Fortuity and Relevant Deformation

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1. Introduction

Black holes as thermodynamic objects

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- **AdS/CFT duality:** [Maldacena '97; Gubser, Klebanov, Polyakov '98; Witten '98]
  Quantum gravity in AdS is equivalent to a CFT living on its boundary:

gravity in AdS		CFT ($N \times N$ matrix fields)
Newton's constant G_N	$\longleftrightarrow$	$1/N^2$
quantum state with energy E	$\longleftrightarrow$	operator with dimension Δ
black hole microstates	$\longleftrightarrow$	**which operators?** (this talk)

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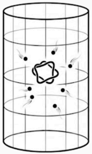

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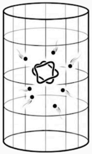

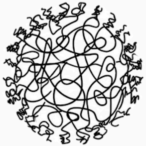

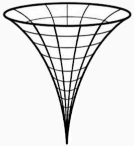

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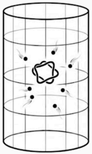

- Everything that defines a heavy object — energy, entropy, size — all scale with N, so it is inherently tied to a **specific finite-N theory**.

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- In AdS/CFT, the microstates are honest states of the **boundary CFT**. Identify and count them there.

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$$Q|\mathrm{BPS}\rangle = Q^\dagger|\mathrm{BPS}\rangle = 0\,, \qquad Z = \mathrm{Tr}\,(-1)^F\cdots$$

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- But the index is a **number**: it tells us how many microstates exist, not what they are.
- Identifying the microstates directly remains difficult, since the condition $Q^\dagger|\psi\rangle = 0$ depends on the full strongly coupled dynamics.

A refined tool: Q-cohomology

- A Hodge-theoretic argument removes $Q^\dagger$ from the problem (cf. harmonic forms):

$$\mathcal{H}_{\text{BPS}} \simeq \mathcal{H}_{Q\text{-closed}}/\mathcal{H}_{Q\text{-exact}} \equiv \boldsymbol{Q\text{-cohomology}}\,.$$

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Goal

Write down **explicitly** the black hole microstates contributing to S_{BH}, as CFT operators.

Caveat: quantum corrections

- Q depends on the coupling,
$$\{Q, Q^\dagger\} \sim H(g_{YM})\,,$$
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- The same caveat applies — but the index protects the counting, and corrections can only be studied on top of the one-loop answer. So we solve the one-loop problem.

2. Fortuity

Trace relations

- In CFT, a BPS state is a gauge-invariant word — heavier state, longer word:

$$\underbrace{\mathrm{Tr}(\phi^3)\,\mathrm{Tr}(\psi\,\partial\phi) + \mathrm{Tr}(F^2\cdots) + \cdots}_{\#\,\text{letters}\,\sim\,E}$$

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- Example: $N = 2$, traceless A

$$\mathrm{Tr}(A^4) = \tfrac{1}{2}\big(\mathrm{Tr}(A^2)\big)^2$$

A word of length 4 is not independent — it collapses into shorter words.

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- $\# \gg N$: relations everywhere.
- The word is shaped by finite-N effects.

Two kinds of BPS operators: monotone vs. fortuitous

- **Monotones** are Q-closed at any value of N and form an infinite family $\{\mathcal{O}_N\}$ that survives at $N = \infty$.

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Conjecture

Monotone BPS states are dual to **smooth horizonless geometries**, and fortuitous BPS states are responsible for **black hole microstates**. [Chang, Lin '24]

3. Fortuity in a 4d $\mathcal{N} = 1$ SCFT

Why an $\mathcal{N} = 1$ SCFT?

- Fortuitous BPS states have been studied in many theories — so far mostly in theories with a weakly coupled/Lagrangian description.

  [Choi, Choi, Kim, Lee, Lee '23; Chang, Chen, Sia, Yang '24; Gadde, Lee, Raj, Tomar '25; Choi, Lee '25; Kim, Lee, Lee, Oh '25; Belin et al. '25; Chang, Lin, Zhang '25; ...]

