot{\vec{S}}_1\cdot\vec{S}_1-3\dot{\vec{S}}_1\cdot\vec{n}\ \vec{S}_1\cdot\vec{n}\big) \\
& +\dot{\vec{S}}_1\cdot\vec{a}_2\times\vec{n}\big(S_1^2-3(\vec{S}_1\cdot\vec{n})^2\big)\Big],
\end{aligned}
\tag{3.7}
$$

$$
\begin{aligned}
\text{Fig. 1(a8)} = & \frac{1}{2}C_{1(BS^3)}\frac{G}{r^4}\frac{m_2}{m_1^2}\Big[\vec{S}_1\cdot\vec{v}_1\times\vec{v}_2\Big(S_1^2\vec{v}_1\cdot\vec{n}+2\vec{S}_1\cdot\vec{v}_1\ \vec{S}_1\cdot\vec{n}\ -5(\vec{S}_1\cdot\vec{n})^2\vec{v}_1\cdot\vec{n}\Big) \\
& +\vec{S}_1\cdot\vec{v}_1\times\vec{n}\Big(S_1^2\big(\vec{v}_1\cdot\vec{v}_2-5\ \vec{v}_1\cdot\vec{n}\ \vec{v}_2\cdot\vec{n}\big)+2\vec{S}_1\cdot\vec{v}_1\ \vec{S}_1\cdot\vec{v}_2 \\
& -10\vec{S}_1\cdot\vec{v}_1\ \vec{S}_1\cdot\vec{n}\ \vec{v}_2\cdot\vec{n}-10\vec{S}_1\cdot\vec{v}_2\ \vec{S}_1\cdot\vec{n}\ \vec{v}_1\cdot\vec{n} \\
& -5(\vec{S}_1\cdot\vec{n})^2\big(\vec{v}_1\cdot\vec{v}_2-7\vec{v}_1\cdot\vec{n}\ \vec{v}_2\cdot\vec{n}\big)\Big)\Big] \\
& -\frac{1}{6}C_{1(BS^3)}\frac{G}{r^3}\frac{m_2}{m_1^2}\Big[2\vec{S}_1\cdot\vec{v}_1\times\vec{v}_2\big(\dot{\vec{S}}_1\cdot\vec{S}_1-3\dot{\vec{S}}_1\cdot\vec{n}\ \vec{S}_1\cdot\vec{n}\big) \\
& -\big(\vec{S}_1\cdot\vec{v}_2\times\vec{a}_1-\dot{\vec{S}}_1\cdot\vec{v}_1\times\vec{v}_2\big)\big(S_1^2-3\big(\vec{S}_1\cdot\vec{n}\big)^2\big)-6\vec{S}_1\cdot\vec{v}_1\times\vec{n}\big(\dot{\vec{S}}_1\cdot\vec{S}_1\ \vec{v}_2\cdot\vec{n} \\
& +\dot{\vec{S}}_1\cdot\vec{v}_2\ \vec{S}_1\cdot\vec{n}+\vec{S}_1\cdot\vec{v}_2\ \dot{\vec{S}}_1\cdot\vec{n}-5\dot{\vec{S}}_1\cdot\vec{n}\ \vec{S}_1\cdot\vec{n}\ \vec{v}_2\cdot\vec{n}\big) \\
& -3\big(\vec{S}_1\cdot\vec{a}_1\times\vec{n}+\dot{\vec{S}}_1\cdot\vec{v}_1\times\vec{n}\big)\big(S_1^2\ \vec{v}_2\cdot\vec{n}+2\vec{S}_1\cdot\vec{v}_2\ \vec{S}_1\cdot\vec{n}-5(\vec{S}_1\cdot\vec{n})^2\vec{v}_2\cdot\vec{n}\big)\Big],
\end{aligned}
\tag{3.8}
$$

