Algebraic Properties of Riemannian Manifolds
(arXiv: 2206.08108)

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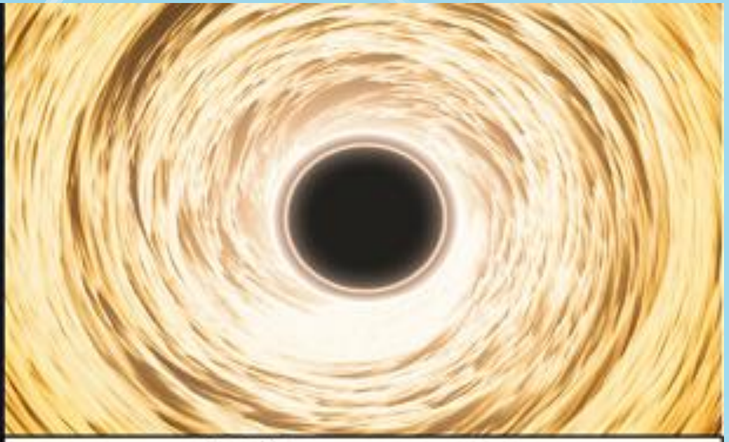

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Center for Quantum SpaceTime

This work is based on the collaboration with

Prof. Youngjoo Chung (GIST) & Prof. Chi-Ok Hwang (GIST)

For this work,

we have extensively used Mathematica and MathSymbolica (www.mathsymbolica.com) developed by Prof. Youngjoo Chung

Wolfram Innovator Award 2017

Riemann Invariants

- Why are scalar invariants of the Riemann tensor $R_{\mu\nu\rho\sigma}$ important?
- Allow a manifestly coordinate invariant characterization of spacetime
- Studying curvature singularities, e.g., Kretschmann scalar
  $K = R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}\left(= \frac{48G^2M^2}{c^4r^6} \text{ for the Schwarzschild solution}\right)$
- Classification of curvature tensors, e.g., Petrov & Segre classifications
- Equivalence problem, i.e.,
  the question of whether two spacetime metrics are equivalent ?
- Understanding the structure of Diff(M), e.g., Riemann normal coordinates
- Effective action for quantum fields with gravitational interaction

Spacetime Invariants

- The invariant characterization of a curved space must be in terms of scalars constructed from $R_{\mu\nu\rho\sigma}$ and $g_{\mu\nu}$.
- # of independent components of Riemann tensor $R_{\mu\nu\rho\sigma}$ in d dimensions

$$\#(\text{Riem}) = \frac{1}{12}\, d^2(d^2 - 1)$$

- # of such scalars (Weinberg, 1972; Penrose & Rindler, 1986)

$$I(d) = \frac{1}{12}\, d^2(d^2 - 1) + \frac{d(d+1)}{2} - d^2 = \frac{d+3}{2}\binom{d}{3} \quad \text{in curved frame}$$

$$= \frac{1}{12}\, d^2(d^2 - 1) - \binom{d}{2} = \frac{d+3}{2}\binom{d}{3} \quad \text{in tangent frame}$$

$$I(4) = 14, \quad I(5) = 40, \quad I(6) = 90$$

Algebraic Invariants

- Is it possible to encode all informations of the Riemann tensor $R_{\mu\nu\rho\sigma}$ in scalar polynomial invariants ?
- No! Why?
- It is known that *pp* waves have all scalar invariants, of all orders, equal to zero. (Ehlers & Kundt, 1962)
- Algebraically independent scalar polynomial (sp) invariants:

  invariants not satisfying any polynomial relation (called a *syzygy*)
- Complete set $\{I_1, I_2, \cdots, I_k\}$ if any other sp invariant can be written as a polynomial in the I_j but no invariant in the set can be so expressed in terms of the others.
- One of the deriving forces in the early development of computer algebra

Riemann Symmetry

- Antisymmetry

$$R_{abcd} = -R_{bacd} = -R_{abdc} = R_{badc}$$

The first Bianchi identity

$$R_{abcd} + R_{acbd} + R_{adbc} = 0$$

Pair symmetry

$$R_{abcd} = R_{cdab}$$

- Of the 4! different possible orderings of the indices, only two are independent. A convenient choice of them is

R_{abcd} and R_{acbd}

Irreducible decomposition of Riemann tensors

- The sp invariants are expressions whose contractions hide very large numbers of individual terms, and so hard to calculate.
- To use the Newman-Penrose complex spinor formalism
- To use a bivector formalism (so-called, a rotor dubbed by mathematicians or 't Hooft symbols by physicists)

