Perturbative Nicolai-Map Diagrammatics: Application to Poincaré Supergravity

Junhyeok Lee (Hanyang U.)

with J. Chae and H. Jang

arXiv:2605.29990 [hep-th] - accepted for publication in JHEP

2026 Workshop on Cosmology and Quantum Spacetime CQUeST
Jul 20 – 24, 2026
St. John's Hotel, Gangneung

Nicolai map

In QFT, the expectation value of a general operator $\mathcal{O}[\phi,\psi]$ is given in the path-integral formalism by

$$\langle \mathcal{O}[\phi,\psi] \rangle = Z^{-1} \int \mathcal{D}\phi \int \mathcal{D}\psi \;\; \mathcal{O}[\phi,\psi] e^{iS[\phi,\psi;\lambda]}.$$

However, if the operator $\mathcal{O}$ depends only on boson field ϕ, its expectation value can be written as

$$\langle \mathcal{O}[\phi] \rangle = Z^{-1} \int \mathcal{D}\phi \int \mathcal{D}\psi \;\; \mathcal{O}[\phi] \; e^{iS[\phi,\psi;\lambda]}.$$

After functionally integrating out ψ, the expectation value reduces to

$$\langle \mathcal{O}[\phi] \rangle = Z^{-1} \int \mathcal{D}\phi \;\; \mathcal{O}[\phi] \; e^{iS_{\phi E}[\phi;\lambda]},$$

where $S_{\phi E}$ is the nonlocal bosonic effective action for ϕ.

Nicolai map

If there exists a map T_λ such that

$$iS_{\phi E}[\phi;\lambda] = iS[T_\lambda\phi, 0; 0] + Trln\left(\frac{\delta T_\lambda \phi}{\delta\phi}\right),$$

then the expectation value reduces to

$$\langle \mathcal{O}[\phi]\rangle = Z^{-1}\int \mathcal{D}\phi\ \ \mathcal{O}[\phi]\ e^{iS_{\phi E}[\phi;\lambda]} = Z^{-1}\int \mathcal{D}\phi\, det\left(\frac{\delta T_\lambda \phi}{\delta\phi}\right)\ \mathcal{O}\left[T_\lambda^{-1}\phi\right]\ e^{iS[T_\lambda\phi,0;0]}.$$

In other words, the expectation value can be computed in the free theory.

$$\langle \mathcal{O}[\phi]\rangle = Z^{-1}\int \mathcal{D}\phi\ \ \mathcal{O}\left[T_\lambda^{-1}\phi\right]\ e^{iS[\phi,0;0]}.$$

Such a map T_λ is called the **Nicolai map.**

Nicolai map

In 1980, Hermann Nicolai first discovered the Nicolai map for the scalar field in the two-dimensional Wess-Zumino model. [H. Nicolai, Phys. Lett. B 89 (1980) 341.]

In 1984, Olaf Lechtenfeld derived a formalism that yields a formal power series of the Nicolai map, which provides an efficient method for constructing the Nicolai map in supersymmetric theories. [O. Lechtenfeld, Ph.D. thesis, Univ. Bonn (1984).]

More recently, Olaf Lechtenfeld and Maximilian Rupprecht formulated this approach in a universal ordered-exponential form, now commonly known as the coupling-flow-operator formalism.

[O. Lechtenfeld and M. Rupprecht, Phys. Rev. D 104 (2021) L021701.]

Nicolai map

Using this approach, Nicolai maps were constructed in various supersymmetric theories, including

- off-shell $\mathcal{N} = 1$ super Yang-Mills theories

  [O. Lechtenfeld and M. Rupprecht, Phys. Lett. B 819 (2021) 136413.]
- on-shell $\mathcal{N} = 1$ super Yang-Mills theory in the Landau gauge in $D = 3,4,6,10$

  [S. Ananth et al., JHEP 10 (2020) 199.]
- $\mathcal{N} = 4$ super Yang-Mills theory in $D = 4$

  [H. Nicolai and J. Plefka, Phys. Rev. D 101 (2020) 125013.]
- supersymmetric nonlinear sigma models with four-fermion interactions in $D = 4$

  [O. Lechtenfeld, JHEP 02 (2025) 065.].