$$
\begin{aligned}
\text{Fig. 1(a9)} = & -\frac{3}{2}C_{1(ES^2)}\frac{G}{r^4}\frac{1}{m_1}\Big[2\vec{S}_1\cdot\vec{S}_2\times\vec{v}_2\Big(\vec{S}_1\cdot\vec{v}_1\ \vec{v}_2\cdot\vec{n}\ +\vec{S}_1\cdot\vec{v}_2\ \vec{v}_1\cdot\vec{n} \\
& +\vec{S}_1\cdot\vec{n}\ \vec{v}_1\cdot\vec{v}_2\ -5\vec{S}_1\cdot\vec{n}\ \vec{v}_1\cdot\vec{n}\ \vec{v}_2\cdot\vec{n}\ \Big)+\vec{S}_2\cdot\vec{v}_1\times\vec{v}_2\Big(S_1^2\vec{v}_2\cdot\vec{n}-2\vec{S}_1\cdot\vec{v}_2\ \vec{S}_1\cdot\vec{n} \\
& +5\big(\vec{S}_1\cdot\vec{n}\big)^2\vec{v}_2\cdot\vec{n}\Big)-\vec{S}_2\cdot\vec{v}_2\times\vec{n}\ \Big(S_1^2\big(\vec{v}_1\cdot\vec{v}_2\ -5\ \vec{v}_1\cdot\vec{n}\ \vec{v}_2\cdot\vec{n}\big)-2\vec{S}_1\cdot\vec{v}_1\ \vec{S}_1\cdot\vec{v}_2 \\
& +10\vec{S}_1\cdot\vec{v}_1\ \vec{S}_1\cdot\vec{n}\ \vec{v}_2\cdot\vec{n}+10\vec{S}_1\cdot\vec{v}_2\ \vec{S}_1\cdot\vec{n}\ \vec{v}_1\cdot\vec{n}\Big) \\
& +5\big(\vec{S}_1\cdot\vec{n}\big)^2\big(\vec{v}_1\cdot\vec{v}_2\ -7\ \vec{v}_1\cdot\vec{n}\ \vec{v}_2\cdot\vec{n}\big)\Big)\Big] \\
& +\frac{1}{2}C_{1(ES^2)}\frac{G}{r^3}\frac{1}{m_1}\Big[2\vec{S}_1\cdot\vec{S}_2\times\vec{v}_2\big(\dot{\vec{S}}_1\cdot\vec{v}_2-3\dot{\vec{S}}_1\cdot\vec{n}\ \vec{v}_2\cdot\vec{n}\big) \\
& -2\Big(\vec{S}_1\cdot\vec{S}_2\times\vec{a}_2+\vec{S}_1\cdot\dot{\vec{S}}_2\times\vec{v}_2\Big)\Big(\vec{S}_1\cdot\vec{v}_1-3\ \vec{S}_1\cdot\vec{n}\ \vec{v}_1\cdot\vec{n}\Big) \\
& +2\dot{\vec{S}}_1\cdot\vec{S}_2\times\vec{v}_2\big(\vec{S}_1\cdot\vec{v}_2-3\vec{S}_1\cdot\vec{n}\ \vec{v}_2\cdot\vec{n}\big) \\
& -\Big(\vec{S}_2\cdot\vec{v}_1\times\vec{a}_2+\dot{\vec{S}}_2\cdot\vec{v}_1\times\vec{v}_2\Big)\Big(S_1^2+3\big(\vec{S}_1\cdot\vec{n}\big)^2\Big)+6\vec{S}_2\cdot\vec{v}_2\times\vec{n}\Big(\dot{\vec{S}}_1\cdot\vec{S}_1\ \vec{v}_2\cdot\vec{n} \\
& -\dot{\vec{S}}_1\cdot\vec{v}_2\ \vec{S}_1\cdot\vec{n}-\vec{S}_1\cdot\vec{v}_2\ \dot{\vec{S}}_1\cdot\vec{n}+5\dot{\vec{S}}_1\cdot\vec{n}\ \vec{S}_1\cdot\vec{n}\ \vec{v}_2\cdot\vec{n}\Big) \\
& -3\Big(\dot{\vec{S}}_2\cdot\vec{v}_2\times\vec{n}+\vec{S}_2\cdot\vec{a}_2\times\vec{n}\Big)\Big(S_1^2\ \vec{v}_1\cdot\vec{n}-2\vec{S}_1\cdot\vec{v}_1\ \vec{S}_1\cdot\vec{n}+5\big(\vec{S}_1\cdot\vec{n}\big)^2\ \vec{v}_1\cdot\vec{n}\Big)\Big] \\
& -C_{1(ES^2)}\frac{G}{r^2}\frac{1}{m_1}\Big[\vec{S}_1\cdot\vec{S}_2\times\vec{a}_2\ \dot{\vec{S}}_1\cdot\vec{n}+\vec{S}_1\cdot\dot{\vec{S}}_2\times\vec{v}_2\ \dot{\vec{S}}_1\cdot\vec{n}+\dot{\vec{S}}_1\cdot\vec{S}_2\times\vec{a}_2\ \vec{S}_1\cdot\vec{n} \\
& +\dot{\vec{S}}_1\cdot\dot{\vec{S}}_2\times\vec{v}_2\ \vec{S}_1\cdot\vec{n}-\Big(\dot{\vec{S}}_2\cdot\vec{v}_2\times\vec{n}+\vec{S}_2\cdot\vec{a}_2\times\vec{n}\Big)\Big(\dot{\vec{S}}_1\cdot\vec{S}_1+3\dot{\vec{S}}_1\cdot\vec{n}\ \vec{S}_1\cdot\vec{n}\Big)\Big],
\end{aligned}
\tag{3.9}
$$

