Scalarized Extremal black holes (scalar hair charge)

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Contents

Scalar hair (Q_s in $\phi_{\text{asym}} = \frac{Q_s}{r} + \cdots$)

- **Primary scalar charge→ independent one**

- **Secondary scalar charge→ dependent on known hairs:**
  BH mass (*M*), charge (*q*) , spin (*a*), and action variables

- **Constant scalar charge (independent of *r*)**
  → Extremal BHs with Constant scalar hair

1. Introduction

BBMB BH→ One known scalarized extremal BH!!

- **Conformally coupled scalar**
  → Evasion of no-hair theorem
  → BBMB black hole with scalar hair [**Hair is secondary and diverges at *r=m***]

- Action: $\int d^4x\sqrt{-g}\left[R-\beta\left(\phi^2 R+6(\partial\phi)^2\right)\right]$ with action variable β
- Conformal transformation for (•••):

$$\hat{\phi}=\frac{\phi}{\Omega}, \qquad \hat{g}_{\mu\nu}=\Omega^2 g_{\mu\nu}$$

- Eqs.: $G_{\mu\nu}=T^{\phi}_{\mu\nu}$

$$\text{with } T^{\phi}_{\mu\nu}=\beta\left[\phi^2 G_{\mu\nu}+g_{\mu\nu}\nabla^2(\phi^2)-\nabla_\mu\nabla_\nu(\phi^2)+6\nabla_\mu\phi\nabla_\nu\phi-3(\nabla\phi)^2 g_{\mu\nu}\right],$$

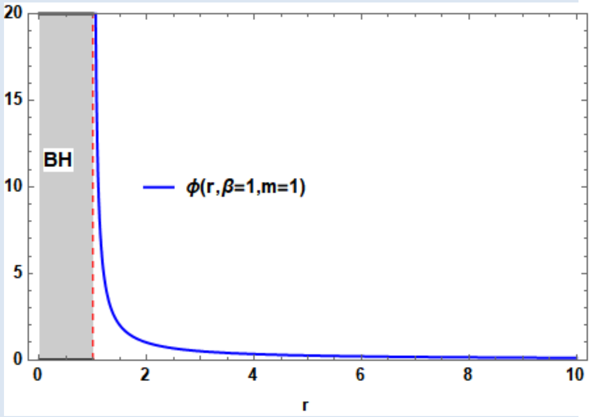
$$\nabla^2\phi-\frac{1}{6}R\phi=0$$

N. M. Bocharova, K. A. Bronnikov and V. N. Melnikov, Vestn. Mosk. Univ. Ser. III Fiz. Astron. , 706, 6 (1970) .
J. D. Bekenstein, Phys. Rev., D51, R6608 (1995).
Thanasis Karakasis, Eleftherios Papantonopoulos, Zi-Yu Tang, Bin Wang, arXiv:2103.14141

BBMB BH→ only scalarized extremal BH!!

- Residual conformal symmtery in (•••) $\Rightarrow \left[R=0 \text{ and } \Box\phi=0\right]$
- Assuming a spherically symmetric spacetime with metric function $f(r)$,

$$\begin{cases} R=0 \Rightarrow r^2 f''(r)+4rf'(r)+2f(r)-2=0 \Rightarrow f(r)=1+\dfrac{a}{r}+\dfrac{b}{r^2} \\ \text{Einstein Eq.: } \phi''\phi-2\phi'^2=0 \Rightarrow \phi=\dfrac{1}{cr+d} \end{cases}$$



- $ds^2_{\text{BBMB}}=-\left(1-\dfrac{m}{r}\right)^2 dt^2+\dfrac{dr^2}{\left(1-\dfrac{m}{r}\right)^2}+r^2 d\Omega_2^2$,

$\phi(r)=\sqrt{\dfrac{1}{\beta}}\dfrac{m}{r-m} \Rightarrow$ Scalar hair is $\boxed{\text{secondary}}$ and $\boxed{\text{it diverges at } r=m}$