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  - How do black hole microstates behave **along an RG flow**?
- A minimal laboratory for both: turn on a mass term in $\mathcal{N}=4$ SYM that triggers an RG flow to an $\mathcal{N}=1$ SCFT in the IR: [Leigh, Strassler '95]

$$W_{\mathrm{UV}} = \mathrm{Tr}\big(\Phi_3[\Phi_1,\Phi_2]\big) + \frac{M}{2}\mathrm{Tr}\big(\Phi_3^2\big) \implies W_{\mathrm{IR}} = -\frac{1}{2M}\mathrm{Tr}\big([\Phi_1,\Phi_2]^2\big)$$

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  - Gravity dual of this RG flow is explicitly constructed. [Pilch, Warner '00]
- The IR SCFT is **intrinsically strongly coupled**, with no weakly coupled point. Nevertheless, the IR BPS data remains tractable via **Q-cohomology**. [Choi, SK '25]

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- **Result 1:** we constructed the complete **graviton** spectrum.
- **Result 2:** explicit **black hole microstate** operators — from the lightest one up to infinite towers.
- **Result 3:** along the RG flow, **gravitons are converted into black hole microstates**.

Result 1: the graviton spectrum

- From the Kaluza–Klein spectrum of the gravity dual [Bobev, Malek, Robinson, Samtleben, van Muiden '20], all single-trace gravitons (monotones) are generated by **three towers**:

$$u_n^{a_1\cdots a_{n+2}} \equiv \mathrm{Tr}\big(\phi^{(a_1}\cdots\phi^{a_{n+2})}\big), \qquad v_n^{a_1\cdots a_{n+2}} \equiv \mathrm{Tr}\big(\phi^{(a_1}\cdots\phi^{a_{n+1}}\psi^{a_{n+2})}\big),$$

$$w_n \equiv \mathrm{Tr}\Big(f(\phi\phi)^{n+1} + \tfrac{1}{4}\textstyle\sum_{k=0}^{n}(\phi\phi)^{n-k}\psi_a(\phi\phi)^k\psi^a\Big).$$

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- Multi-trace products then give the full graviton spectrum.

Result 2: explicit black hole operators $(SU(2))$

- The lightest black hole (fortuitous) operator: (at the threshold q^7 of the index)

$$\mathcal{O}_7^{abc} = \phi^{(a} \cdot \phi^b \phi^{c)} \cdot f - \frac{1}{2} \phi^{(a} \cdot \psi^b \times \psi^{c)}$$

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- We also found a graviton of **non-Coulomb type**, w^2 — it vanishes on diagonal configurations although w does not.
- And the first **non-Coulomb fortuitous** operator:

$$\begin{aligned}\mathcal{O}_{16} = {} & \tfrac{1}{3}\big(\psi_a\cdot\psi_b\times\psi_c\big)\big(\psi^a\cdot\psi^b\times\phi^c\big) + \big(f\cdot\phi_a\times\phi^a\big)\big(f\cdot\phi_b\big)\big(\psi^b\cdot\phi_c\times\phi^c\big) \\ & + \big(f\cdot\phi_a\big)\big(\psi^a\cdot\phi_b\times\phi^b\big)\big(\psi_c\cdot\psi^c\big) + 2\big(f\cdot\psi_a\big)\big(\phi^a\cdot\psi_b\big)\big(\psi^b\cdot\phi_c\times\phi^c\big).\end{aligned}$$

Result 3: UV origin — microstates along the RG flow

- Under the deformation, the supercharge splits:

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- **The RG flow converts gravitons into black hole microstates** — being a microstate is not RG-invariant.