$$
\begin{aligned}
\text{Fig. 1(a10)} = & \frac{3}{2}C_{1(ES^2)}\frac{G}{r^4}\frac{1}{m_1}\Big[2\vec{S}_1\cdot\vec{S}_2\times\vec{v}_1\Big(\vec{S}_1\cdot\vec{v}_1\ \vec{v}_2\cdot\vec{n}+\vec{S}_1\cdot\vec{v}_2\ \vec{v}_1\cdot\vec{n}+\vec{S}_1\cdot\vec{n}\vec{v}_1\cdot\vec{v}_2 \\
& -5\vec{S}_1\cdot\vec{n}\ \vec{v}_1\cdot\vec{n}\ \vec{v}_2\cdot\vec{n}\Big)-\vec{S}_2\cdot\vec{v}_1\times\vec{v}_2\Big(S_1^2\vec{v}_1\cdot\vec{n}-2\vec{S}_1\cdot\vec{v}_1\ \vec{S}_1\cdot\vec{n} \\
& +5\big(\vec{S}_1\cdot\vec{n}\big)^2\vec{v}_1\cdot\vec{n}\Big)-\vec{S}_2\cdot\vec{v}_1\times\vec{n}\Big(S_1^2\big(\vec{v}_1\cdot\vec{v}_2-5\ \vec{v}_1\cdot\vec{n}\ \vec{v}_2\cdot\vec{n}\big) \\
& -2\vec{S}_1\cdot\vec{v}_1\ \vec{S}_1\cdot\vec{v}_2+10\vec{S}_1\cdot\vec{v}_1\ \vec{S}_1\cdot\vec{n}\vec{v}_2\cdot\vec{n}+10\vec{S}_1\cdot\vec{v}_2\ \vec{S}_1\cdot\vec{n}\vec{v}_1\cdot\vec{n} \\
& +5\big(\vec{S}_1\cdot\vec{n}\big)^2\big(\vec{v}_1\cdot\vec{v}_2+7\vec{v}_1\cdot\vec{n}\ \vec{v}_2\cdot\vec{n}\big)\Big)\Big] \\
& -\frac{1}{2}C_{1(ES^2)}\frac{G}{r^3}\frac{1}{m_1}\Big[\vec{S}_1\cdot\vec{S}_2\times\vec{v}_1\big(\dot{\vec{S}}_1\cdot\vec{v}_2-3\dot{\vec{S}}_1\cdot\vec{n}\ \vec{v}_2\cdot\vec{n}\big) \\
& +2\vec{S}_1\cdot\vec{S}_2\times\vec{a}_1\big(\vec{S}_1\cdot\vec{v}_2-3\vec{S}_1\cdot\vec{n}\ \vec{v}_2\cdot\vec{n}\big)+\dot{\vec{S}}_1\cdot\vec{S}_2\times\vec{v}_1\big(\vec{S}_1\cdot\vec{v}_2-3\vec{S}_1\cdot\vec{n}\ \vec{v}_2\cdot\vec{n}\big) \\
& -2\vec{S}_1\cdot\dot{\vec{S}}_2\times\vec{v}_1\Big(\vec{S}_1\cdot\vec{v}_1-3\vec{S}_1\cdot\vec{n}\ \vec{v}_1\cdot\vec{n}\Big)-\vec{S}_1\cdot\vec{S}_2\times\vec{v}_2\big(\dot{\vec{S}}_1\cdot\vec{v}_1-3\dot{\vec{S}}_1\cdot\vec{n}\ \vec{v}_1\cdot\vec{n}\big) \\