Moreover, the inverse Nicolai map enabled the $\mathcal{O}(g^6)$ computation of the infinite straight-line Maldacena–Wilson loop in $\mathcal{N}$=4 SYM without explicit fermion or ghost loops.

[H. Malcha, Phys. Lett. B 833 (2022) 137377.]

Nicolai map

Attempts to construct the supergravity Nicolai map within the coupling-flow-operator formalism have continued. However, efforts to obtain the supergravity Nicolai map within this formalism have so far been obstructed by various issues arising from local supersymmetry.
[F. Arrighi, S. Khandelwal and O. Lechtenfeld, JHEP 02 (2026) 185.]

We therefore adopt a brute-force approach.

1. Construct a general ansatz $T_\kappa c$ of perturbation vielbein field c with undetermined coefficients.
2. Substitute the ansatz into the defining conditions of the Nicolai map,

$$S[c;\kappa] = S[T_\kappa c, 0,0;0] - iTrln\left(\frac{\delta T_\kappa c}{\delta c}\right)$$

where $S[c;\kappa]$ is bosonic effective action of supergravity, $S[T_\kappa c, 0,0;0]$ is kinetic term of vielbein.

3. Solve the resulting equations for the coefficients.

Set up

We now generalize the discussion to a class of gravitational theories coupled to massless fermionic fields, whose only coupling constant is κ and whose purely bosonic sector is fixed to the Einstein–Hilbert action.

We perturbatively expand the vielbein around a four-dimensional Minkowski background.

$$e^a{}_\mu = \eta^a{}_\mu + \kappa c^a{}_\mu$$

$$e_a{}^\mu = \eta^\mu{}_a - \kappa c^\mu{}_a + \kappa^2 c^\mu{}_c c^c{}_a \cdots$$

The Minkowski background allows Lorentz and spacetime indices to be identified.

The local Lorentz symmetry is fixed by the symmetric vielbein gauge condition, which enforces

$$c_{ab} = c_{ba}.$$

Set up

Also, we impose a harmonic gauge fixing term,

$$-\frac{1}{4}\eta_{\alpha\beta}\partial_\mu(2c^{\mu\alpha}-\eta^{\mu\alpha}c)\partial_\upsilon\left(2c^{\upsilon\beta}-\eta^{\upsilon\beta}c\right)$$

to eliminate the diffeomorphism gauge redundancy.

Combining this gauge fixing term with the original vielbein kinetic term,

$$-2\partial_\mu c^{\upsilon\gamma}\partial_\gamma c^\mu{}_\upsilon+\eta^{\gamma\mu}\partial_\mu c^\upsilon{}_a\partial_\gamma c^a{}_\upsilon+2\partial_\mu c\partial_\gamma c^{\mu\gamma}-\eta^{\gamma\mu}\partial_\mu c\partial_\gamma c$$

we obtain the effective gauge fixed vielbein kinetic term,

$$c\boxtimes c\coloneqq\int \mathrm{d}^4\mathrm{x}\left(-c_{ab}\Box c^{ab}+\frac{1}{2}c\Box c\right).$$

Set up

The bosonic effective action consists of two parts: the Einstein–Hilbert contribution inherited directly from the original action and the loop contribution generated by integrating out the fermionic and ghost fields.

For example, at order κ, the loop contribution is given by the sum of two independent structures:

$$N_1 i \int d^4x\, c_a^b(x)\, \partial^b \partial_a G(0) + N_2 i \int d^4x\, c(x)\, \delta(0)$$

where $G(x-y)$ is massless scalar propagator and N_i are theory-dependent coefficients.

For pure $\mathcal{N}=1$ supergravity,

$$N_1 = +4, \qquad N_2 = -1.$$

Set up

Since $S[c;\kappa]$ contains loop corrections, we make the $\hbar$ -dependence explicit by writing the path integral as

$$Z = \int \mathcal{D}\mathbf{\Phi}\, e^{\frac{i}{\hbar}S[\mathbf{\Phi}]}.$$

Accordingly, we expand the effective action $S[c;\kappa]$ in powers of $\hbar$ and κ:

$$S[c;\kappa] = \sum_{r=0}^{\infty}\sum_{\mathrm{n}=0}^{\infty} \hbar^r \kappa^n S^{(r,n)}[c]\,.$$