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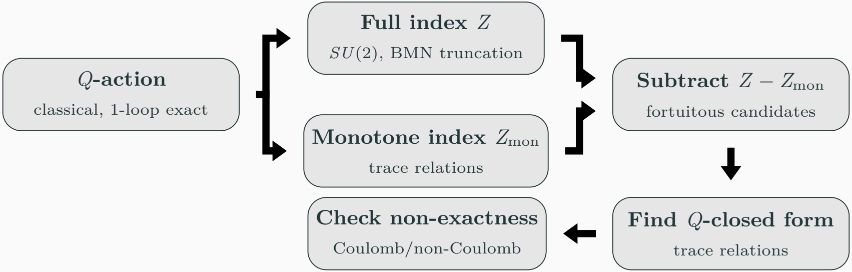
  - Conical singularity in the D-brane (Coulomb branch) moduli space? [Johnson, Lovis, Page '00]
  - Polarized D-brane (Myers effect)? [Polchinski, Strassler '00]

Thank you!

Backup

Backup: toolkit and the Q-action



- Q-action:

$$[Q, \phi^a] = 0, \qquad \{Q, \psi_a\} = -2[\phi_a, \phi^b \phi_b], \qquad [Q, f] = -[\phi^a, \psi_a] \qquad (a = 1, 2)$$

  - This is the **BMN sector**, a consistent truncation in the classical interacting theory.
  - This Q-action can be obtained by **cohomologically** integrating out $\Phi_3 = (\phi_3, \psi_3)$.
    The cubic term encodes the quartic W_{IR}: $\phi^a \phi_a = \epsilon_{ab} \phi^a \phi^b = -[\phi_1, \phi_2] \equiv (\phi\phi)$.

Backup: index computations

- Index of the IR SCFT (receives contributions from both monotone and fortuitous states):

$$\begin{aligned} Z_{SU(2)}(q,s) &= \mathrm{Tr}\,(-1)^F q^{4(j+R)} s^{2r_F} \\ &= 1 + q^2\chi_2 + q^4(\chi_4 - \chi_2) + q^6(\chi_6 - \chi_4 - \chi_2 + 1) + q^7\chi_3 + \cdots \end{aligned}$$

- Eigenvalue (Coulomb) method: set all fields diagonal, $A = \mathrm{diag}(a, -a)$. The Coulomb part of the monotone index is then analytic, and

$$Z_{SU(2)} - Z_{SU(2)}^{\text{Coulomb-mon}} = q^7\chi_3 + q^9(\chi_5 - \chi_3) + q^{11}(\chi_7 - \chi_5 - \chi_3) + q^{12}(\chi_4 + 1) + \cdots$$

- Including the towers built on $\mathcal{O}_{7,9,12,14}$, e.g. $\mathcal{O}_{7,u^k}^{a_1,\ldots,a_{2k+3}} \equiv u^{(a_1 a_2} \cdots u^{a_{2k-1} a_{2k}} \mathcal{O}_7^{a_{2k+1} a_{2k+2} a_{2k+3})}$, the residual is

$$Z - Z_{\text{Coulomb},\,\mathcal{O}_{7,9,12,14}} = q^{12} - q^{16} + q^{17}(\chi_5 + \chi_1) + q^{18} + \cdots$$

- q^{12}: the non-Coulomb graviton $w^2 = \left(\phi_1 \times \phi_2 \cdot f + \psi_1 \cdot \psi_2\right)^2$; $-q^{16}$: the non-Coulomb fortuitous $\mathcal{O}_{16}$ (next slide).

Backup: the non-Coulomb fortuitous operator $\mathcal{O}_{16}$

$$\mathcal{O}_{16} = \tfrac{1}{3}\big(\psi_a\cdot\psi_b\times\psi_c\big)\big(\psi^a\cdot\psi^b\times\phi^c\big) + \big(f\cdot\phi_a\times\phi^a\big)\big(f\cdot\phi_b\big)\big(\psi^b\cdot\phi_c\times\phi^c\big)$$
$$+\big(f\cdot\phi_a\big)\big(\psi^a\cdot\phi_b\times\phi^b\big)\big(\psi_c\cdot\psi^c\big) + 2\big(f\cdot\psi_a\big)\big(\phi^a\cdot\psi_b\big)\big(\psi^b\cdot\phi_c\times\phi^c\big).$$

- Not Q-exact: the flavor-singlet $\phi\,\psi^5$ term cannot arise from $Q\Lambda$ — there is no flavor singlet with letter content $\psi^4 f$.
- The same argument shows it is not a graviton.

