Will quantum cosmology resolve the cosmological singularities?

(Big Bang and Late-time Singularities)

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BB Singularity Theorem for Cosmology

- Hawking-Penrose singularity theorem [Hawking, Penrose, PRSL A 314 ('70)]: Under the weak energy condition

$$T_{\mu\nu}W^{\mu}W^{\nu} \geq 0 \Leftrightarrow \rho \geq 0, \rho + p_i \geq 0 \text{ for } (i = 1,2,3),$$

for any timelike (null) vector $W \in T_p$ at each point $p \in M$,

all timelike (null) geodesics end at a singularity and the universe has a beginning.

- Singularity of inflationary spacetimes [Borde, Guth, Vilenkin, PRL 90 ('03)]: Many inflation models violate the weak energy condition and under the averaged expansion condition $H_{\mathrm{av}} > 0$, the universe has a beginning.

Classification of Late-time Singularities

[Nojiri, Odintsov, Tsujikawa, PRD 71 ('05); Kamenshchik, CQG 30 ('13)]

- FRW geometry and Friedman equation

$$ds^2 = -dt^2 + a^2(t)d\vec{x}^2$$

$$\left(\frac{\dot{a}}{a}\right)^2 = H^2 = \varrho\,, \quad \frac{\ddot{a}}{a} = \dot{H} + H^2 = -\frac{1}{2}(\varrho + 3p)\,, \quad \dot{\varrho} + 3H(\varrho + p) = 0$$

- Riemann curvature tensor

$$R^{\alpha}{}_{0\beta 0} = -\frac{\ddot{a}}{a}\delta^{\alpha}_{\beta} = -\left(\dot{H} + H^2\right)\delta^{\alpha}_{\beta}$$

$$R^{1}{}_{212} = R^{1}{}_{313} = R^{2}{}_{323} = \dot{a}^2$$

- Geodesic deviation equation and comoving observer

$$\frac{D^2\eta^i}{ds^2} = R^{i}{}_{klm}u^k u^l \eta^m$$

$$\frac{D^2\eta^\alpha}{ds^2} = R^{\alpha}{}_{00\beta}u^0 u^0 \eta^\alpha = \frac{\ddot{a}}{a}\eta^\alpha$$

Late-time Singularities

[I. Albarran ('21)]

Singularity	t	a	H	$\dot{H}$	$\ddot{H}$	$V(\phi)$	Models	Tipler/ Krolak
Big Rip	t_{br}	∞	∞	∞	∞	$e^{\alpha\phi}$	DE EoS $w < -1$	S
Big Freeze	t_{bf}	a_{bf}	∞	∞	∞	ϕ^{γ_1}	Generalized C haplygin Gas	W/S
Sudden Singularity	t_{ss}	a_{ss}	H_{ss}	∞	∞	ϕ^{γ_2}	Standard type GCG	W
Type IV	t_{iv}	a_{iv}	H_{iv}	$\dot{H}_{iv}$	∞	ϕ^{γ_3}	GCG F(R)	W
Little Rip	∞	∞	∞	∞	∞	ϕ^4	F(R)	W
Little Sibling of BR	∞	∞	∞	$\dot{H}_{ls}$	0	ϕ^2		W

$\gamma_1, \gamma_2 < -2$, $\gamma_3 \neq 2(p-1)/(p+1)$ for integer p

Why Quantum Gravity?

- General Relativity (GR) contains spacetime singularities, such the Big Bang and black hole singularities.
- GR breaks down near a singularity, where large quantum fluctuations are expected.
- Quantum theory near the horizon, though not the singularity itself, predicts the Hawking radiation and vacuum polarization.
- The question is then how to quantize the spacetime?
- Wheeler-DeWitt equation approach (other approaches: loop quantum gravity or string theory)

Wheeler-DeWitt Equation

Geometrodynamical Approach

- The Wheeler-DeWitt (WDW) equation [PR **160, 162** (‘67)]
  - Variables: metrics on 3-surfaces invariant under diffeomorphism (geometrodynamics) and extrinsic curvature for momenta.
  - WDW equation: the super-Hamiltonian constraint and/or the super-momentum constraints via Dirac quantization.
  - Merit: wave functions depend on the metrics on the 3-surfaces and allow a direct interpretation.
  - Demerit: do not know the infinite dimensional symplectic manifold of solutions of the Einstein equation, probably true for other QG.

ADM Formalism

- Arnwitt-Deser-Misner formalism: foliate a globally hyperbolic spacetime manifold by spacelike 3-surfaces

$$ds^2 = -(N^2 - N_i N^i)dt^2 + 2N_i dt dx + h_{ij} dx^i dx^j$$

$$N = \text{lapse function} \ \& \ N_i = \text{shift vector}$$

- The Hamiltonian for gravity and matter fields

$$H = \int d^3x [N H_0 + N_i H^i]$$

$$H_0 = \text{super}-\text{Hamiltonian} \ \& \ H_i = \text{super}-\text{momentum}$$