- $T_\mu^{\phi\ \nu}=\dfrac{2m^2}{r^4}\text{diag}\,[-1,-1,1,1] \Rightarrow$ Conformal scalar provides mass energy-momentum T

beyond Horndeski (bH) theory with primary scalar hair

- Shift ($\phi \to \phi + \text{const.}$) and $Z_2(\phi \to -\phi)$-symmetric second order scalar-tensor theory:

$\boxed{\text{beyond Horndeski gravity}}$

$$\mathcal{S}_{bH} = \frac{1}{16\pi}\int d^4x\sqrt{-g}\left(\mathcal{L}_2^H + \mathcal{L}_4^H + \mathcal{L}^{bH}\right), \quad X = -\partial^\mu\phi\partial_\mu\phi/2$$

$$\text{with} \begin{cases} \mathcal{L}_2^H = G_2(X), \\ \mathcal{L}_4^H = G_4(X)R + G_{4X}\left[(\Box\phi)^2 - (\nabla_\mu\nabla_\nu\phi)(\nabla^\mu\nabla^\nu\phi)\right], \\ \mathcal{L}^{bH} = F_4(X)\epsilon^{\mu\nu\rho\sigma}\epsilon^{\alpha\beta\gamma}{}_\sigma\partial_\mu\phi\partial_\alpha\phi(\nabla_\nu\nabla_\beta\phi)(\nabla_\rho\nabla_\gamma\phi) \end{cases}$$

- $G_2(X) = -\frac{2\eta}{\lambda^2}X^{5/2}, \quad G_4(X) = 1 - \eta X^{5/2}, \quad F_4(X) = \eta\sqrt{X}$ with action variables η and λ

Miguel Zumalacárregui and Juan García-Bellido, arXiv:1308.4685

- Analytic bH black hole solution with primary scalar charge q

with scalar hair $\phi = qt + \psi(r)$ and its kinetic term $X = \frac{q^2}{2}\frac{1}{1+(r/\lambda)^2}$:

$$ds^2 = -h(r)dt^2 + \frac{dr^2}{h(r)} + r^2 d\Omega^2, \quad \boxed{h(r) = 1 - \frac{2M}{r} + \frac{\sqrt{2}\lambda\eta}{3}\frac{q^5}{r}\left(1 - \frac{r^3}{(r^2+\lambda^2)^{3/2}}\right)}$$

- Shift symmetry $\to$ Noether current $\to$ Shift-conserved charge Q_{sh}:

$$\mathbf{J} = J_\mu dx^\mu = \frac{1}{\sqrt{-g}}\frac{\delta \mathcal{S}}{\delta(\partial_\mu \phi)}, \quad J^\mu = -\frac{4X}{q}G_{2X}\delta_0^\mu \Rightarrow \boxed{Q_{sh} \propto \int \star\mathbf{J} = \frac{20\pi}{3\sqrt{2}}\lambda\eta q^4}$$

- Fixing $q^5 = \frac{3\sqrt{2}M}{\eta\lambda}$, one finds Bardeen (regular) BH with secondary scalar charge $q(M) = \left(\frac{3\sqrt{2}M}{\eta\lambda}\right)^{\frac{1}{5}}$:

$$ds^2 = -\bar{h}(r)dt^2 + \frac{dr^2}{\bar{h}(r)} + r^2\Omega^2, \quad \boxed{\bar{h}(r) = 1 - \frac{2Mr^2}{(r^2+\lambda^2)^{3/2}}}, \quad \bar{Q}_{sh}(M) = 20\pi\frac{M}{q} = 20\pi M^{4/5}\left(\frac{\eta\lambda}{3\sqrt{2}}\right)^{1/5}$$

Athanasios Bakopoulos, Christos Charmousis, Panagiota Kanti, Nicolas Lecoeur, Theodoros Nakas, arXiv:2310.11919
YSMyung, Theodoros Nakas, arXiv:2505.02368

Scalarized extremal bH BH with primary scalar charge?