Likewise, we expand the ansatz $T_\kappa c$ in powers of $\hbar$ and κ:

$$T_\kappa c = \sum_{r=0}^{\infty}\sum_{\mathrm{n}=0}^{\infty} \hbar^r \kappa^n T^{(r,n)} c\,.$$

Substituting these expansions into the defining condition for the Nicolai map, we obtain …

$$S^{(0,0)} = c \boxtimes c,$$

$$S^{(0,1)} = 2c \boxtimes T^{(0,1)}c,$$

$$S^{(1,1)} = 2c \boxtimes T^{(1,1)}c - i\,\mathrm{Tr}\left(\frac{\delta T^{(0,1)}c}{\delta c}\right),$$

$$S^{(0,2)} = 2c \boxtimes T^{(0,2)}c + T^{(0,1)}c \boxtimes T^{(0,1)}c,$$

$$S^{(1,2)} = 2c \boxtimes T^{(1,2)}c + 2T^{(0,1)}c \boxtimes T^{(1,1)}c - i\,\mathrm{Tr}\left(\frac{\delta T^{(0,2)}c}{\delta c}\right) + \frac{1}{2}i\,\mathrm{Tr}\left(\left(\frac{\delta T^{(0,1)}c}{\delta c}\right)^2\right),$$

$$S^{(2,2)} = T^{(1,1)}c \boxtimes T^{(1,1)}c - i\,\mathrm{Tr}\left(\frac{\delta T^{(1,2)}c}{\delta c}\right).$$

(up to κ^2 order)

Ansatz

At each order in $\hbar$ and κ, we take the Nicolai map expansion ansatz for $T^{(r,n)}c$ to be

$$T^{(r,n)} = \sum_A M_A^{(r,n)} \cdot \mathcal{T}^A .$$

where A is a collective index, $M_A^{(r,n)}$ is a A-th undetermined coefficient of the r-th order in $\hbar$ and n-th order in κ, and $\mathcal{T}^A$ are defined to consist of

- derivatives ∂,
- integration measures $\int$,
- vielbein perturbation fields c,
- propagators $G(x-y)$ satisfying $\Box G(x-y) = \delta(x-y)$.

$$ex) \quad \int d^4y\, G(x-y)\, \partial^\mu \partial^\rho\, c^{\nu\sigma}(y)\, c_{\rho\sigma}(y)$$

$\mathcal{T}^A$ can be classified according to their basic structure such as

- the number of derivatives,
- the number of fields located at each coordinate,
- the coordinates connected by the propagator,

where basic structure determines all aspects of $\mathcal{T}^A$ except for ① position of ∂ and ② index of ∂, c

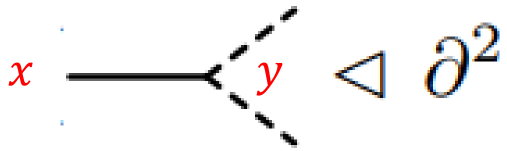
Ansatz

We employed diagrammatics to determine which basic structures should be included in the ansatz, using a notation inspired by Casarin, Lechtenfeld, and Rupprecht.

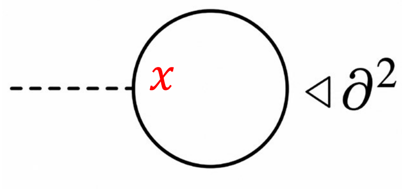
[L. Casarin, O. Lechtenfeld and M. Rupprecht, JHEP 12 (2023) 132.]

- Dotted lines represent vielbein perturbations, while solid lines represent propagators.
- $\triangleleft\, \partial^n$ to the right of each diagram indicates the number n of derivatives ∂.

Example, basic structure of $\int d^4y\, G(x-y)\, \partial^\mu \partial^\rho\, c^{\nu\sigma}(y)\, c_{\rho\sigma}(y)$ is represented