- Wheeler-DeWitt equation via Dirac quantization

$$\hat{H}_0 \left(\hat{\pi}_{ij}, \hat{h}_{ij}, \hat{\pi}_{\varphi}, \hat{\varphi} \right) \Psi(h_{ij}, \varphi) = 0$$

$$\hat{H}^i \left(\hat{\pi}_{ij}, \hat{h}_{ij}, \hat{\pi}_{\varphi}, \hat{\varphi} \right) \Psi(h_{ij}, \varphi) = 0$$

ADM Formalism for FRW Geometry

- The metric for Friedmann-Robertson-Walker universe

$$ds^2 = -N^2 dt^2 + a^2(t) d\Omega_3^2$$

- The Lagrangian density for the Einstein-Hilbert action with the cosmological constant ($\sigma = 16\pi G$)

$$L_G = \frac{1}{\sigma}\sqrt{-g}(R - 2\Lambda) = -\frac{6a\dot{a}^2}{N\sigma} + \frac{6kNa}{\sigma} - \frac{2N\Lambda a^3}{\sigma} + \frac{6}{N\sigma}\frac{d}{dt}(a^2\dot{a})$$

- The conjugate momentum $\pi_a = \partial L_G/\partial \dot{a} = -(12/N\sigma)a\dot{a}$
- The Hamiltonian for gravity

$$H_G = \int d^3x\, N\left[-\frac{\sigma}{8}\frac{\pi_a^2}{a} + \frac{6}{\sigma}ka - \frac{2\Lambda}{\sigma}a^3\right]$$

- The WDW equation for a FRW universe minimally coupled to a homogeneous scalar field (inflaton):

$$\left[a^{-q}\frac{\partial}{\partial a}\left(a^q\frac{\partial}{\partial a}\right) - \frac{1}{a^2}\frac{\partial^2}{\partial \varphi^2} - V_G(a) + V_M(\varphi)\right]\Psi(a,\varphi) = 0$$

$$V_G(a) = ka^2 - \frac{\Lambda}{3}a^4, \qquad V_M(\varphi) = 2a^4 V(\varphi), \qquad \left(\hbar = c = l_P^2 = \frac{16\pi}{m_P^2} = 1\right)$$

Quantum FRW Universe

- The WDW equation for a FRW universe minimally coupled to a homogeneous scalar field (inflaton):

$$\left[a^{-q}\frac{\partial}{\partial a}\left(a^{q}\frac{\partial}{\partial a}\right)-\frac{1}{a^{2}}\frac{\partial^{2}}{\partial\varphi^{2}}-V_{G}(a)+V_{M}(\varphi)\right]\Psi(a,\varphi)=0$$

$$V_{G}(a)=ka^{2}-\frac{\Lambda}{3}a^{4},\left(\hbar=c=l_{P}^{2}=\frac{16\pi}{m_{P}^{2}}=1\right)$$

$$V_{M}(\varphi)=2a^{4}V(\varphi)$$

- The WDW equation for a FRW universe coupled to a phantom field:

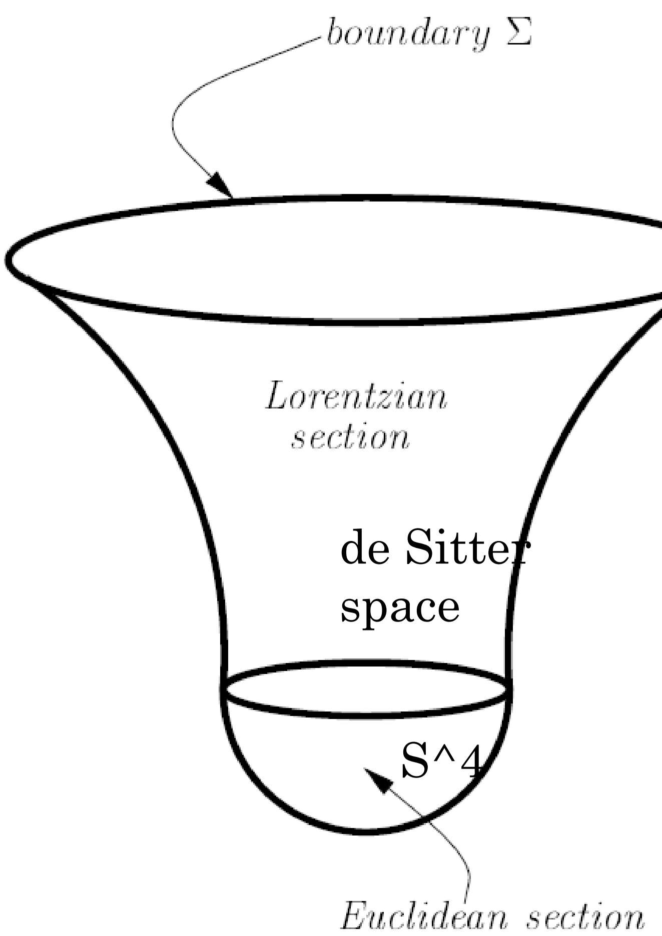
$$\left[a^{-q}\frac{\partial}{\partial a}\left(a^{q}\frac{\partial}{\partial a}\right)+\frac{1}{a^{2}}\frac{\partial^{2}}{\partial\varphi^{2}}-V_{G}(a)+V_{M}(i\varphi)\right]\Psi(a,\varphi)=0$$

How to Interpret the Wave Function of the Universe?