- Extremal point of bH BH chooses with $\xi = \eta q^5$ and i-th outer horizon radius $r_{+i}(M,\xi,\lambda)$

$$\Rightarrow \xi = \frac{(r_{+i}^2(M,\xi,\lambda)+\lambda^2)^{5/2}}{\sqrt{2}\lambda^3 r_{+i}^2(M,\xi,\lambda)} \Rightarrow q: \text{dependent} \Rightarrow \text{secondary scalar charge}$$

Five examples for scalarized BHs with secondary scalar charge in the modified gravity theory without Ostrogradski instability and ghosts

1. **Einstein-dilaton-Gauss-Bonnet theory with dilatonic coupling function (inspired by string theories:** $\alpha \rightarrow$ GB coupling constant, $\kappa \rightarrow$ dilaton coupling constant **)**

$$f(\phi) = \alpha e^{\kappa\phi} \rightarrow \boxed{\text{class I: } f(0) \neq 0,\ f'(0) \neq 0,\ \text{secondary}}$$

$$\rightarrow \text{Dilatonic scalarization, but no analytic solution}$$

2. **Einstein-Gauss-Bonnet-scalar (EGBS) theory with nonminimal coupling function (a special case of Horndeski theory:** $\lambda \rightarrow$ GB coupling constant, $f(\phi) \rightarrow \mathbb{Z}_2$-symmetric coupling function**)**

$$f(\phi) = \frac{\phi^2}{2\lambda^2},\ \ \frac{1-e^{-6\phi^2}}{12\lambda^2} \rightarrow \boxed{\text{Class IIA: } f(0) = 0,\ f'(0) = 0,\ f''(0) \neq 0,\ \text{secondary}}$$

P. Kanti, N.E. Mavromatos, J. Rizos, K. Tamvakis, E. Winstanley, hep-th/9511071
P. Kanti, N.E. Mavromatos, J. Rizos, K. Tamvakis and E. Winstanley, hep-th/9703192
J. L. Blazquez-Salcedo, F. S. Khoo, and Jutta Kunz, arXiv: 1706.03262

3. **Einstein-Maxwell-dilaton theory with dilatonic coupling function**

$$f(\phi) = e^{2\alpha\phi} \left\{ \alpha=1 (\text{low energy strings}),\ \alpha=\sqrt{3} (\text{KK theory}) \right\}$$

$$\rightarrow \boxed{\text{class I: } f(0) \neq 0,\ f'(0) \neq 0,\ \text{secondary}}$$

$$\rightarrow \text{Dilatonic scalarization} \rightarrow \text{Analytic solution } (\alpha=1 \text{GMGHS})$$

- $ds^2 = -a(r)dt^2 + \dfrac{dr^2}{a(r)} + b(r)d\Omega_2^2$ with $a(r) = \left(1-\dfrac{r_+}{r}\right)\left(1-\dfrac{r_-}{r}\right)^{\frac{1-\alpha^2}{1+\alpha^2}}$ and $b(r) = r^2\left(1-\dfrac{r_-}{r}\right)^{\frac{2\alpha^2}{1+\alpha^2}}$,

$$A_t(r) = -\frac{\sqrt{r_+ r_-}}{\sqrt{1+\alpha^2}\ r}, \quad \boxed{\phi(r,M,Q,\alpha) = \frac{\alpha}{1+\alpha^2}\ln\left(1-\frac{r_-}{r}\right)}, \quad r_\pm = \frac{1+\alpha^2}{1\pm\alpha^2}\left(M \pm \sqrt{M^2-(1-\alpha^2)Q^2}\right)$$

$\Rightarrow r_+$: event horizon and r_- : spacelike singularity

G. W. Gibbons and K. I. Maeda, Nucl. Phys. B 298 (1988) 741
D. Garfinkle, G. T. Horowitz and A. Strominger, Phys. Rev. D 43 (1991) 3140 Erratum: [Phys. Rev. D 45 (1992) 3888]

4. **Einstein-Maxwell-scalar (EMS) theory with nonminimal coupling function (a special case of EMS theory)**

$$f(\phi)=1+\alpha\phi^2,\ e^{\alpha\phi^2}\ \rightarrow \boxed{\text{Class IIA}: f(0)\neq 0, f'(0)=0, f''(0)\neq 0,\ \text{secondary}}$$