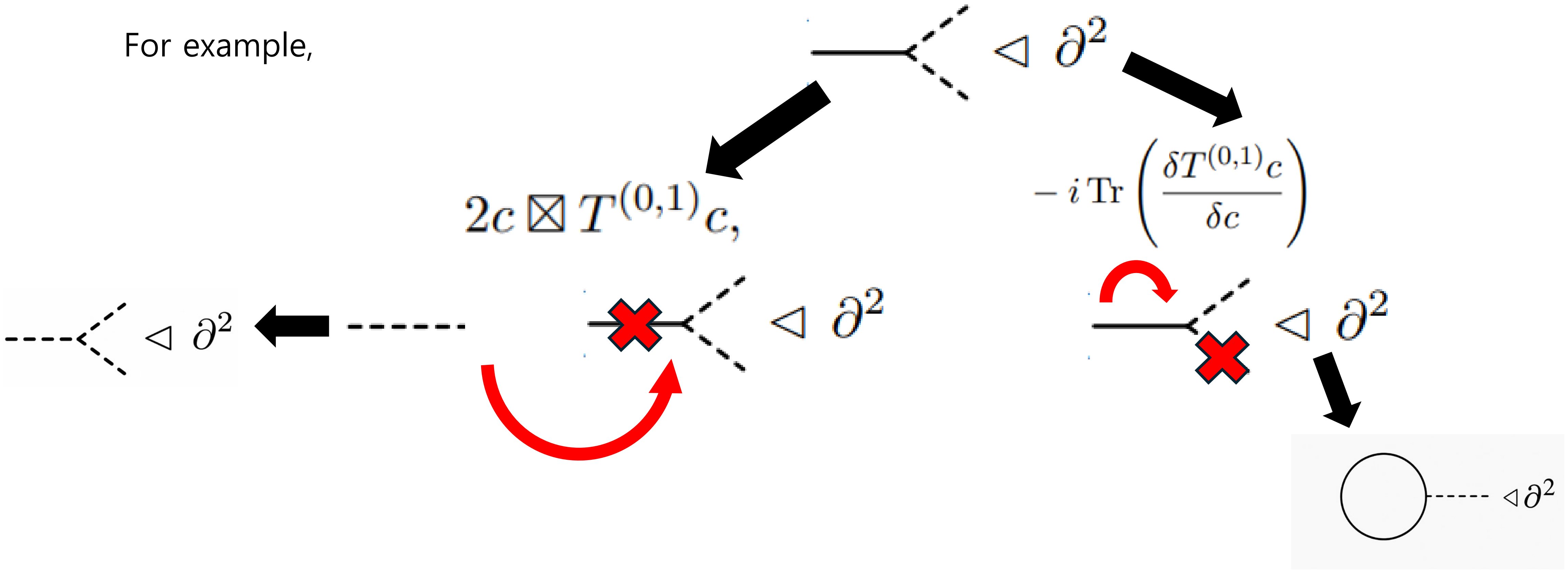
The basic structures appearing in the bosonic effective action can also be represented using the same diagrammatic notation. For example, $\int d^4x\, c_a^b(x)\, \partial^b \partial_a G(0)$ is represented diagrammatically as



Ansatz

The main advantage of the diagrammatic approach is that it makes transparent how a given basic structure in the ansatz generates a corresponding basic structure in the bosonic effective action when substituted into the defining condition.