- Born' probability interpretation of wave function assumes an ensemble of systems with a given condition while we live inside the universe and the wave function of our universe is just unique. (On the contrary, there may be copies of our universe in the multiverse.)
- Everett's many world interpretation of the wave function assumes that the wave function splits off through the evolution (like branches of a tree).
- Bohm's pilot theory assumes that the wave packet is peaked along a trajectory with a quantum potential included.

Hartle-Hawking's No-Boundary Wave Function

Hartle-Hawking's No-Boundary Wave Function of the Universe



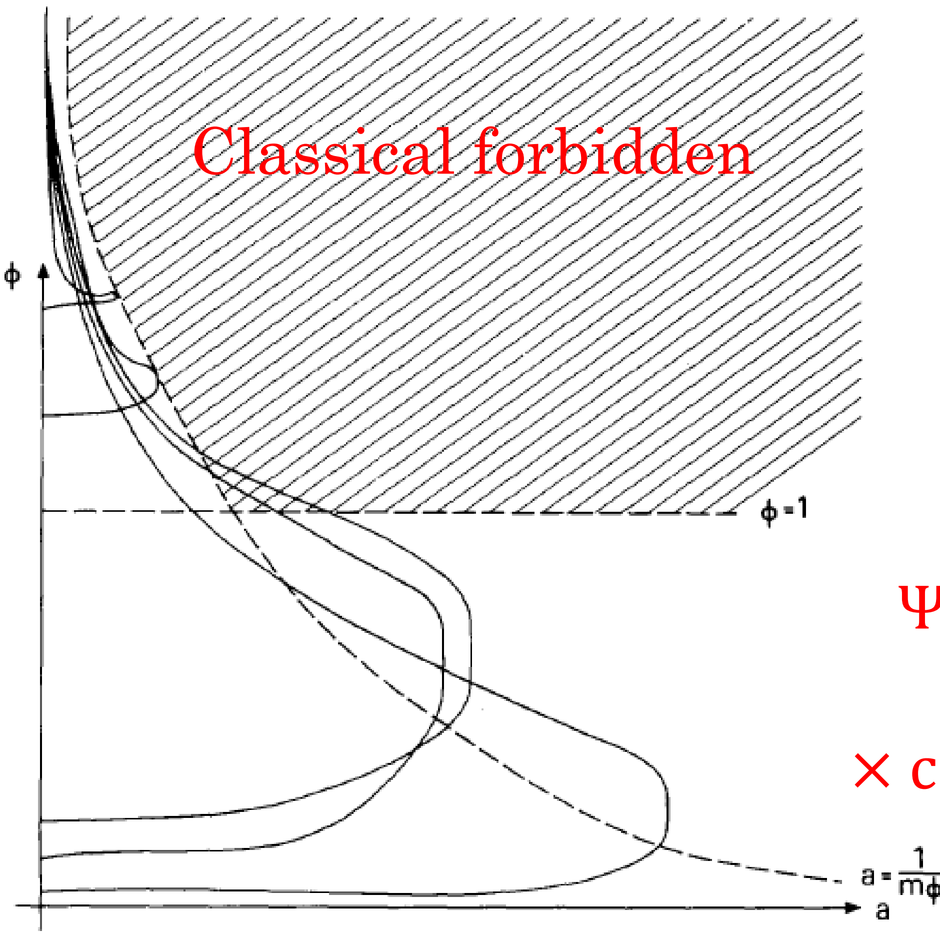
[Fig. Bousso, Hawking, gr-qc/9608009]

- The sum over all 4-compact Euclidean manifolds with the 3-surface boundary

$$\Psi[h_{ij},\varphi] = N\int D[g]D[\varphi]\exp(-I_E[g,\varphi])$$

- The HH wave function prescribes the boundary condition, i.e. "no-boundary" but one-boundary at the 3-surface.

Wave functions for a FRW coupled to a Massive Scalar [Hawking, NPB 239 ('84)]



$$\Psi_{HH}(a,\phi) = N_k e^{1/3m^2\phi^2}$$
$$\times \cos\left(\frac{(m^2\phi^2a^2-1)^{\frac{3}{2}}}{3m^2\phi^2} - \frac{\pi}{4}\right)$$

Predictions of HH Wave Function

[as summarized by D. N. Page, hep-th/0610121]

- Lorentzian spacetime can emerge in a WKB limit of an analytical continuation [Hartle, Hawking, PRD 28 ('83); Hawking, NPB 239 ('84)].
- The universe can inflate to large size [Hawking, NPB 239 ('84)].
- Models can predict near-critical density [Hawking, NPB 239 ('84); Hawking, Page, NPB 264 ('86)].
- Models can predict low anisotropies [Hawking, Luttrell, PLB 143 ('84)].
- Inhomogeneities start in ground states and so can fit CMB data [Halliwell, Hawking, PRD 31 ('85)].
- Entropy starts low and grows with time [Hawking, PRD 32 ('85); Page, PRD 32 ('85); Hawking, Laflamme, Lyons, PRD 47 ('93); SPK, S. W. Kim, PRD 49 ('94); SPK, S. W. Kim, PRD 51 ('95); …]