5. **Nonlinear scalarization for EGBS & EMS theories**

→ $\boxed{\text{Class IIB}: f'(0)=0,\ f''(0)=0,\ \text{secondary}}$

→ **No tachyonic instability →Nonlinear instability →Scalarized phase**

→ **A few branches of scalarized black holes for fixed** α

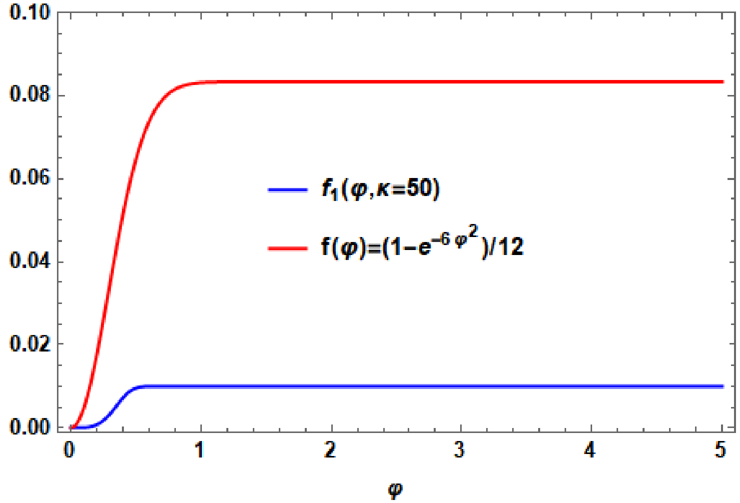
$$\boxed{V_{eff}=-f(\phi)R^2_{GB}<0\,/\,f(\phi)F^2<0\rightarrow \text{Evasion of no-hair theorem}}$$

D. D. Doneva and S. S. Yazadjiev, arXiv: 2107.01738
A. M. Pombo and D. D. Doneva, arXiv: 2310.08638
J. L. Blazquez-Salcedo, D. D. Doneva, J. Kunz and S. S. Yazadjiev, arXiv:2203.00709
D. D. Doneva, L. G. Collodel and S. S. Yazadjiev, arXiv: 2208.02077
M. Y. Lai, D. C. Zou, R. H. Yue and Y. S. Myung, arXiv: 2304.08012

2. Nonlinear scalarization in the EGBS theory

- $S_{\rm EGBS} = \dfrac{1}{16\pi}\int d^4x\sqrt{-g}\left[R - 2\partial_\mu\varphi\partial^\mu\varphi - V_{eff}(\varphi)\right]$
- $V_{eff}(\varphi) = -\lambda^2 R^2_{\rm GB}\, f_1(\varphi) < 0 \rightarrow$ Evasion of no-hair theorem
- $f_1(\varphi) = \dfrac{1}{2\kappa}\left(1 - e^{-\kappa\varphi^4}\right) > 0 \rightarrow \boxed{\text{Class IIB: } f_1(0)=0,\ f_1{}'(0)=0,\ f_1{}''(0)=0;\ \text{secondary scalar}}$
- $\square\varphi + \dfrac{\lambda^2}{4} R^2{}_{\rm GB} f_1{}'(\varphi) = 0 \Rightarrow$ Linearized theory : No tachyonic instability $\rightarrow$ SBH: stable

  $\Rightarrow \boxed{\text{Nonlinear instability}} \rightarrow \boxed{\text{scalarized phase}}$
- Solving full eqs $\rightarrow$ Nonlinear scalarized black holes $\rightarrow$ a few branches depending on κ