H. A. Buchdahl, On rotor calculus I, J. Aust. Math. Soc. **6** (1966) 402.

H. A. Buchdahl, On rotor calculus II, J. Aust. Math. Soc. **6** (1966) 424.

- Curvature form of any bundle with connection over an oriented four-manifold: $F \in C^\infty(\mathfrak{g} \otimes \Omega^2)$ = 2-form in Ω^2 taking values in a Lie algebra $\mathfrak{g}$
- Riemann curvature tensor $R \in C^\infty(\mathfrak{g} \otimes \Omega^2)$ where $\mathfrak{g} = spin(4)$ or $so(4)$
- Thus the curvature tensor is associated with two vector spaces $\mathfrak{g}$ and Ω^2

Irreducible decomposition of Riemann tensors

- The splitting of vector spaces is induced by the existence of the projection operators

$$P_\pm = \frac{1}{2}(1 \pm *), \qquad P_\pm = \frac{1}{2}(1 \pm \gamma_5)$$

acting on the vector spaces Ω^2 and the $so(4)$ generators $J_{ab} = \frac{1}{4}[\gamma_a, \gamma_b]$, respectively.

- $$\Omega^2 \equiv \Lambda^2 T^* M = \Omega^2_+ \oplus \Omega^2_-,$$

where $\Omega^2_\pm$ is the ± 1 eigenspaces of the Hodge star operator $* : \Omega^2 \to \Omega^2$.

- $so(4) = su_+(2) \oplus su(2)_-$. with $J_{ab} = J^+_{ab} \oplus J^-_{ab}$

where $J^\pm_{ab} \equiv P_\pm J_{ab}$

Irreducible decomposition of Riemann tensors

$$J^+_{ab} = \frac{i}{2}\eta^i_{ab}\tau^i \in su(2)_+, \qquad J^-_{ab} = \frac{i}{2}\bar{\eta}^i_{ab}\tau^i \in su(2)_-,$$

where the expansion coefficients, the so-called 't Hooft symbols, are given by[1]

$$\eta^i_{ab} = -i\mathrm{Tr}\left(J^+_{ab}\tau^i\right), \qquad \bar{\eta}^i_{ab} = -i\mathrm{Tr}\left(J^-_{ab}\tau^i\right).$$

- Combining of the decomposition of two vector spaces $\mathfrak{g}$ and Ω^2, it leads to an irreducible decomposition of the general Riemann curvature tensor

$$R_{abcd} = f^{ij}_{(++)}\eta^i_{ab}\eta^j_{cd} + f^{ij}_{(+-)}\eta^i_{ab}\bar{\eta}^j_{cd} + f^{ij}_{(-+)}\bar{\eta}^i_{ab}\eta^j_{cd} + f^{ij}_{(--)}\bar{\eta}^i_{ab}\bar{\eta}^j_{cd},$$

$R_{abcd} = R_{cdab}$: $\quad f^{ij}_{(++)} = f^{ji}_{(++)}, \quad f^{ij}_{(--)} = f^{ji}_{(--)}, \quad f^{ij}_{(+-)} = f^{ji}_{(-+)}.$

$\varepsilon^{abcd}R_{abcd} = 0.$: $\quad f^{ij}_{(++)}\delta^{ij} = f^{ij}_{(--)}\delta^{ij}$

Weyl and Ricci tensors

$$W_{abcd} = f^{ij}_{(++)}\eta^i_{ab}\eta^j_{cd} + f^{ij}_{(--)}\bar\eta^i_{ab}\bar\eta^j_{cd} - \frac{1}{3}\left(f^{ij}_{(++)}\delta^{ij} + f^{ij}_{(--)}\delta^{ij}\right)(\delta_{ac}\delta_{bd} - \delta_{ad}\delta_{bc})$$
$$= \widetilde{f}^{ij}_{(++)}\eta^i_{ab}\eta^j_{cd} + \widetilde{f}^{ij}_{(--)}\bar\eta^i_{ab}\bar\eta^j_{cd},$$

where $\widetilde{f}^{ij}_{(++)} \equiv f^{ij}_{(++)} - \frac{1}{3}\delta^{ij}(f^{kl}_{(++)}\delta^{kl})$ and $\widetilde{f}^{ij}_{(--)} \equiv f^{ij}_{(--)} - \frac{1}{3}\delta^{ij}(f^{kl}_{(--)}\delta^{kl})$ are symmetric, traceless 3×3 matrices. In the end the Riemann tensor is decomposed as