& -\dot{\vec{S}}_1\cdot\vec{S}_2\times\vec{v}_2\Big(\vec{S}_1\cdot\vec{v}_1-3\vec{S}_1\cdot\vec{n}\ \vec{v}_1\cdot\vec{n}\Big)+3\vec{S}_1\cdot\vec{S}_2\times\vec{n}\Big(\dot{\vec{S}}_1\cdot\vec{v}_1\ \vec{v}_2\cdot\vec{n} \\
& +\dot{\vec{S}}_1\cdot\vec{v}_2\ \vec{v}_1\cdot\vec{n}+\dot{\vec{S}}_1\cdot\vec{n}\ \vec{v}_1\cdot\vec{v}_2-5\dot{\vec{S}}_1\cdot\vec{n}\ \vec{v}_1\cdot\vec{n}\ \vec{v}_2\cdot\vec{n}\Big) \\
& +3\dot{\vec{S}}_1\cdot\vec{S}_2\times\vec{n}\Big(\vec{S}_1\cdot\vec{v}_1\ \vec{v}_2\cdot\vec{n}+\vec{S}_1\cdot\vec{v}_2\ \vec{v}_1\cdot\vec{n}+\vec{S}_1\cdot\vec{n}\ \vec{v}_1\cdot\vec{v}_2 \\
& -5\vec{S}_1\cdot\vec{n}\ \vec{v}_1\cdot\vec{n}\ \vec{v}_2\cdot\vec{n}\Big)-2\vec{S}_2\cdot\vec{v}_1\times\vec{v}_2\big(\dot{\vec{S}}_1\cdot\vec{S}_1+3\dot{\vec{S}}_1\cdot\vec{n}\ \vec{S}_1\cdot\vec{n}\big) \\
& -\vec{S}_2\cdot\vec{a}_1\times\vec{v}_2\Big(S_1^2+3\big(\vec{S}_1\cdot\vec{n}\big)^2\Big)+6\vec{S}_2\cdot\vec{v}_1\times\vec{n}\Big(\dot{\vec{S}}_1\cdot\vec{S}_1\ \vec{v}_2\cdot\vec{n}-\vec{S}_1\cdot\vec{v}_2\dot{\vec{S}}_1\cdot\vec{n} \\
& -\dot{\vec{S}}_1\cdot\vec{v}_2\ \vec{S}_1\cdot\vec{n}+5\dot{\vec{S}}_1\cdot\vec{n}\ \vec{S}_1\cdot\vec{n}\ \vec{v}_2\cdot\vec{n}\Big)+3\vec{S}_2\cdot\vec{a}_1\times\vec{n}\Big(S_1^2\vec{v}_2\cdot\vec{n}-2\vec{S}_1\cdot\vec{v}_2\ \vec{S}_1\cdot\vec{n} \\
& +5\big(\vec{S}_1\cdot\vec{n}\big)^2\vec{v}_2\cdot\vec{n}\Big)+3\dot{\vec{S}}_2\cdot\vec{v}_1\times\vec{n}\big(3S_1^2\ \vec{v}_1\cdot\vec{n}+2\vec{S}_1\cdot\vec{v}_1\ \vec{S}_1\cdot\vec{n}-5(\vec{S}_1\cdot\vec{n})^2\vec{v}_1\cdot\vec{n}\big)\Big] \\