Quantum FRW Cosmology

Structure in Quantum Cosmology

[Halliwell, Hawking, PRD 31 (‘85)]

- Small perturbations of FRW-geometry

$$h_{ij} = a^2(\Omega_{ij} + \varepsilon_{ij})$$

- Super-Hamiltonian at quadratic order

$$\begin{aligned}
{}^{S}H^{n}_{|2} = \tfrac{1}{2} e^{-3\alpha} \Bigg[& \left[\tfrac{1}{2} a_n^2 + \frac{10(n^2-4)}{(n^2-1)} b_n^2 \right] \pi_\alpha^2 + \left[\tfrac{15}{2} a_n^2 + \frac{6(n^2-4)}{(n^2-1)} b_n^2 \right] \pi_\phi^2 \\
& - \pi_{a_n}^2 + \frac{(n^2-1)}{(n^2-4)} \pi_{b_n}^2 + \pi_{f_n}^2 + 2 a_n \pi_{a_n} \pi_\alpha + 8 b_n \pi_{b_n} \pi_\alpha - 6 a_n \pi_{f_n} \pi_\phi \\
& - e^{4\alpha} \left[\tfrac{1}{3}(n^2 - \tfrac{5}{2}) a_n^2 + \frac{(n^2-7)}{3} \frac{(n^2-4)}{(n^2-1)} b_n^2 + \tfrac{2}{3}(n^2-4) a_n b_n - (n^2-1) f_n^2 \right] \\
& + e^{6\alpha} m^2 (f_n^2 + 6 a_n f_n \phi) + e^{6\alpha} m^2 \phi^2 \left[\tfrac{3}{2} a_n^2 - \frac{6(n^2-4)}{(n^2-1)} b_n^2 \right] \Bigg] ,
\end{aligned}$$

$${}^{V}H^{n}_{|2} = \tfrac{1}{2} e^{-3\alpha} \left[(n^2-4) c_n^2 (10\pi_\alpha^2 + 6\pi_\phi^2) + \frac{1}{(n^2-4)} \pi_{c_n}^2 + 8 c_n \pi_{c_n} \pi_\alpha + (n^2-4) c_n^2 (2 e^{4\alpha} - 6 e^{6\alpha} m^2 \phi^2) \right]$$

$${}^{T}H^{n}_{|2} = \tfrac{1}{2} e^{-3\alpha} \{ d_n^2 (10\pi_\alpha^2 + 6\pi_\phi^2) + \pi_{d_n}^2 + 8 d_n \pi_{d_n} \pi_\alpha + d_n^2 [(n^2+1) e^{4\alpha} - 6 e^{6\alpha} m^2 \phi^2] \} .$$

Parametrically Interacting Relativistic 'Particle' in Superspace

[SPK, Page, PRD 45 ('92); SPK, PRD 46 ('92)]

- The supermetric for FRW geometry and a minimal scalar

$$ds^2 = -da^2 + a^2 d\varphi^2$$

- The Hamiltonian constraint and the WDW equation

$$H(a,\varphi) = -\underbrace{\left(\pi_a^2 + V_G(a)\right)}_{\text{gravity part}} + \underbrace{\frac{1}{a^2}\left(\pi_\varphi^2 + 2a^6 V(\varphi)\right)}_{\text{matter part}} = 0$$

$$[-\nabla^2 - V_G(a) + 2a^4 V(\varphi)]\Psi(a,\varphi) = 0$$

$$\nabla^2 = -a^{-1}\frac{\partial}{\partial a}\left(a\frac{\partial}{\partial a}\right) + \frac{1}{a^2}\frac{\partial^2}{\partial \varphi^2}, \qquad V_G(a) = ka^2 - 2\Lambda a^4$$

- The universe scatters from an initial surface to a final one in the superspace of the 3-geometry. A prescription of the boundary condition?

Inflation with Negative Λ

[Hartle, Hawking, Hertog, arXiv:1205.3807; JCAP01('14)]

- Negative Λ and massive scalar field with negative mass and the WKB wave function $\Psi(a,\varphi) = e^{-S/\hbar}$

$$\pi_a^2 = \underbrace{(2\Lambda + m^2\varphi^2)}_{\text{negative}} a^4 - ka^2 + \frac{1}{a^2}\pi_\varphi^2$$

- Continuation to $a = ib$ and $\Psi(a,\varphi) = e^{iI/\hbar}$

$$\pi_b^2 = \underbrace{-(2\Lambda + m^2\varphi^2)}_{\text{positive}} b^4 - ka^2 + \frac{1}{a^2}\pi_\varphi^2$$

- Phase transition from AdS to dS in a complex superspace.