$$R_{abcd} = W_{abcd} + \frac{1}{12}R(\delta_{ac}\delta_{bd} - \delta_{ad}\delta_{bc}) + f^{ij}_{(+-)}(\eta^i_{ab}\bar\eta^j_{cd} + \bar\eta^j_{ab}\eta^i_{cd})$$

where the last one corresponds to the traceless Ricci tensor $S_{ab} \equiv R_{ab} - \frac{1}{4}\delta_{ab}R = 2f^{ij}_{(+-)}\eta^i_{ac}\bar\eta^j_{bc}$ or, conversely, $f^{ij}_{(+-)} = \frac{1}{8}S_{ab}\eta^i_{ac}\bar\eta^j_{bc}$.

$$R_{ab} = (f^{ij}_{(++)}\delta^{ij} + f^{ij}_{(--)}\delta^{ij})\delta_{ab} + 2f^{ij}_{(+-)}\eta^i_{ac}\bar\eta^j_{bc},$$
$$R = 4(f^{ij}_{(++)}\delta^{ij} + f^{ij}_{(--)}\delta^{ij}).$$

Quadratic (pseudo-)scalar invariants

- $$\begin{aligned} R_{abcd}R_{abcd} &= 16\Big(f^{ij}_{(++)}f^{ij}_{(++)} + 2f^{ij}_{(+-)}f^{ij}_{(+-)} + f^{ij}_{(--)}f^{ij}_{(--)}\Big), \\ R_{ab}R_{ab} &= 4\big(f^{ij}_{(++)}\delta^{ij} + f^{ij}_{(--)}\delta^{ij}\big)^2 + 16f^{ij}_{(+-)}f^{ij}_{(+-)}, \\ R^2 &= 16\big(f^{ij}_{(++)}\delta^{ij} + f^{ij}_{(--)}\delta^{ij}\big)^2, \\ \varepsilon^{abcd}\varepsilon^{efgh}R_{abef}R_{cdgh} &= 64\Big(f^{ij}_{(++)}f^{ij}_{(++)} - 2f^{ij}_{(+-)}f^{ij}_{(+-)} + f^{ij}_{(--)}f^{ij}_{(--)}\Big), \\ \varepsilon^{cdef}R_{abcd}R_{abef} &= 32\Big(f^{ij}_{(++)}f^{ij}_{(++)} - f^{ij}_{(--)}f^{ij}_{(--)}\Big). \end{aligned}$$

- Gauss-Bonnet relation

$$R_{abcd}R_{abcd} - 4R_{ab}R_{ab} + R^2 = \frac{1}{4}\varepsilon^{abcd}\varepsilon^{efgh}R_{abef}R_{cdgh}.$$

Parity transformation: even ($P = +1$) vs. odd ($P = -1$)

- Let us introduce 3×3 matrices defined by

$$(A_\pm)_{ij} \equiv f^{ij}_{(\pm\pm)}, \qquad (B)_{ij} \equiv f^{ij}_{(+-)}, \qquad (B^T)_{ij} \equiv f^{ij}_{(-+)}, \qquad i,j = 1,2,3.$$

Then the matrices $A_\pm$ are symmetric, i.e. $A_\pm^T = A_\pm$, but the matrix B is a general 3×3 matrix with no explicit symmetry. The parity transformation (3.23) denoted by P acts on the matrices as

$$P : A_\pm \leftrightarrow A_\mp, \qquad P : B \leftrightarrow B^T.$$

$$Q_1 \equiv R = 4(\mathrm{Tr}A_+ + \mathrm{Tr}A_-).$$

$$Q_2 = \{R^2, \mathrm{Tr}\left(A_+^2 + A_-^2\right), \mathrm{Tr}(BB^T)\},$$

$$\widetilde{Q}_1 = 4(\mathrm{Tr}A_+ - \mathrm{Tr}A_-) = 0,$$

$$\widetilde{Q}_2 = \mathrm{Tr}\left(A_+^2 - A_-^2\right),$$

$$\varepsilon_{abcd}\eta^{(\pm)i}_{de} = \mp(\delta_{ec}\eta^{(\pm)i}_{ab} + \delta_{ea}\eta^{(\pm)i}_{bc} - \delta_{eb}\eta^{(\pm)i}_{ac}), \quad \text{where } \eta^{(+)i}_{ab} \equiv \eta^i_{ab} \text{ and } \eta^{(-)i}_{ab} \equiv \bar{\eta}^i_{ab}.$$

Cubic scalar invariants

S. A. Fulling, R. C. King, B. G. Wybourne and C. J. Cummins, Normal forms for tensor polynomials: I. The Riemann tensor, Class. Quantum Grav. **9** (1992) 1151.