& +\frac{1}{2}C_{1(ES^2)}\frac{G}{r^2}\frac{1}{m_1}\Big[\vec{S}_1\cdot\dot{\vec{S}}_2\times\vec{v}_1\ \dot{\vec{S}}_1\cdot\vec{n}+\dot{\vec{S}}_1\cdot\dot{\vec{S}}_2\times\vec{v}_1\ \vec{S}_1\cdot\vec{n} \\
& +\vec{S}_1\cdot\vec{S}_2\times\vec{v}_2\ \ddot{\vec{S}}_1\cdot\vec{n}+2\dot{\vec{S}}_1\cdot\vec{S}_2\times\vec{v}_2\ \dot{\vec{S}}_1\cdot\vec{n}+\ddot{\vec{S}}_1\cdot\vec{S}_2\times\vec{v}_2\ \vec{S}_1\cdot\vec{n} \\
& +\vec{S}_1\cdot\vec{S}_2\times\vec{n}\big(\ddot{\vec{S}}_1\cdot\vec{v}_2-3\ddot{\vec{S}}_1\cdot\vec{n}\ \vec{v}_2\cdot\vec{n}\big)-\vec{S}_1\cdot\dot{\vec{S}}_2\times\vec{n}\Big(\dot{\vec{S}}_1\cdot\vec{v}_1-3\dot{\vec{S}}_1\cdot\vec{n}\ \vec{v}_1\cdot\vec{n}\Big) \\
& +2\dot{\vec{S}}_1\cdot\vec{S}_2\times\vec{n}\big(\dot{\vec{S}}_1\cdot\vec{v}_2-3\dot{\vec{S}}_1\cdot\vec{n}\ \vec{v}_2\cdot\vec{n}\big)-\dot{\vec{S}}_1\cdot\dot{\vec{S}}_2\times\vec{n}\Big(\vec{S}_1\cdot\vec{v}_1-3\vec{S}_1\cdot\vec{n}\ \vec{v}_1\cdot\vec{n}\Big) \\
& +\ddot{\vec{S}}_1\cdot\vec{S}_2\times\vec{n}\big(\vec{S}_1\cdot\vec{v}_2-3\vec{S}_1\cdot\vec{n}\ \vec{v}_2\cdot\vec{n}\big)+2\vec{S}_1\cdot\dot{\vec{S}}_2\times\vec{a}_1\ \vec{S}_1\cdot\vec{n} \\
& +2\ \dot{\vec{S}}_2\cdot\vec{v}_1\times\vec{n}\Big(\dot{\vec{S}}_1\cdot\vec{S}_1+3\dot{\vec{S}}_1\cdot\vec{n}\ \vec{S}_1\cdot\vec{n}\Big)+\dot{\vec{S}}_2\cdot\vec{a}_1\times\vec{n}\Big(S_1^2+3\big(\vec{S}_1\cdot\vec{n}\big)^2\Big)\Big] \\
& -\frac{1}{2}C_{1(ES^2)}\frac{G}{r}\frac{1}{m_1}\Big[\vec{S}_1\cdot\dot{\vec{S}}_2\times\vec{n}\ \ddot{\vec{S}}_1\cdot\vec{n}+2\dot{\vec{S}}_1\cdot\dot{\vec{S}}_2\times\vec{n}\ \dot{\vec{S}}_1\cdot\vec{n}+\ddot{\vec{S}}_1\cdot\dot{\vec{S}}_2\times\vec{n}\ \vec{S}_1\cdot\vec{n}\Big].
\end{aligned}
$$