Backup: higher Coulomb-type fortuitous operators

$$-q^9\chi_3: \quad \mathcal{O}_9^{abc} = \phi^{(a}\cdot\psi^b\phi^{c)}\cdot f - \tfrac{1}{6}\,\psi^{(a}\cdot\psi^b\times\psi^{c)}$$

$$q^{12}\chi_4: \quad \mathcal{O}_{12}^{abcd} = \phi^{(a}\cdot\phi^b\phi^c\cdot f\phi^{d)}\cdot f - \phi^{(a}\cdot f\phi^b\cdot\psi^c\times\psi^{d)}$$

$$-q^{14}\chi_4: \quad \mathcal{O}_{14}^{abcd} = f\cdot\phi^{(a}f\cdot\phi^b\psi^c\cdot\phi^{d)} - f\cdot\psi^{(a}\psi^b\cdot\psi^c\times\phi^{d)}$$

Single-trace monotone towers (from the KK spectrum), schematically:

tower	BPS operator	KK-level
u_n	$\mathrm{Tr}\big(\phi^{(a_1}\cdots\phi^{a_{n+2})}\big)$	$n \geq 0$
v_n	$\mathrm{Tr}\big(\phi^{(a_1}\cdots\phi^{a_{n+1}}\psi^{a_{n+2})}\big)$	$n \geq 0$
w_n	$\mathrm{Tr}\big(f(\phi\phi)^{n+1} + \frac{1}{4}\sum_{k=0}^{n}(\phi\phi)^{n-k}\psi_a(\phi\phi)^k\psi^a\big)$	$n \geq 0$

The full KK table (multiplets $B_1\bar{L}$, $A_1\bar{L}$, $A_2\bar{L}$, $\ldots$): next slide.