- One curvature-scalar term: R^3,
- Two Ricci terms: $RR_{ab}R_{ab}$, $R_{ab}R_{bc}R_{ca}$,
- Five Riemann terms: $R_{ac}R_{bd}R_{abcd}$, $RR_{abcd}R_{abcd}$, $R_{ab}R_{acde}R_{bcde}$, $R_{abcd}R_{cdef}R_{efab}$, $R_{acbd}R_{cedf}R_{eafb}$.

- Cubic basis elements

$$Q_3 = \{R^3, R\,\mathrm{Tr}\left(A_+^2 + A_-^2\right), R\,\mathrm{Tr}(BB^T), \mathrm{Tr}\left(A_+^3 + A_-^3\right), \mathrm{Tr}\left(B^T A_+ B + B A_- B^T\right), \det B\}.$$

- ∃ two algebraic relations (D. Xu, 1987)

$$R_{ab}R_{acde}R_{bcde} = \frac{1}{4}R^3 - 2RR_{ab}R_{ab} + 2R_{ab}R_{bc}R_{ca} + 2R_{ac}R_{bd}R_{abcd} + \frac{1}{4}RR_{abcd}R_{abcd}, \qquad ($$

$$R_{acbd}R_{cedf}R_{eafb} = -\frac{5}{8}R^3 + \frac{9}{2}RR_{ab}R_{ab} - 4R_{ab}R_{bc}R_{ca} - 3R_{ac}R_{bd}R_{abcd} - \frac{3}{8}RR_{abcd}R_{abcd} + \frac{1}{2}R_{abcd}R_{cdef}R_{efab}. \qquad ($$

Cubic second-rank tensors

- S. A. Fulling, R. C. King, B. G. Wybourne and C. J. Cummins, Normal forms for tensor polynomials: I. The Riemann tensor, Class. Quantum Grav. **9** (1992) 1151.

$$RR_{ac}R_{bc} = R\left(\frac{R^2}{16}\delta_{ab} - Rf^{ij}_{(+-)}\left(\eta^i\bar{\eta}^j\right)_{ab} + 4f^{ij}_{(+-)}f^{ij}_{(+-)}\delta_{ab} - 4\varepsilon^{i_1i_2i_3}\varepsilon^{j_1j_2j_3}f^{i_1j_1}_{(+-)}f^{i_2j_2}_{(+-)}\left(\eta^{i_3}\bar{\eta}^{j_3}\right)_{ab}\right)$$

A: R^2R_{ab}	I: $R_{ab}R_{cdef}R_{cdef}$
B: $RR_{ac}R_{bc}$	J: $R_{ac}R_{bedf}R_{cedf}$
C: $R_{ab}R_{cd}R_{cd}$	K: $R_{aecd}R_{bfcd}R_{ef}$
D: $R_{ac}R_{bd}R_{cd}$	L: $R_{acbd}R_{cedf}R_{ef}$
E: $RR_{acbd}R_{cd}$	M: $R_{acde}R_{bcdf}R_{ef}$
F: $R_{acbd}R_{ce}R_{de}$	N: $R_{agcd}R_{bgef}R_{cdef}$
G: $R_{ac}R_{becd}R_{de}$	O: $R_{aecg}R_{bfdg}R_{cdef}$
H: $RR_{aecd}R_{becd}$	P: $R_{aebg}R_{cdef}R_{cdfg}$