PART III

[Hanson,Regge 1974] [Fedoruk,Lukierski 2014]

The Relativistic Spherical Top as a Massive Twistor

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"Spherical Top" in special relativity

$$S = \int dt \left(\frac{1}{2} m \vec{v}^2 \right)$$

"spin" →

$$S = \int dt \left(\frac{1}{2} m \vec{v}^2 + \frac{1}{2} I \vec{\omega}^2 \right)$$

(1+3)-dim space-time ↓

$$S = - \int ds \sqrt{-\eta_{ab} \dot{x}^a \dot{x}^b}$$

→

$$S = (?)$$

[Frenkel 1926][Thomas 1926]
... [Hanson,Regge 1974]

3 translation d.o.f. + 3 rotation d.o.f.

$$x, p \in \mathbb{R}^{1,3}$$
$$\Lambda \in \mathrm{SO}(1,3)$$
$$S \in \mathfrak{so}(1,3)$$

$$\{x^\mu, p_\nu\} = \delta^\mu{}_\nu\,,$$
$$\{\Lambda^\rho{}_A, S_{\mu\nu}\} = -(\delta^\rho{}_\mu \Lambda_{\nu A} - \delta^\rho{}_\nu \Lambda_{\mu A})\,,$$
$$\{S_{\mu\nu}, S_{\rho\sigma}\} = -(\eta_{\nu\rho} S_{\mu\sigma} - \eta_{\mu\rho} S_{\nu\sigma} - \eta_{\nu\sigma} S_{\mu\rho} + \eta_{\mu\sigma} S_{\nu\rho})\,.$$

Translation: $4 \to 3$ $\quad p^2 + m^2 = 0$ $\quad$ "mass shell constraint"

Rotation: $6 \to 3$ $\quad (?)$ $\quad$ "spin constraints"

Spin constraints

Intuition : remove the "boosts" from $\Lambda^{\mu}{}_{A}$.

But, boosts do not form a closed subalgebra.

$$\Lambda^{\mu}{}_{0} S_{\mu\nu} = 0 \qquad \text{(temporal)}$$

$$\hat{p}^{\mu} S_{\mu\nu} = 0 \qquad \text{(covariant)}$$

$$(\hat{p}^{\mu} + \Lambda^{\mu}{}_{0}) S_{\mu\nu} = 0 \qquad \text{(Pryce-Newton-Wigner)}$$

[Pryce 1948][Newton, Wigner 1949]

Our proposal

(mass shell) (modified PNW) [Steinhoff 2015]

$$\phi_0 = \frac{1}{2}(p^2 + m^2)\,, \qquad \phi_a = \frac{1}{2}(\hat{p}^\mu + \Lambda^\mu{}_0)S_{\mu\nu}\Lambda^\nu{}_a\,,$$

$$\chi^0 = \frac{1}{p^2}x^\mu p_\mu\,, \qquad \chi^a = \hat{p}_\mu \Lambda^{\mu a}\,.$$

Canonically conjugate pairs of

"gauge generators" and "gauge-fixing conditions"

$$\{\chi^A, \phi_B\} \approx \delta^A{}_B \qquad \{\phi_A, \phi_B\} \approx 0$$

Dirac bracket

$$\{f,g\}_* = \{f,g\} - \{f,\phi_A\}\{\chi^A,g\} + \{f,\chi^A\}\{\phi_A,g\}$$

The only dynamical information
not governed by symmetry is ...

the "Regge trajectory"

$$p^2 + m^2(\tilde{S}^2) = 0$$

[Hanson,Regge 1974]

Equations of motion

Lagrange multipliers

$$\mathcal{S} = \int d\sigma \left(p_\mu \dot{x}^\mu + \frac{1}{2} S_{\mu\nu} \Omega^{\mu\nu} - \kappa \phi_0 - \kappa^a \phi_a \right)$$

$$= \int d\sigma \left(p_\mu \dot{x}^\mu + \frac{1}{2} S_{\mu\nu} \Omega^{\mu\nu} - \frac{\kappa}{2} (p^2 + m^2) - \frac{\kappa^a}{2} (\hat{p}^\mu + \Lambda^\mu{}_0) S_{\mu\nu} \Lambda^\nu{}_a \right) .$$

$$\frac{dp_\mu}{d\tau} = 0 , \quad p^\mu = m \frac{dx^\mu}{d\tau} ,$$

$$\frac{dS_{\mu\nu}}{d\tau} = 0 , \quad S^{\mu\nu} = \frac{1}{2m'} \bar{\Omega}^{\mu\nu} .$$

$$\left(\bar{\Omega}^{\mu\nu} = \Lambda^{\mu A} \frac{d\Lambda^\nu{}_A}{d\tau} \right)$$

Massive twistor

Twistor coordinates :