Backup: full single-trace graviton (KK) spectrum

Multiplet	BPS operator	KK-level
$B_1\bar{L}[\frac{9+3n}{4},0,\frac{1}{2},-\frac{n+3}{2}]\otimes[\frac{n+1}{2}]^{(-\frac{n+3}{2})}$	$\mathrm{Tr}(\lambda_\alpha\phi^{(a_1}\cdots\phi^{a_{n+1})})$	$n\geq 0$
$B_1\bar{L}[\frac{6+3n}{4},0,0,-\frac{n+2}{2}]\otimes[\frac{n+2}{2}]^{(-\frac{n+2}{2})}$	$\mathrm{Tr}(\phi^{(a_1}\cdots\phi^{a_{n+2})})$	$n\geq 0$
$B_1\bar{L}[\frac{12+3n}{4},0,0,-\frac{n+4}{2}]\otimes[\frac{n}{2}]^{(-\frac{n+4}{2})}$	$\mathrm{Tr}(\lambda^\alpha\lambda_\alpha\phi^{(a_1}\cdots\phi^{a_n)})$	$n\geq 0$
$A_1\bar{L}[\frac{6+3n}{2},\frac{1}{2},\frac{1}{2},-n]\otimes[0]^{(0)}$ $A_1\bar{A}_1[3,\frac{1}{2},\frac{1}{2},0]\otimes[0]^{(0)}$	$\mathrm{Tr}([(\phi\phi)^n,\lambda_\alpha]f-[(\phi\phi)^n,\psi^a]D_\alpha\phi_a-[(\phi\phi)^n,\phi^a]D_\alpha\psi_a)$ $\mathrm{Tr}(f\lambda_\alpha-\frac{3}{4}\psi_a D_\alpha\phi^a+\frac{1}{4}\phi^a D_\alpha\psi_a)$	$n\geq 1$ $n=0$
$A_1\bar{L}[\frac{6+3n}{2},\frac{1}{2},0,-n]\otimes[0]^{(1)}$	$\mathrm{Tr}\big(\lambda^\alpha\lambda_\alpha f-\psi^a\lambda^\alpha D_\alpha\phi_a+\frac{1}{2}\lambda^\alpha\psi^a D_\alpha\phi_a+\frac{1}{2}\phi^a\lambda^\alpha D_\alpha\psi_a$ $-\phi^a\phi^b D^\alpha\phi_a D_\alpha\phi_b+2\phi^b(D^\alpha\phi^a)\phi_a D_\alpha\phi_b\big)\quad(n=1)$	$n\geq 1$
$A_1\bar{L}[\frac{6+3n}{2},\frac{1}{2},0,-n]\otimes[0]^{(-1)}$	$\mathrm{Tr}(f(\phi\phi)^{n+1}+\frac{1}{4}\sum_{k=0}^{n}(\phi\phi)^{n-k}\psi^a(\phi\phi)^k\psi_a)$	$n\geq 0$
$A_2\bar{L}[\frac{11+3n}{4},0,\frac{1}{2},-\frac{n+1}{2}]\otimes[\frac{n+1}{2}]^{(-\frac{n+1}{2})}$	$\mathrm{Tr}(\lambda_\alpha\phi^{(a_1}\cdots\phi^{a_n}\psi^{a_{n+1})}-2\phi^{(a_1}\cdots\phi^{a_n}(\phi\phi)D_\alpha\phi^{a_{n+1})})$	$n\geq 0$
$A_2\bar{L}[\frac{8+3n}{4},0,0,-\frac{n}{2}]\otimes[\frac{n+2}{2}]^{(-\frac{n}{2})}$ $A_2\bar{A}_2[2,0,0,0]\otimes[1]^{(0)}$	$\mathrm{Tr}(\phi^{(a_1}\cdots\phi^{a_{n+1}}\psi^{a_{n+2})})$ $\mathrm{Tr}(\phi^{(a}\psi^{b)})$	$n\geq 1$ $n=0$
$A_2\bar{L}[\frac{14+3n}{4},0,0,-\frac{n+2}{2}]\otimes[\frac{n}{2}]^{(-\frac{n+2}{2})}$	$\mathrm{Tr}(\lambda^\alpha\lambda_\alpha\phi^{(a_1}\cdots\phi^{a_{n-1}}\psi^{a_n)}-2\{\lambda^\alpha,\phi^{(a_1}\cdots\phi^{a_{n-1}}(\phi\phi)\}D_\alpha\phi^{a_n)})$	$n\geq 1$

Backup: from the UV $\mathcal{Q}$-action to the IR $\mathcal{Q}$-action

Start with the classical $\mathcal{Q}$-action of mass-deformed $\mathcal{N}=4$ SYM ($m=1,2,3$):

$$[\mathcal{Q},\phi_m]=0, \qquad \{\mathcal{Q},\psi_m\}=\tfrac{1}{2}\epsilon_{mnp}[\phi_n,\phi_p]+M\delta_{m,3}\phi_3, \qquad [\mathcal{Q},f]=[\phi_m,\psi_m]\,.$$

Solve the second equation for the massive field:

$$\phi_3=\tfrac{1}{M}\big(\{\mathcal{Q},\psi_3\}-[\phi_1,\phi_2]\big)\,,$$

and plug into $\{\mathcal{Q},\psi_a\}=-[\phi_a,\phi_3]$, $[\mathcal{Q},f]=[\phi_a,\psi_a]+[\phi_3,\psi_3]$ $(a=1,2)$:

$$\Big\{\mathcal{Q},\Big(\psi_a+\tfrac{1}{M}[\phi_a,\psi_3]\Big)\Big\}=-\tfrac{1}{M}\big[\phi_a,[\phi_1,\phi_2]\big], \qquad \Big[\mathcal{Q},\Big(f-\tfrac{1}{M}\psi_3\psi_3\Big)\Big]=\Big[\phi_a,\Big(\psi_a+\tfrac{1}{M}[\phi_a,\psi_3]\Big)\Big].$$