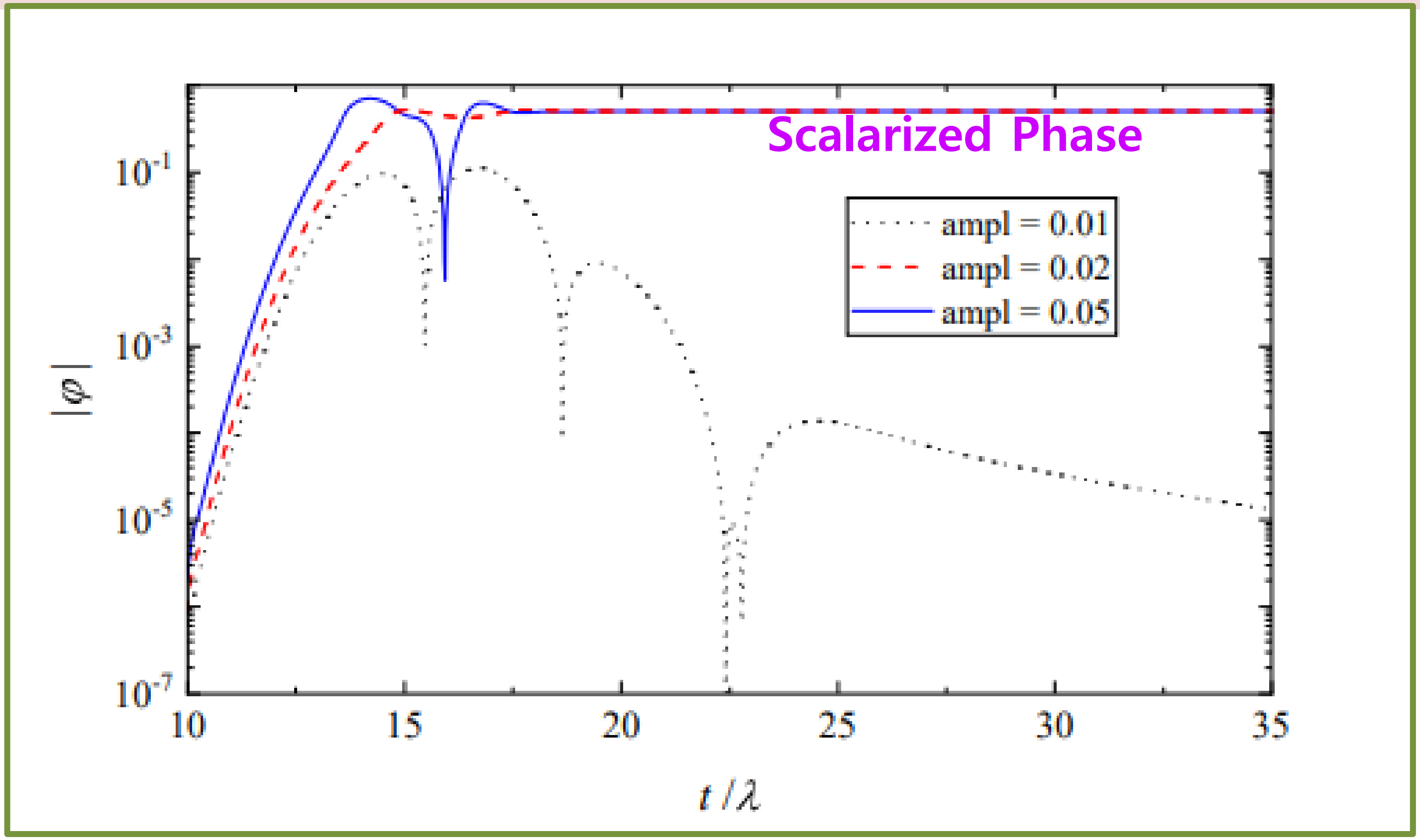
D. D. Doneva and S. S. Yazadjiev, arXiv: 2107.01738
Meng-Yun Lai, De-Cheng Zou, Rui-Hong Yue, YSMyung, arXiv:2304.08012

- $\boxed{-\partial_t^2\varphi+\partial_x^2\varphi+\frac{2\Delta}{r^3}\partial_x\varphi\ +\lambda^2\frac{12M^2\Delta}{r^8}f_1{}'(\varphi)=0}$ based on SBH with $\boxed{R_{\rm GB}^2=\frac{48M^2}{r^6}}$