Quantum Phantom Cosmology

[Dabrowski, Kiefer, Sandhöfer, PRD 74 (‘06)]

- The Hamiltonian constraint and the WDW equation

$$H(a,\varphi) = -\underbrace{\left(\pi_a^2 + V_G(a)\right)}_{\text{gravity part}} + \underbrace{\frac{1}{a^2}\left(-\pi_\varphi^2 + 2a^6 V(\varphi)\right)}_{\text{phantom part}} = 0$$

$$[-\nabla^2 - V_G(a) + 2a^4 V(\varphi)]\Psi(a,\varphi) = 0$$

$$\nabla^2 = -a^{-1}\frac{\partial}{\partial a}\left(a\frac{\partial}{\partial a}\right) - \frac{1}{a^2}\frac{\partial^2}{\partial \varphi^2}, \qquad V_G(a) = ka^2 - 2\Lambda a^4$$

- It was argued that the quantum effects dominate the region of classical BR singularity and the wave packets that follow the classical trajectory disperse in the quantum region.

Quantum-Classical Transition

From QG to SQG to CG

Quantum Gravity

$\hat{G}_{\mu\nu} = 8\pi G \hat{T}_{\mu\nu}$	WDW, HH wave function, tunneling wave function

$$G = 1/m_{\mathrm{P}}^2 \rightarrow 0$$

Semiclassical Quantum Gravity

$G_{\mu\nu}^{\mathrm{C}} + G_{\mu\nu}^{\mathrm{Q}}[G] = 8\pi G \langle \hat{T}_{\mu\nu} \rangle$	QFT in curved spacetime, Hawking radiation, pair production

$$\hbar \rightarrow 0$$

Classical Gravity

$G_{\mu\nu}^{\mathrm{C}} + G_{\mu\nu}^{\mathrm{Q}}[G] = 8\pi G \left(T_{\mu\nu}^{\mathrm{C}} + T_{\mu\nu}^{\mathrm{Q}}[\hbar] \right)$	Inflationary models

de Broglie-Bohm Pilot-Wave Theory

- In the causal interpretation, a particle has a definite (suitable) path that is affected by the wave function

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi + V\psi$$

- In a semiclassical regime, where $\psi = Fe^{\frac{i}{\hbar}S}$

Hamilton–Jacobi equation (real part of QM)

$$\frac{\partial S}{\partial t} + \frac{1}{2m}(\nabla S)^2 + V + V_{\mathrm{Q}} = 0, \left(V_{\mathrm{Q}} = -\frac{\hbar^2}{2m}\frac{\nabla^2 F}{F}\right)$$

continuity equation (imaginary part of QM)

$$\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\vec{v})^2 = 0, \left(\rho = F^2, \vec{v} = \frac{\nabla S}{m}\right)$$

Quantum FRW Universe

(minisuperspace model)

- The metric for Friedmann-Robertson-Walker universe

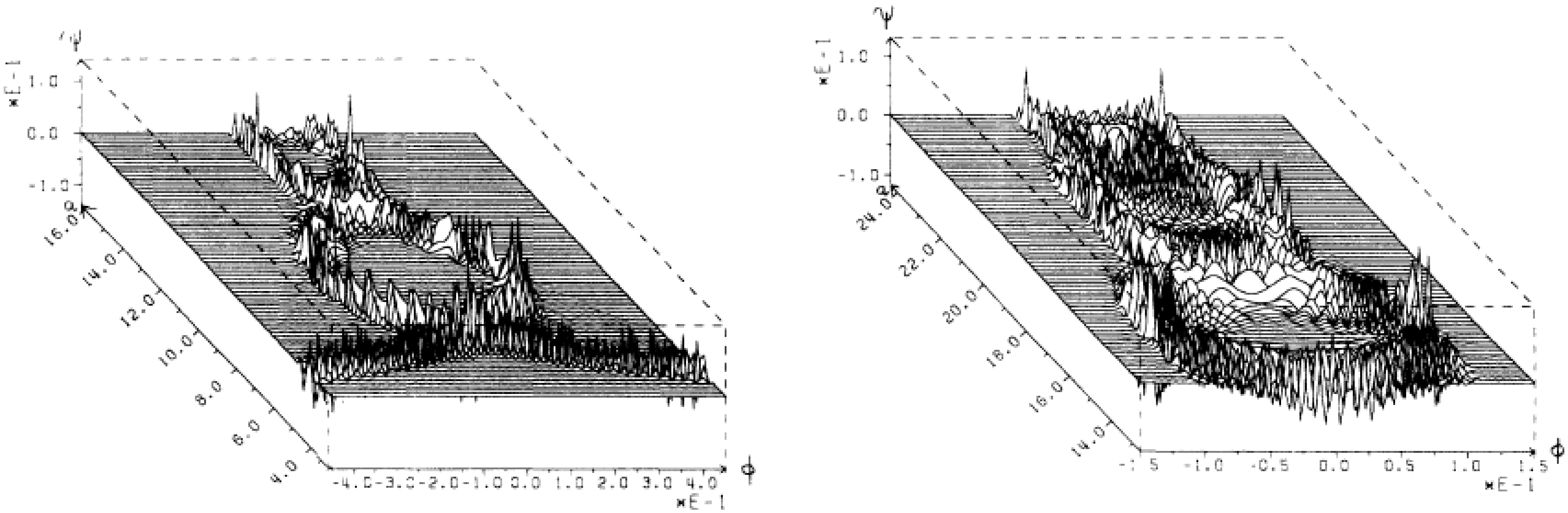
$$ds^2 = -N^2 dt^2 + a^2(t) d\Omega_3^2$$