Prescribe the IR $\mathcal{Q}$-action via the field redefinition

$$\phi_a^{\mathrm{IR}}\equiv M^{-1/4}\phi_a^{\mathrm{UV}}, \qquad \psi_a^{\mathrm{IR}}\equiv\sqrt{2}\,M^{1/4}\Big(\psi_a^{\mathrm{UV}}+\tfrac{1}{M}[\phi_a^{\mathrm{UV}},\psi_3]\Big), \qquad f^{\mathrm{IR}}\equiv-\sqrt{2}\Big(f^{\mathrm{UV}}-\tfrac{1}{M}\psi_3\psi_3\Big).$$

"Integrating out" ϕ_3 and ψ_3 gives (subscript IR omitted)

$$[\mathcal{Q},\phi^a]=0, \qquad \{\mathcal{Q},\psi_a\}=-2[\phi_a,\phi_b\phi^b], \qquad [\mathcal{Q},f]=-[\phi^a,\psi_a]\,.$$

$SU(2)_F$ indices raised/lowered with ϵ^{ab}, ϵ_{ab}; $\epsilon^{ab}\phi_a\phi_b=\phi_a\phi^a=[\phi_1,\phi_2]\equiv(\phi\phi)$.

Backup: tracing UV origins (spectral sequence)

Inverting the field redefinition and expanding an IR operator in M:

$$\mathcal{O}_{\mathrm{IR}} \sim \mathcal{O}^{(0)} + M\,\mathcal{O}^{(1)} + \cdots + M^m \mathcal{O}^{(m)}, \qquad Q = Q_0 + M\,Q_1, \qquad Q_1 = \phi_3 \frac{\partial}{\partial \psi_3}\,.$$

Q-closure gives order-by-order conditions:

$$Q_0\,\mathcal{O}^{(0)} = 0, \qquad M\big(Q_0\,\mathcal{O}^{(1)} + Q_1\,\mathcal{O}^{(0)}\big) = 0, \qquad \cdots$$

- Naively taking the leading Q_0-closed operator does not directly give the UV cohomology from which the IR cohomology descended.
- The decomposition is ambiguous up to Q-exact shifts $\mathcal{O}_{\mathrm{IR}} \to \mathcal{O}_{\mathrm{IR}} + Q\Lambda$ with $Q\Lambda = Q_0\Lambda + MQ_1\Lambda$:

$$\mathcal{O}^{(0)} \to \mathcal{O}^{(0)} + Q_0\Lambda^{(0)}, \qquad \mathcal{O}^{(1)} \to \mathcal{O}^{(1)} + Q_0\Lambda^{(1)} + Q_1\Lambda^{(0)}, \ \cdots$$

Backup: IR monotones and Coulomb fortuitous from UV gravitons

UV single-trace gravitons of $\mathcal{N}=4$ SYM:

$$u^{m_1\cdots m_{n+2}} = \operatorname{Tr}\phi^{(m_1}\cdots\phi^{m_{n+2})}, \qquad v^{m_1\cdots m_k}{}_{m_{k+1}} = \operatorname{Tr}\phi^{(m_1}\cdots\phi^{m_k)}\psi_{m_{k+1}} - (\text{trace}),$$

$$w_m = \operatorname{Tr}\big(\phi_m f + \tfrac{1}{2}\epsilon_{mnp}\psi_n\psi_p\big).$$