Table 1: The 16 second-rank tensors at cubic order

- ∃ 14 linearly independent basis elements,
  only two syzygy relations exist

$$R^2R_{ab} - 2RR_{ac}R_{bc} + 4R_{ac}R_{bd}R_{cd} - 4R_{ab}R_{cd}^2 - 2RR_{cd}R_{acbd} + 4R_{ce}R_{de}R_{acbd}$$
$$+4R_{ac}R_{de}R_{becd} + R_{ab}R_{cdef}^2 + 4R_{ef}R_{acbd}R_{cedf} - 2R_{ac}R_{bedf}R_{cedf} + 2R_{aebg}R_{cdef}R_{cdfg} = 0,$$

$$3R^2R_{ab} - 4RR_{ac}R_{bc} + 8R_{ac}R_{bd}R_{cd} - 12R_{ab}R_{cd}^2 - 4RR_{cd}R_{acbd} + 8R_{ce}R_{de}R_{acbd}$$
$$-2RR_{aecd}R_{becd} + 8R_{ef}R_{acde}R_{bcdf} + 4R_{ef}R_{aecd}R_{bfcd} + 3R_{ab}R_{cdef}^2 + 8R_{ef}R_{acbd}R_{cedf}$$
$$-4R_{agcd}R_{bgef}R_{cdef} + 8R_{aecg}R_{bfdg}R_{cdef} + 4R_{aebg}R_{cdef}R_{cdfg} = 0.$$

Quartic scalar invariants

I: R^4	VIII: $\mathrm{Tr}(BB^T)\mathrm{Tr}(BB^T)$
II: $R^2\mathrm{Tr}\,(BB^T)$	IX: $\mathrm{Tr}(BB^T)\mathrm{Tr}\,(A_+^2 + A_-^2)$
III: $R^2\mathrm{Tr}\,(A_+^2 + A_-^2)$	X: $\mathrm{Tr}A_+^2\mathrm{Tr}A_+^2 + \mathrm{Tr}A_-^2\mathrm{Tr}A_-^2$
IV: $R\mathrm{Tr}\,(A_+^3 + A_-^3)$	XI: $\mathrm{Tr}\,(BB^TBB^T)$
V: $R\mathrm{Tr}\,(B^TA_+B + BA_-B^T)$	XII: $\mathrm{Tr}\,(A_+BA_-B^T)$
VI: $R\det B$	XIII: $\mathrm{Tr}\,(B^TA_+^2B + BA_-^2B^T)$
VII: $\mathrm{Tr}A_+^2\mathrm{Tr}A_-^2$	XIV: $\mathrm{Tr}\,(A_+^4 + A_-^4)$

Table 3: Matrix representation of the quartic basis elements

- $\exists\ 12 = 26 - 14$ algebraic relations relation
- A. Harvey found 6 such relations (1995) based on the fact that any object antisymmetrized over $(d+1)$ indices in d dimensions identically vanishes.

A: R^4	N: $R_{ab}R_{cd}R_{aecf}R_{bedf}$
B: $R^2R_{ab}R_{ab}$	O: $RR_{abcd}R_{cdef}R_{efab}$
C: $RR_{ab}R_{bc}R_{ca}$	P: $RR_{acbd}R_{aebf}R_{cedf}$
D: $(R_{ab}R_{ab})^2$	Q: $R_{ab}R_{acbd}R_{efgc}R_{efgd}$
E: $R_{ab}R_{bc}R_{cd}R_{da}$	R: $R_{ab}R_{cdef}R_{agef}R_{bgcd}$
F: $RR_{ab}R_{cd}R_{acbd}$	S: $R_{ab}R_{cedf}R_{egfa}R_{gcbd}$
G: $R_{ab}R_{ce}R_{ed}R_{acbd}$	T: $(R_{abcd}R_{abcd})^2$
H: $R^2R_{abcd}R_{abcd}$	U: $R_{abcd}R_{abce}R_{fghd}R_{fghe}$
I: $RR_{ab}R_{acde}R_{bcde}$	V: $R_{abcd}R_{cdef}R_{efgh}R_{ghab}$
J: $R_{ab}R_{ab}R_{cdef}R_{cdef}$	W: $R_{abcd}R_{abef}R_{cegh}R_{dfgh}$
K: $R_{ab}R_{bc}R_{defa}R_{defc}$	X: $R_{abcd}R_{efab}R_{gche}R_{gdhf}$
L: $R_{ab}R_{cd}R_{acef}R_{bdef}$	Y: $R_{acbd}R_{cedf}R_{egfh}R_{gahb}$
M: $R_{ab}R_{cd}R_{aebf}R_{cedf}$	Z: $R_{acbd}R_{eafb}R_{gehc}R_{fgdh}$