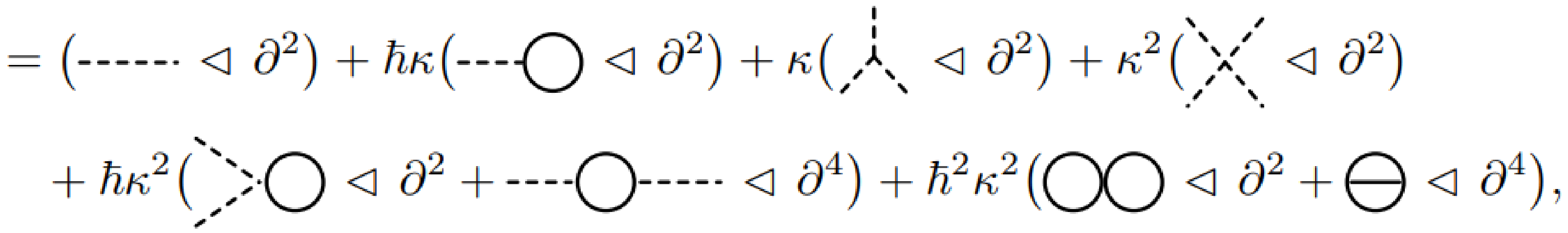
For example,



Ansatz

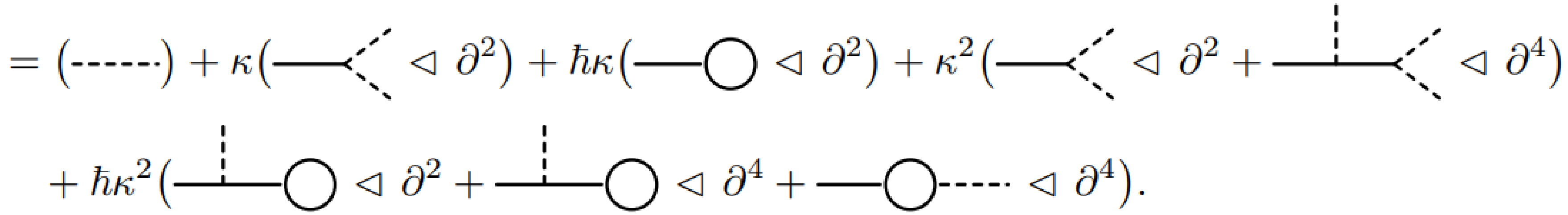
The effective action we consider is represented diagrammatically as follows:

$$S = S^{(0,0)} + \kappa S^{(0,1)} + \hbar\kappa S^{(1,1)} + \kappa^2 S^{(0,2)} + \hbar\kappa^2 S^{(1,2)} + \hbar^2\kappa^2 S^{(2,2)}$$



When substituted into the Nicolai map condition, the ansatz capable of reproducing the effective action above are represented by the following diagrams:

$$T_\kappa c = c + \kappa T^{(0,1)}c + \hbar\kappa T^{(1,1)}c + \kappa^2 T^{(0,2)}c + \hbar\kappa^2 T^{(1,2)}c$$