IR monotones descend from a subset:

$$u_{\mathrm{IR}}^{a_1\cdots a_{n+2}} = M^{-\frac{n+2}{4}}\, u^{a_1\cdots a_{n+2}}, \qquad v_{\mathrm{IR}}^{a_1\cdots a_{n+2}} = \sqrt{2}\, M^{-\frac{n}{4}}\, \epsilon^{b3(a_1}\, v^{a_2\cdots a_{n+2})}{}_b\,,$$

$$w_{\mathrm{IR},0} = -\frac{4}{3M^{3/2}}\, Q\operatorname{Tr}(\psi_3)^3 + \frac{\sqrt{2}}{M^{1/2}}\, Q\operatorname{Tr}(f\psi_3) - \sqrt{2}\, M^{1/2}\operatorname{Tr}\big(\phi_3 f + \psi_1\psi_2\big)$$

using $(\phi\phi) = M\phi_3 - Q\psi_3$; the first two terms are Q-exact. Coulomb-type IR fortuitous operators map to composites of UV gravitons, e.g.

$$\mathcal{O}_7^{abc} = M^{-3/4}\Big(\phi^{(a}\cdot\phi^{b}\big(\phi^{c)}\cdot f + \psi^{c)}\cdot\psi_3\big) + \phi^{(a}\cdot\psi_3\,\phi^{b}\cdot\psi^{c)}\Big)$$

$$+\, O(M^{1/4}) \;\sim\; u^{(ab}w^{c)} + v^{(a}{}_3 v^{bc)}\,.$$

Backup: $\mathcal{O}_{16}$ descends from the lightest UV fortuitous operator

One can argue $\mathcal{O}_{16}$ corresponds to the **lightest fortuitous operator** of $SU(2)$ $\mathcal{N}=4$ SYM $(q=t^{3/2})$:

$$\mathcal{O}_{16} = M^{-2}\mathcal{O}_{t28} + M^{-1}\mathcal{O}_{t26} + \mathcal{O}_{t24} + M\,\mathcal{O}_{t22}\,, \qquad Q_0\,\mathcal{O}_{t28} = 0, \quad Q_1\mathcal{O}_{t28} + Q_0\mathcal{O}_{t26} = 0\,.$$

1. No fortuitous BPS state in the t^{28} sector [Chang, Lin '22; Choi, Kim, Lee, Lee, Park '23] $\Rightarrow$ $\mathcal{O}_{t28} = Q_0\mathcal{O}'_{t28}$, so $\mathcal{O}_{16} - M^{-2}Q\mathcal{O}'_{t28} = M^{-1}(\mathcal{O}_{t26} - Q_1\mathcal{O}'_{t28}) + \mathcal{O}_{t24} + M\mathcal{O}_{t22}$.
2. No fortuitous state in the t^{26} sector either; a graviton piece (even length L) could only come from $Q_1\mathcal{O}'_{t28}$ since $\mathcal{O}_{t26}$ is non-Coulomb — removable by a suitable Λ: $\mathcal{O}_{t26} - Q_1\mathcal{O}'_{t28} = Q_0\mathcal{O}'_{t26}$.
3. Then $\mathcal{O}_{t24} - Q_1\mathcal{O}'_{t26}$ is Q_0-closed; not Q_0-exact (else $\mathcal{O}_{16}$ would be Q-exact), not a graviton (odd length). In the t^{24} sector there exists exactly one fortuitous Q_0-cohomology in the UV!

	$\mathcal{O}_{t28}$	$\mathcal{O}_{t26}$	$\mathcal{O}_{t24}$	$\mathcal{O}_{t22}$	$\mathcal{O}_{16}$
r^{UV}	$-\frac{13}{3}$	$-\frac{11}{3}$	-3	$-\frac{7}{3}$	-3
j	$\frac{5}{2}$	$\frac{5}{2}$	$\frac{5}{2}$	$\frac{5}{2}$	$\frac{5}{2}$
L^{UV}	9	8	7	6	