Table 2: The 26 quartic scalars

Quartic scalar invariants

- Harvey relation $R^{ab}{}_{[ab}R^{cd}{}_{cd}R^{ef}{}_{ef}R^{gh}{}_{gh]} = 0$

$$(19):\ -\frac{5}{48}R^4+\frac{1}{2}R^2R_{ab}^2-\frac{1}{3}RR_{ab}R_{bc}R_{ca}+\frac{1}{16}R_{abcd}^4-R_{ab}R_{cdef}R_{efga}R_{gbcd}$$
$$+2R_{ab}R_{cedf}R_{egfa}R_{gcbd}+R_{ab}R_{bcad}R_{efgc}R_{gdef}+R_{acbd}R_{cedf}R_{egfh}R_{agbh}$$
$$-2R_{abcd}R_{abef}R_{cgeh}R_{dgfh}-2R_{acbd}R_{aebf}R_{cheg}R_{dhfg}+\frac{1}{8}R_{abcd}R_{cdef}R_{efgh}R_{ghab}$$
$$+R_{abcd}R_{abef}R_{cegh}R_{ghdf}-R_{abcd}R_{abce}R_{dhfg}R_{fgeh}=0. \qquad \text{(D.4)}$$

$$-\frac{1}{4}R^4+R^2R_{ab}R_{ab}+R_{ab}^4-2R_{ab}R_{bc}R_{cd}R_{da}-2R_{ab}R_{cd}R_{efac}R_{efbd}+2R_{acbd}R_{cedf}R_{egfh}R_{gahb}$$
$$-4R_{acbd}R_{eafb}R_{fgdh}R_{gehc}+\frac{3}{2}R_{abcd}R_{abef}R_{cegh}R_{ghdf}-R_{abcd}R_{abce}R_{fghd}R_{hefg}=0. \qquad \text{(D.5)}$$

- Using the notation (4.25), Eqs. (D.4) and (D.5) are equally written as

$$\frac{R^4}{16}-12R^2\mathrm{Tr}\left(A_+^2+A_-^2\right)+128R\mathrm{Tr}\left(A_+^3+A_-^3\right)-768\mathrm{Tr}\left(A_+^4+A_-^4\right)$$
$$+384\left(\mathrm{Tr}(A_+^2)\mathrm{Tr}(A_+^2)+\mathrm{Tr}(A_-^2)\mathrm{Tr}(A_-^2)\right)=0. \qquad \text{(D.6)}$$

$$P-6S+3X=RR_{abcd}R_{cedf}R_{eafb}-6R_{ab}R_{cedf}R_{egfa}R_{gcbd}+3R_{abcd}R_{efab}R_{gche}R_{gdhf}=0$$

Independent quintic basis elements

The quintic scalars have 90 basis elements, denoted by $\mathcal{R}^0_{10,5}$ in Appendix A of Ref. [35], among which the number of Weyl tensor monomials, denoted by $\mathcal{C}^0_{10,5}$ in Appendix D, is 19. But their explicit expression has not been displayed in Ref. [35]. Thus we do not know the explicit form of quintic monomials. Nevertheless, our method gives a hint as to how many independent quintic basis elements can exist. It is necessary to count how many quintic matrix polynomials made of the matrices in (4.25) which are parity even. To do this, let us

$S^5_1 \sim S^5_{13}:\ Q_1 \otimes Q_4$	$S^5_{19}:\ \mathrm{Tr}BB^T\mathrm{Tr}\left(B^TA_+B + BA_-B^T\right)$
$S^5_{14}:\ \mathrm{Tr}A_+^2\mathrm{Tr}A_+^3 + \mathrm{Tr}A_-^2\mathrm{Tr}A_-^3$	$S^5_{20}:\ \mathrm{Tr}BB^T\det B$
$S^5_{15}:\ \mathrm{Tr}A_+^2\mathrm{Tr}A_-^3 + \mathrm{Tr}A_-^2\mathrm{Tr}A_+^3$	$S^5_{21}:\ \mathrm{Tr}\left(A_+^5 + A_-^5\right)$
$S^5_{16}:\ \mathrm{Tr}\left(A_+^2 + A_-^2\right)\mathrm{Tr}\left(B^TA_+B + BA_-B^T\right)$	$S^5_{22}:\ \mathrm{Tr}\left(B^TA_+^3B + BA_-^3B^T\right)$
$S^5_{17}:\ \mathrm{Tr}BB^T\mathrm{Tr}\left(A_+^3 + A_-^3\right)$	$S^5_{23}:\ \mathrm{Tr}\left(A_+^2BA_-B^T\right) + \mathrm{Tr}\left(A_-^2BA_+B^T\right)$
$S^5_{18}:\ \mathrm{Tr}\left(A_+^2 + A_-^2\right)\det B$	$S^5_{24}:\ \mathrm{Tr}\left(BB^TA_+BB^T + B^TBA_-B^TB\right)$