$(8_{\mathbb{C}} = 16_{\mathbb{R}})$

$$Z_A{}^I = \begin{pmatrix} \lambda_\alpha{}^I \\ i\mu^{\dot{\alpha}I} \end{pmatrix}, \quad \bar{Z}_I{}^B = (Z_A{}^I)^\dagger A^{\bar{A}B} = \begin{pmatrix} -i\bar{\mu}_I{}^\beta & \bar{\lambda}_{I\dot{\beta}} \end{pmatrix}.$$

conformal symmetry
to be broken by
the mass shell condition

$SU(2,2) \times SU(2)$

interpreted as the
"little group" symmetry
of a massive particle

Poisson bracket algebra :

$$\{Z_A{}^I, \bar{Z}_J{}^B\} = -i\,\delta_A{}^B \delta^I{}_J \quad \Longrightarrow \quad \{\lambda_\alpha{}^I, \bar{\mu}_J{}^\beta\} = \delta_\alpha{}^\beta \delta^I{}_J, \quad \{\bar{\lambda}_{I\dot{\alpha}}, \mu^{\dot{\beta}J}\} = \delta_{\dot{\alpha}}{}^{\dot{\beta}} \delta_I{}^J.$$

Constraints

$$\phi_0 = \frac{1}{2}(p^2 + m^2)\,, \quad \chi^0 = \frac{1}{p^2} x^\mu p_\mu\,.$$

Reduction from 16 to 12.

Spin constraints are **not** needed.

Mass-shell constraint should be "complexified".

$$\phi = -\frac{1}{2}\left(\det(\lambda) - m(\tilde{S}^2)\right)\,, \quad \bar{\chi} = \frac{1}{\det(\lambda)}\langle\bar{\mu}\lambda\rangle\,,$$
$$\bar{\phi} = -\frac{1}{2}\left(\det(\bar{\lambda}) - m(\tilde{S}^2)\right)\,, \quad \chi = \frac{1}{\det(\bar{\lambda})}[\bar{\lambda}\mu]\,.$$

Twistor model "solves" the spin constraints.

Map to the spherical top

$$p_{\alpha\dot\alpha} = (\sigma^\mu)_{\alpha\dot\alpha} p_\mu = \begin{pmatrix} p_0 + p_3 & p_1 - ip_2 \\ p_1 - ip_2 & p_0 - p_3 \end{pmatrix}$$

The map (assuming constraints)

$$p_{\alpha\dot\alpha} = -\lambda_\alpha{}^I \bar\lambda_{I\dot\alpha}\,, \quad x^{\dot\alpha\alpha} = \frac{1}{2}\left(\frac{1}{\det(\lambda)}\mu^{\dot\alpha I}\lambda^{\alpha J}\epsilon_{IJ} + \frac{1}{\det(\bar\lambda)}\epsilon^{IJ}\bar\lambda_I{}^{\dot\alpha}\bar\mu_J{}^\alpha\right)\,,$$

$$(\Lambda_{\alpha\dot\alpha})^I{}_J = \frac{2\lambda_\alpha{}^I\bar\lambda_{J\dot\alpha}}{|\det(\lambda)|}\,, \quad S_{\alpha\beta} = \frac{1}{2}\left(\bar\mu_{I(\alpha}\lambda_{\beta)}{}^I - \frac{1}{\det(\lambda)}\lambda_{(\alpha}{}^I\lambda_{\beta)}{}^J(\mu^{\dot\gamma}{}_I\bar\lambda_{J\dot\gamma})\right)\,.$$

incidence relation

complex Minkowski space

Hints

$$\mu^{\dot\alpha I} = -z^{\dot\alpha\beta}\lambda_\beta{}^I\,, \quad z^\mu = x^\mu + iy^\mu\,.$$

$$J_{\mu\nu} = x_\mu p_\nu - x_\nu p_\mu + S_{\mu\nu}\,, \quad S_{\mu\nu} = \frac{1}{2}\varepsilon_{\mu\nu}{}^{\rho\sigma}(y_\rho p_\sigma - y_\sigma p_\rho)\,.$$