- The WDW equation for a FRW universe with a minimal scalar field (inflaton) up to a factor ordering

$$\left[-\frac{\hbar^2}{2M}\frac{\partial^2}{\partial a^2} - MV_G(a) + \hat{H}_m(\pi_\varphi, \varphi)\right]\Psi(a, \varphi) = 0$$

$$V_G(a) = \frac{1}{2}ka^2 - \Lambda a^4, \left(c = 1, M = \frac{3m_P^2}{4\pi}\right)$$

Wave Packet for FRW with a Minimal Scalar



A closed universe (k=1), m = 6, and n =120 (harmonic quantum number)
[Fig. from Kiefer, PRD 38 ('88)]

Pilot-Wave Theory and Born-Oppenheimer Idea

- The wave functions are peaked around some trajectories (wave packets) and allow the pilot-wave theory

$$\left[-\frac{\hbar^2}{2M}\nabla^2 - MV_{\mathrm{G}}(h_a) + \widehat{H}\left(\varphi, -i\frac{\delta}{\delta\varphi}, h_a\right)\right]\Psi(h_a,\varphi) = 0, \left(h_a = h_{ij}\right)$$

- Apply the Born-Oppenheimer idea that separates a slow moving massive particle (M=Planck mass squared) from a fast moving light particle (matter field) and then expand the quantum state for the fast moving variable by a certain basis to be determined

$$|\Psi(h_a,\varphi)\rangle = \psi(h_a)|\Phi(\varphi,h_a)\rangle$$

$$|\Phi(\varphi,h_a)\rangle = \sum_k c_k(h_a)|\Phi_k(\varphi,h_a)\rangle$$

Semiclassical Quantum Gravity

[SPK, PRD 52 ('95); CQG 13 ('96); PRD 55 ('97)]

- Apply the de Broglie-Bohm pilot-wave theory to the gravity part only

$$\psi(h_a) = F(h_a)e^{iS(h_a)/\hbar}$$

- Then, in a semiclassical regime, the WDW equation is equivalent to

$$\frac{1}{2M}(\nabla S)^2 - MV_{\mathrm{G}}(h_a) + H_{nn} - \frac{\hbar^2}{2M}\frac{\nabla^2 F}{F} - \frac{\hbar^2}{M}\mathrm{Re}(Q_{nn}) = 0$$

$$\frac{1}{2}\nabla^2 S + \frac{\nabla F}{F}\cdot\nabla S + \mathrm{Im}(Q_{nn}) = 0$$

$$H_{nk}(h_a) := \langle\Phi_n(\varphi, h_a)|\widehat{H}|\Phi_k(\varphi, h_a)\rangle; \quad \vec{A}_{nk}(h_a) := i\langle\Phi_n(\varphi, h_a)|\nabla|\Phi_k(\varphi, h_a)\rangle$$

$$Q_{nn}(h_a) := \frac{\nabla F}{F}\cdot\left(\frac{\nabla c_n}{c_n} - i\sum_k \vec{A}_{nk}\frac{c_k}{c_n}\right)$$

Semiclassical Quantum Gravity

- In quantum gravity, time is NOT *a priori* given since the WDW equation is a constraint equation (problem of time). Thus, time should be defined from the wave function itself.
- In the semiclassical quantum gravity, time emerges from the wave packet and the cosmological time is defined as the directional derivative of the action, not necessarily a classical one, along the trajectory

$$\frac{\delta}{\delta\tau} := \frac{1}{M} \nabla S(h_a) \cdot \nabla$$

Semiclassical Quantum Gravity

- The matter field obeys the Heisenberg (matrix) equation

$$i\hbar\frac{\delta}{\delta\tau}c_n = \sum_{k\neq n} H_{nk}\, c_k - \frac{\hbar}{M}\nabla S\cdot\sum_k \vec{A}_{nk}\, c_k - \frac{\hbar^2}{2M}\sum_{k\neq n}\Omega_{nk}\, c_k$$

$$H_{nk}(h_a) := \langle\Phi_n(\varphi,h_a)|\hat{H}|\Phi_k(\varphi,h_a)\rangle; \quad \vec{A}_{nk}(h_a) := i\langle\Phi_n(\varphi,h_a)|\nabla|\Phi_k(\varphi,h_a)\rangle$$

$$\Omega_{nk}(h_a) := \nabla^2\delta_{nk} - 2i\vec{A}_{nk}\cdot\nabla + \langle\Phi_n(\varphi,h_a)|\nabla^2|\Phi_k(\varphi,h_a)\rangle$$

- The unitarity of the quantum state of matter field is preserved.

$$C^{+}(\tau)\cdot C(\tau) = 1, C(\tau) = \begin{pmatrix} c_1(\tau) \\ c_2(\tau) \\ \vdots \end{pmatrix}$$