Table 4: Matrix representation of the quintic basis elements

Including parity odd basis elements

$$\widetilde{Q}_1 = 4\left(\mathrm{Tr}A_+ - \mathrm{Tr}A_-\right) = 0, \qquad \text{by Eq. (3.14)},$$

$$\widetilde{Q}_2 = \mathrm{Tr}\left(A_+^2 - A_-^2\right),$$

$$\widetilde{Q}_3 = \{R\,\mathrm{Tr}\left(A_+^2 - A_-^2\right), \mathrm{Tr}\left(A_+^3 - A_-^3\right), \mathrm{Tr}\left(B^T A_+ B - B A_- B^T\right)\},$$

$$\widetilde{Q}_4 = \{\widetilde{III}, \widetilde{IV}, \widetilde{V}, \widetilde{IX}, \widetilde{X}, \widetilde{XIII}, \widetilde{XIV}\} = R\widetilde{Q}_3 \bigcup \{\widetilde{IX}, \widetilde{X}, \widetilde{XIII}, \widetilde{XIV}\},$$

For the quintic order, the situation becomes a bit complicated. Besides the 23 independent quintic scalar invariants in Table 4, it is necessary to include the new invariants from the products, $Q_1 \otimes \widetilde{Q}_4$, $Q_2 \otimes \widetilde{Q}_3$, $Q_3 \otimes \widetilde{Q}_2$, $\widetilde{Q}_2 \otimes \widetilde{Q}_3$ and $\widetilde{Q}_5^W$ where the last one is the set of quintic Weyl monomials. It is not difficult to determine newly generated basis elements; 6 elements from $Q_1 \otimes \widetilde{Q}_4$, 7 elements from $Q_2 \otimes \widetilde{Q}_3$ and $Q_3 \otimes \widetilde{Q}_2$ and 4 elements from $\widetilde{Q}_5^W$ while one even element from $\widetilde{Q}_2 \otimes \widetilde{Q}_3$, $\mathrm{Tr}\left(A_+^2 - A_-^2\right)\mathrm{Tr}\left(B^T A_+ B - B A_- B^T\right)$. Hence 17 odd elements and 1 even elemеmt are newly generated after including the pseudo-scalars in $\widetilde{Q}_n$. However, the 17 odd elements may not be completely independent since it is expected that the quintic order will also give rise to an identity similar to Eq. (5.10) according to the Caley-Hamilton theorem [54, 55, 56]. Therefore we claim that there are totally 40 = 23 + 17 linearly independent quintic (pseudo-)scalar invariants.

Generalizations to several directions

Going to higher dimensions

In six dimensions, there also exists a global isomorphism between a six-dimensional Lorentz group and a classical Lie group:

$$Spin(6) \cong SU(4). \tag{5.16}$$

Therefore it is possible to devise a six-dimensional version of the 't Hooft symbols using the isomorphism between the chiral $so(6)$ Lorentz algebra and the $su(4)$ Lie algebra [61]. Since the chiral and anti-chiral (or the fundamental and anti-fundamental) representations of $so(6)$ (or $su(4)$) must be distinct, there are two kinds of 't Hooft symbols, η^a_{AB} and $\bar{\eta}^a_{AB}$, with $a = 1, \cdots, 15,\ A, B = 1, \cdots, 6$, and they satisfy algebraic identities similar to the four-dimensional case. The six-dimensional Riemann curvature tensor can be classified into two classes:

$$\mathbb{A} : R^{(+)}_{ABCD} = f^{ab}_{(++)} \eta^a_{AB} \eta^b_{CD}, \tag{5.17}$$

$$\mathbb{B} : R^{(-)}_{ABCD} = f^{ab}_{(--)} \bar{\eta}^a_{AB} \bar{\eta}^b_{CD}. \tag{5.18}$$

This expression may be very useful for studying the algebraic properties of six-dimensional scalar invariants. In particular, it was argued [61] that the existence of two classes and their splitting into $\mathbb{A}$ and $\mathbb{B}$ would be related to the mirror symmetry of Calabi-Yau manifolds. It will be interesting to find any relationship between the scalar invariants of each class. We hope to address this issue in the near future too.