Equations of motion

$$\phi = -\frac{1}{2}\left(\det(\lambda) - m(\tilde{S}^2)\right) ,$$
$$\bar{\phi} = -\frac{1}{2}\left(\det(\bar{\lambda}) - m(\tilde{S}^2)\right) .$$

$$\mathcal{S} = \int d\sigma \left(\frac{i}{2}(\bar{Z}_I{}^A \dot{Z}_A{}^I - Z_A{}^I \dot{\bar{Z}}_I{}^A) - \bar{\kappa}\phi - \kappa\bar{\phi} \right)$$

$$\dot{\lambda}_\alpha{}^I = -i(\mathrm{Re}\,\kappa) m' \lambda_\alpha{}^J W_J{}^I ,$$
$$\dot{\mu}^{\dot{\alpha} I} = -i(\mathrm{Re}\,\kappa) m' \mu^{\dot{\alpha} J} W_J{}^I + \frac{\kappa}{2} \bar{\lambda}^{\dot{\alpha} I} .$$

Perfect agreement with the spherical top model !

Quantization

$$\bar{\mu}_I{}^{\alpha} = -i\frac{\partial}{\partial \lambda_\alpha{}^I}\,, \quad \mu^{\dot{\alpha} I} = -i\frac{\partial}{\partial \bar{\lambda}_{I\dot{\alpha}}}\,.$$

Quantize first, and then impose constraints

On-shell fields and Fierz-Pauli/Bargmann-Wigner equation:

$$\Phi^{(2s,0)}_{\alpha_1\alpha_2\ldots\alpha_{2s}}(x) = c_{I_1\ldots I_{2s}} \int d[\lambda,\bar{\lambda}]\, \lambda^{I_1}_{\alpha_1}\lambda^{I_2}_{\alpha_2}\cdots\lambda^{I_{2s}}_{\alpha_{2s}}\, \exp\left(ix^{\dot{\beta}\beta}\bar{\lambda}_{\dot{\beta}J}\lambda^J_\beta\right)\,,$$

$$\Phi^{(2s-1,1)}_{\dot{\alpha}_1|\alpha_2\ldots\alpha_{2s}}(x) = c_{I_1\ldots I_{2s}} \int d[\lambda,\bar{\lambda}]\, \bar{\lambda}^{I_1}_{\dot{\alpha}_1}\lambda^{I_2}_{\alpha_2}\cdots\lambda^{I_{2s}}_{\alpha_{2s}}\, \exp\left(ix^{\dot{\beta}\beta}\bar{\lambda}_{\dot{\beta}J}\lambda^J_\beta\right)\,,$$

$$\vdots \qquad\qquad \vdots$$

$$\Phi^{(0,2s)}_{\dot{\alpha}_1\dot{\alpha}_2\ldots\dot{\alpha}_{2s}}(x) = c_{I_1\ldots I_{2s}} \int d[\lambda,\bar{\lambda}]\, \bar{\lambda}^{I_1}_{\dot{\alpha}_1}\bar{\lambda}^{I_2}_{\dot{\alpha}_2}\cdots\bar{\lambda}^{I_{2s}}_{\dot{\alpha}_{2s}}\, \exp\left(ix^{\dot{\beta}\beta}\bar{\lambda}_{\dot{\beta}J}\lambda^J_\beta\right)\,,$$

where $c_{I_1\ldots I_{2s}}$ is a totally symmetrized SU(2) tensor, and the integration measure is

$$d[\lambda,\bar{\lambda}] = d^4\lambda d^4\bar{\lambda}\left[\delta(\det(\lambda)-m)\delta(\det(\bar{\lambda})-m)\right]\,.$$

Newman-Janis shift

[Newman, Janis 1965]

[Guevara, Maybee, O'Connell, Ochirov, Vines 2012.11570]

Complex Minkowski spacetime with the"spin-length" vector :

$$x^\mu \quad \to \quad x^\mu \pm i y^\mu \,, \qquad y_\mu = \frac{1}{m^2} W_\mu = \frac{1}{m^2} \varepsilon_{\mu\nu\rho\sigma} p^\nu S^{\rho\sigma}$$

opens a way to couple the twistor model

to background fields.

[J. Kim, SL, work in progress]

Summary and Outlook

Spinor-helicity variables are useful.

- QFT (QED, QCD, ...)
- GR (post-Minkowskian expansion, Newman-Janis shift, etc.)

Perhaps, we should rewrite some chapters of textbooks ?