with $\Delta=r^2-2M\ r$ and $dx=(r^2/\Delta)dr$

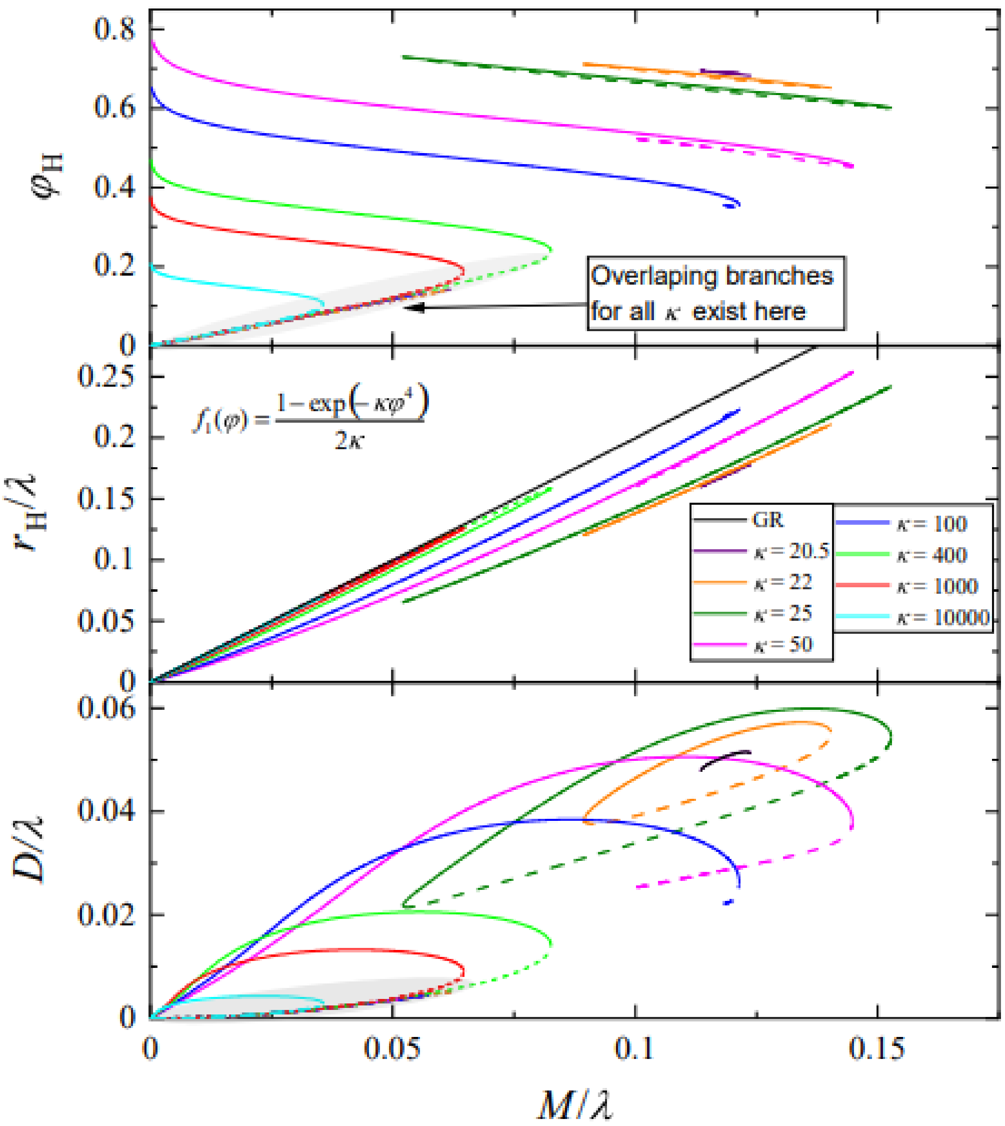
- Nonlinear instability $\rightarrow$ Time evoultion of $|\varphi|$ on the SBH with $M/\lambda=0.1$ and $\kappa=50$

$\Rightarrow$ Depending on the initial amplitude φ_0 of Gaussian pulse: $\varphi(t=0,x)=\varphi_0\exp\left(-\frac{(x-a)^2}{2\sigma^2}\right)$



D. D. Doneva and S. S. Yazadjiev, arXiv: 2107.01738

Nonlinear scalarized BHs with exponential coupling $f_1(\varphi)$: Horizon scalar (φ_H) and scalar charge ($Q_s = D$) as functions of M/λ