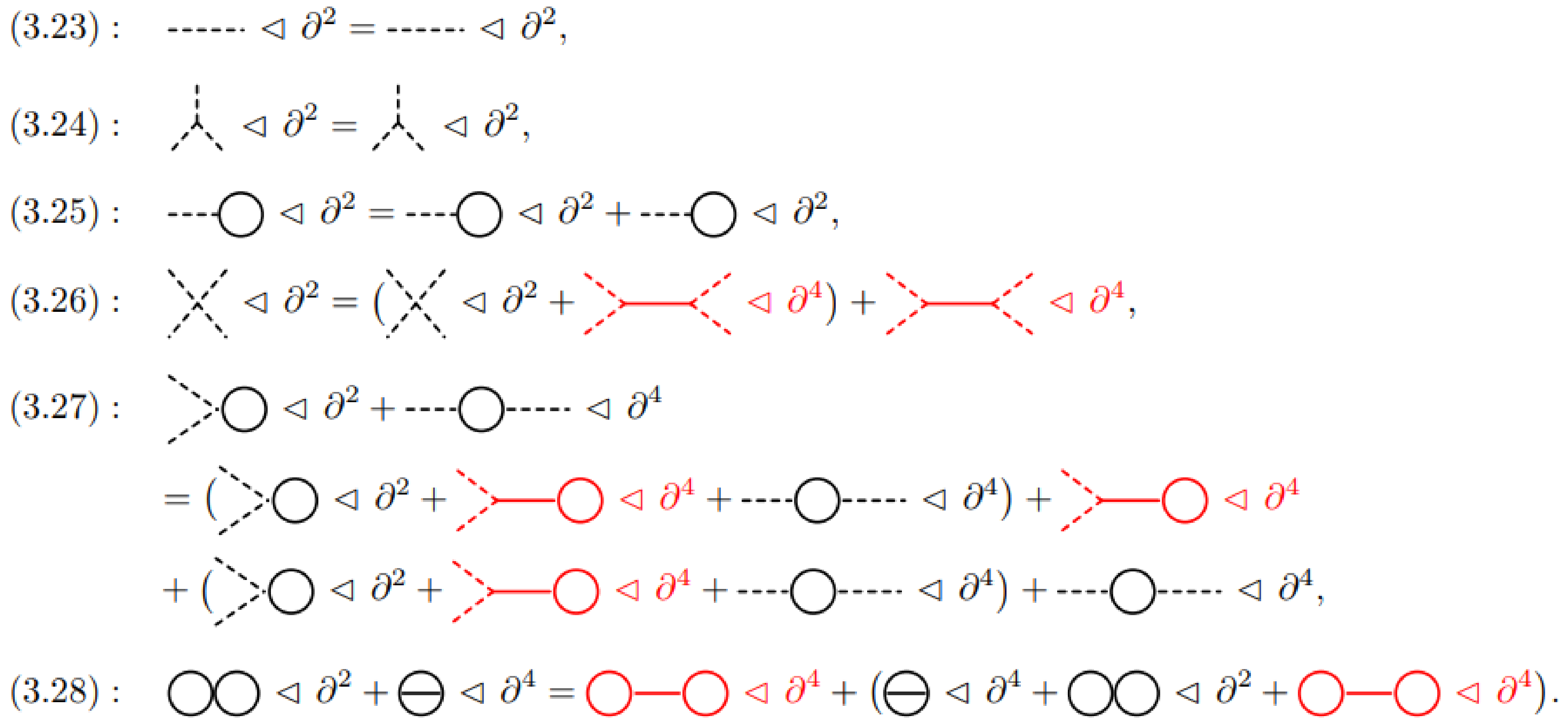


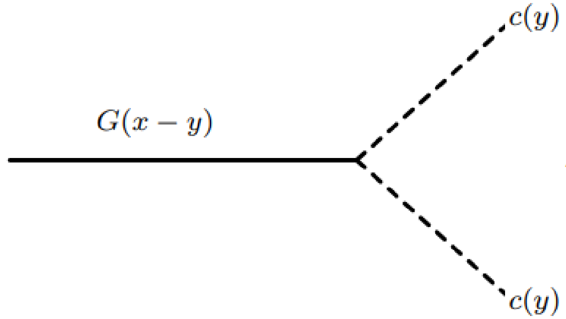
Figure 6: Diagrammatic representation of Nicolai map defining conditions in Eqs. (3.22)–(3.27). The left side of the equality comes from the effective action, while the right comes from the Nicolai-map expansion ansatz. The diagrams in red are the anomalous ones to be canceled out by one another.

Ansatz

For each basic structure $\mathcal{B}^{(r,n),X}$, we apply all possible index assignments I_j and derivative arrangements D_i to construct the explicit Nicolai-map expansion ansatz,

$$T^{(a,b)}c = \sum_X \sum_i \sum_j M_{i,j}^{(a,b),X}\,(D_i \circ I_j)\mathcal{B}^{(a,b),X} = \sum_X \mathcal{B}^{(a,b),X} \triangleleft \partial^{N_\partial}.$$

For example, basic structure of $T^{(0,1)}c$ is

$T^{(0,1)}c =$ 

$\triangleleft\, \partial^2.$

admits 9 derivative arrangements D_i and 12 index assignments I_j , yielding $9 \times 12 = 108$ terms.

After removing duplicates, 68 terms remain.

Result

First result : Substituting the ansatz into the Nicolai map conditions yields 752 equations involving 17,735 undetermined ansatz coefficients M_A and 209 theory dependent coefficients N_B parameterizing the loop contribution to the bosonic effective action.

Expectation : Since the number of unknowns greatly exceeds the number of equations, one might expect a straightforward procedure:

1. Compute the loop contribution to the bosonic effective action for the selected theory.
2. Substitute the values of the theory-dependent coefficients N_B.
3. Solve the resulting equations for a set of ansatz coefficients M_A defining the Nicolai map.

Result

Second result : However, this procedure does not apply to every theory. Solving the full system of 752 equations for the 17,735 coefficients M_A and 209 coefficients N_B, we find that, unlike the other variables, two of the bosonic effective-action coefficients are uniquely fixed:

$$N_1 = +4, \qquad N_2 = -1.$$

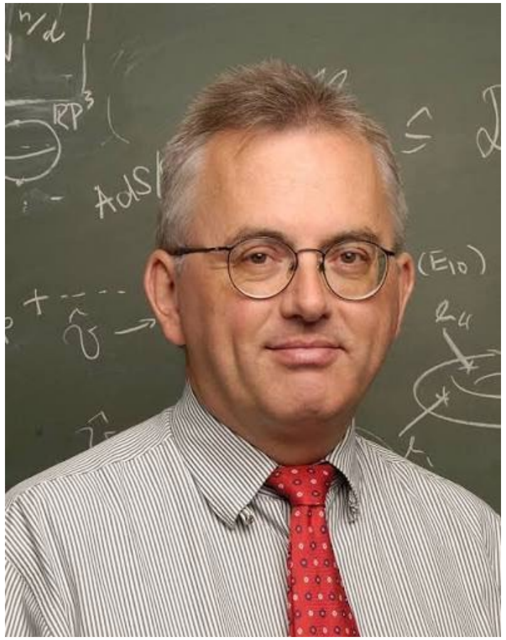
These are the coefficients of the loop contribution to the bosonic effective action at order κ.
Conclusion : Therefore, a theory admits a Nicolai map only if its order-κ loop contribution takes the required form

$$4i\,\kappa \int d^4x\, c_a^b(x)\,\partial^b\partial_a G(0) - i\,\kappa \int d^4x\, c(x)\,\delta(0).$$

Remarkably, these values coincide precisely with those of pure $\mathcal{N} = 1$ supergravity.

Summary

- We developed a perturbative diagrammatic method for constructing the Nicolai map.
- The Nicolai-map conditions reduce to a system of 752 equations involving 17,735 ansatz coefficients.
- For the Nicolai-map conditions to admit a solution, the order-κ loop coefficients must match those of pure $\mathcal{N}=1$ supergravity.



Hermann Nicolai (Max Planck Institute for Grav. Phys., Germany)

"Dear Hun,
Just to let you know that I am *very* impressed by your recent work
on the O(₩kappa^2) map! You have accomplished something that for many ye
ars (really decades!) I thought was absolutely impossible.....You should now def
initely advertise these results, as 99,9% percent of the supergravity community
seems to think this just cannot work."
"With your results this idea can now be put on much
firmer ground, as it also shows that quantization of gravity cannot work
without supersymmetry (independently of the question of UV divergences!)."

- June 28th, 2026

Olaf Lechtenfeld (Leibniz University Hannover, Germany)

"Dear Hun,
it is great to see that you have gotten back to the Nicolai map! I remember w
ell our attempts in 2022 on the case of supergravity and on the large-N expan
sion. Unfortunately, at the time nothing came out of it. So your recent paper
2605.29990 came as a pleasant surprise...Let me congratulate you on the heroi
c effort to construct the Nicolai map to second order. I can hardly imagine the
diligence that went into this. "

- June 5th, 2026

Open question

What other theories satisfy this condition?

We have indications that pure $\mathcal{N} \geq 2$ supergravity theories may satisfy the condition for a modified Nicolai map, in which all boson fields except the vielbein perturbation field are integrated out.

Indication

At order κ, the loop contribution to the effective action is generated by the kinetic terms of the physical fields.

$$Scalar: +\frac{1}{2} i\kappa \int d^4x\, c\delta(0) - i\kappa \int d^4x\, c^{\mu\nu} \partial_\mu \partial_\nu G(0)$$

$$Majorana: -2i\kappa \int d^4x\, c\delta(0) + 2i\kappa \int d^4x\, c^{\mu\nu} \partial_\mu \partial_\nu G(0)$$

$$U(1)\ Gauge: +\frac{1}{2} i\kappa \int d^4x\, c\delta(0) - 2i\kappa \int d^4x\, c^{\mu\nu} \partial_\mu \partial_\nu G(0)$$

$$Gravitino: -2i\kappa \int d^4x\, c\delta(0) + 2i\kappa \int d^4x\, c^{\mu\nu} \partial_\mu \partial_\nu G(0)$$

on-shell physical degrees of freedom

Open question

- At present, it remains unclear whether the condition is satisfied for the term $\int d^4x\, c(x)\, \delta(0)$.
- BRST invariance of the functional measure requires an additional factor involving an appropriate power of g. This factor generates an extra contribution proportional to $\int d^4\, x\, c\, \delta(0)$.
- For pure $\mathcal{N} = 1$ supergravity, the Nicolai map condition is satisfied when this contribution is included.
- Our preliminary calculation indicates that an appropriate power of g may also exist for pure $\mathcal{N} = 2$ supergravity.
- The remaining question is whether this mechanism extends to pure $\mathcal{N} > 2$ supergravity.