Scalar Field Cosmology

- The extended superspace for a FRW with a minimal scalar and the cosmological time:

$$ds^2 = -ada^2 + a^3 d\varphi^2$$

$$\frac{\partial}{\partial \tau} = -\frac{1}{Ma}\frac{\partial S(a)}{\partial a}\frac{\partial}{\partial a}, \left(\frac{\partial a(\tau)}{\partial \tau} = -\frac{1}{Ma}\frac{\partial S(a)}{\partial a}\right)$$

- The Heisenberg matrix equation for the scalar field

$$i\hbar \frac{\partial c_n}{\partial \tau} = \sum_{k \neq n} H_{nk}\, c_k - \hbar \sum_{k} B_{nk}\, c_k - \frac{\hbar^2}{2Ma}\sum_{k \neq n} \Omega_{nk}\, c_k$$

$$H_{nk}(a(\tau)) := \langle \Phi_n | \hat{H} | \Phi_k \rangle; \quad B_{nk}(a(\tau)) := i \langle \Phi_n | \frac{\partial}{\partial \tau} | \Phi_k \rangle$$

$$\Omega_{nk}(a(\tau)) := -\frac{1}{\dot{a}^2}\left[\left(\frac{\partial^2}{\partial \tau^2} - \frac{\ddot{a}}{\dot{a}}\frac{\partial}{\partial \tau}\right)\delta_{nk} - 2iB_{nk}\frac{\partial}{\partial \tau} + \langle \Phi_n | \frac{\partial^2}{\partial \tau^2} - \frac{\ddot{a}}{\dot{a}}\frac{\partial}{\partial \tau} | \Phi_k \rangle\right]$$

Scalar Field Cosmology

- The semiclassical Friedmann equation

$$\left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2} - \Lambda = \frac{8\pi}{3m_P^2 a^3}\left[H_{nn} - \frac{4\pi\hbar^2}{3m_P^2 a\dot{a}} U_{nn}\,\mathrm{Re}(R_{nn}) + \frac{2\pi\hbar^2}{3m_P^2 a}\left(U_{nn}^2 + \frac{1}{\dot{a}}\dot{U}_{nn}\right)\right]$$

$$R_{nn} = \frac{\dot{c}_n}{c_n} - i\sum_k B_{nk}\frac{c_k}{c_n}$$

$$U_{nn} := \frac{\partial F/\partial a}{F} = -\frac{1}{2}\frac{(a\dot{a})^{\cdot}}{a\dot{a}^2 + (4\pi\hbar/3m_P^2)\,\mathrm{Im}(R_{nn})}$$

- The effective energy density

$$\rho_{nn} = H_{nn} - \frac{4\pi\hbar^2}{3m_P^2 a\dot{a}} U_{nn}\,\mathrm{Re}(R_{nn}) + \frac{2\pi\hbar^2}{3m_P^2 a}\left(U_{nn}^2 + \frac{1}{\dot{a}}\dot{U}_{nn}\right)$$

Massive Scalar Field Cosmology

- The semiclassical Friedmann equation with a massive scalar field at the lowest order of $\hbar/M$

$$\hat{H} = -\frac{\hbar^2}{2a^3}\frac{\partial^2}{\partial\varphi^2} + \frac{m^2 a^3}{2}\varphi^2$$

$$\left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2} - \Lambda = \frac{8\pi}{3m_P^2 a^3}\left[H_{nn} + \frac{2\pi\hbar^2}{3m_P^2 a}\left(U_{nn}^{(0)2} + \frac{1}{\dot{a}}\dot{U}_{nn}^{(0)}\right)\right]$$

$$R_{nn}^{(0)} = 0\,; \quad U_{nn}^{(0)} = -\frac{1}{2}\frac{(a\dot{a})^{\cdot}}{a\dot{a}^2}$$

$$H_{nn} = \hbar a^3\left(n + \frac{1}{2}\right)\left[\dot{\phi}^*\dot{\phi} + m^2\phi^*\phi\right]; \quad \ddot{\phi} + 3\frac{\dot{a}}{a}\dot{\phi} + m^2\phi = 0$$

- Near the BB singularity, ultra-relativistic theory and $a \approx t^{\frac{1}{3}}$, the quantum effect (RHS) has $-\left({}^{2\pi\hbar}\!/_{3m_p^2 a^3}\right)^2$ and protects the universe from the BB singularity.

Second Quantized Universes

Scalar Field Quantum Cosmology

- The ADM formalism for a FRW geometry

$$ds^2 = -N^2 dt^2 + a^2(t) d\Omega_3^2$$

- The WDW equation (Dirac quantization of Hamiltonian constraint) for a FRW universe minimally coupled to a single-field inflaton (scalar field); the WDW equation is already second quantized

$$\left[\frac{\partial^2}{\partial a^2} - \frac{1}{a^2}\frac{\partial^2}{\partial \varphi^2} + 2a^4 V(\varphi) - V_G(a)\right]\Psi(a,\varphi) = 0$$

$$V_G(a) = ka^2 - 2\Lambda a^4, \left(\hbar = c = l_P^2 = \frac{16\pi}{m_P^2} = 1\right)$$

Quantum Universes in the Superspace

- The supermetric for FRW geometry and a minimal scalar

$$ds^2 = -da^2 + a^2 d\varphi^2$$

- The Hamiltonian constraint and the WDW equation

$$H(a,\varphi) = \underbrace{-(\pi_a^2 + V_G(a))}_{\text{gravity part } H_G} + \frac{1}{a^2}\underbrace{\left(\pi_\varphi^2 + 2a^6 V(\varphi)\right)}_{\text{scalar field part } H_M} = 0$$

$$[-\nabla^2 - V_G(a) + 2a^4 V(\varphi)]\Psi(a,\varphi) = 0$$

$$\nabla^2 = -\frac{\partial^2}{\partial a^2} + \frac{1}{a^2}\frac{\partial^2}{\partial \varphi^2}, V_G(a) = ka^2 - 2\Lambda a^4$$

- A Cauchy initial value problem w.r.t. the scale factor a and a prescription for the boundary condition and the KG inner product is well defined for all a, including singularities $(H, \dot{H}, \ddot{H} = \infty)$.

Quantum Universes in the Superspace

[SPK, Page, PRD 45 ('92); SPK, PRD 46 ('92)]

- The scalar field for single-field inflation model

$$V(\varphi) = \lambda_{2p}\varphi^{2p}/(2p)$$

- The eigenfunctions and the Symanzik scaling law

$$H_M(\varphi, a)\Phi_n(\varphi, a) = E_n(a)\Phi_n(\varphi, a)$$

$$E_n(a) = \left(\lambda_{2p}a^6/p\right)^{1/(p+1)}\varepsilon_n$$

$$\Phi_n(\varphi, a) = \left(\lambda_{2p}a^6/p\right)^{1/4(p+1)}F_n\left(\left(\lambda_{2p}a^6/p\right)^{1/(p+1)}\varphi\right)$$

- The coupling matrix among the energy eigenfunctions

$$\frac{\partial}{\partial a}\vec{\Phi}(\varphi, a) = \Omega(a)\vec{\Phi}(\varphi, a)$$

$$\Omega_{mn}(a) = (3/4(p+1)a)(\varepsilon_m - \varepsilon_n)\int d\varsigma F_m(\varsigma)\, F_n(\varsigma)\varsigma^2$$

Quantum Universes in the Superspace

- The gravitational part of the WDW equation

$$\Psi(a,\varphi) = \vec{\Phi}^T(\varphi,a)\cdot\vec{\psi}(a)$$

$$\left[\frac{d^2}{da^2} - V_G(a) + \frac{E(a)}{a^2} - \left(2\Omega(a)\frac{d}{da} - \Omega^2(a) - \frac{1}{a}\Omega(a)\right)\right]\vec{\psi}(a) = 0$$

- The transition matrix and the Cauchy problem

$$\Psi(a,\varphi) = \vec{\Phi}^T(\varphi,a)T(a)\vec{\psi}(a)\,; \quad T(a) = \exp\left[\int^a da'\Omega(a')\right]$$

$$\frac{d}{da}\begin{pmatrix}\vec{\psi}(a)\\ d\vec{\psi}(a)/da\end{pmatrix} = \begin{pmatrix}0 & I\\ T^{-1}\left(V_G - \frac{E}{a^2}\right)T & 0\end{pmatrix}\begin{pmatrix}\vec{\psi}(a)\\ d\vec{\psi}(a)/da\end{pmatrix}$$

Quantum Universes in the Superspace

- The two-component wave function

$$\begin{pmatrix} \Psi(a,\varphi) \\ \partial\Psi(a,\varphi)/\partial a \end{pmatrix} = \begin{pmatrix} \vec{\Phi}^T(\varphi,a) & 0 \\ 0 & \vec{\Phi}^T(\varphi,a) \end{pmatrix} T\exp\left[\int \begin{pmatrix} \Omega(a') & I \\ V_G(a') - E/a'^2 & \Omega(a') \end{pmatrix} da'\right]$$

$$\times \begin{pmatrix} \int d\varphi' \vec{\Phi}(\varphi', a_0)\, \Psi(a_0, \varphi') \\ \int d\varphi' \, \vec{\Phi}(\varphi', a_0) \partial\Psi(a_0, \varphi)/\partial a_0 \end{pmatrix}$$

- The off-diagonal components are the gravitational part equation only with $V_G(a) - E/a^2$.
- The continuous transitions among energy eigenfunctions [Kiefer, CQG 4 (‘87); PRD 38 (‘88)].
- Hartle-Hawkig wave function $\Psi_{HH}(a_0, \phi)$ at a_0 will give the two-component wave function at any a.
- All matter fields regardless of interactions become ultral-relativistic for $a_0 \ll 1$.

Summary

- Quantum cosmology may be a consistent framework for studying quantum fluctuations of spacetime and matter fields.
- Can quantum cosmology resolve the BB and late-time singularity? Quantum cosmology with N-massless fields (or tachyons or phantom fields) has the rigged Hilbert space (normalized wave functions). The wave functions have constant dispersions at the BB singularity and an exponentially suppressed dispersion at the BR.
- How to interpret the wave function of the universe? Conventional wisdom is the probability interpretation (regular at singularities, normalizability) [Alberran ('21)]. Still another interpretation: "Is third quantization necessary"? KG inner product is well defined at singularities.