A few branches and their stability

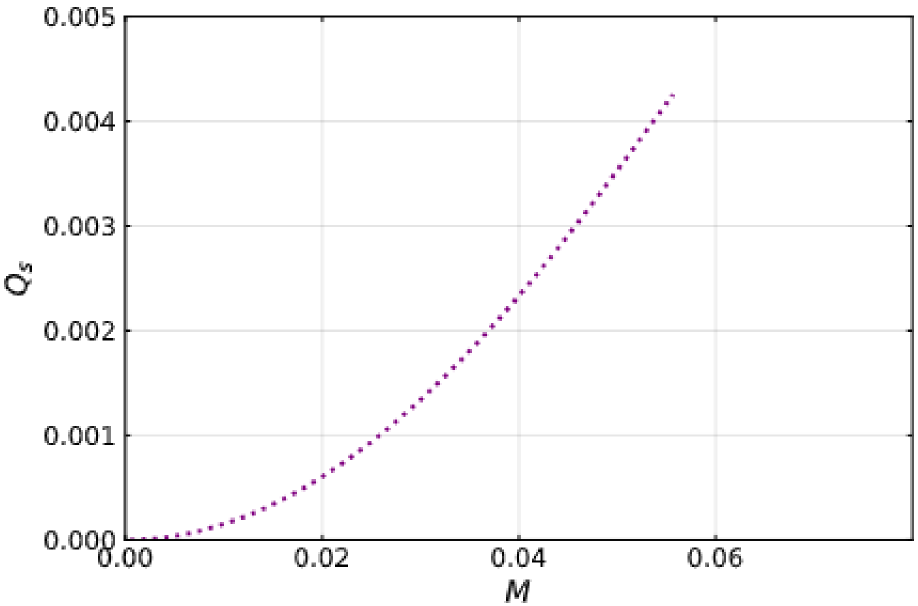
- $\kappa = 400 \rightarrow$ two branches
  $\rightarrow$ solid: stable
- $\kappa = 100 \rightarrow$ three branches
  $\rightarrow$ solid: stable
- $\kappa = 50 \rightarrow$ three branches
  $\rightarrow$ solid: stable

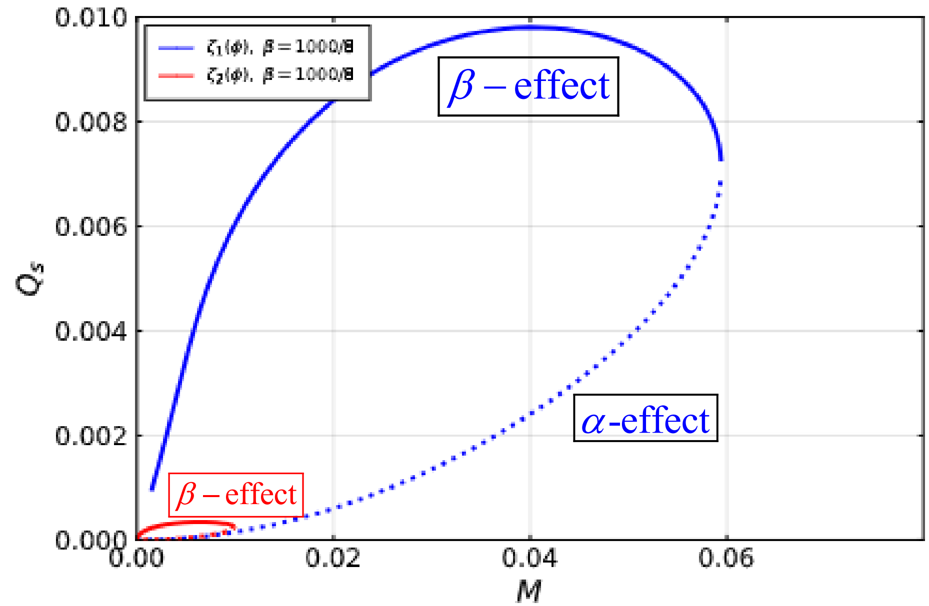
D. D. Doneva and S. S. Yazadjiev, arXiv: 2107.01738

Polynomial nonlinear coupling functions to GB term with α=1/4

$$\varsigma_3(\phi)=\alpha\phi^4, \qquad \varsigma_2(\phi)=\alpha\phi^4-\beta\phi^6, \qquad \varsigma_1(\phi)=\alpha\phi^4-\beta\phi^8$$

$\Downarrow$ $\Downarrow$ $\Downarrow$





→Single branch →Two branches/Two branches

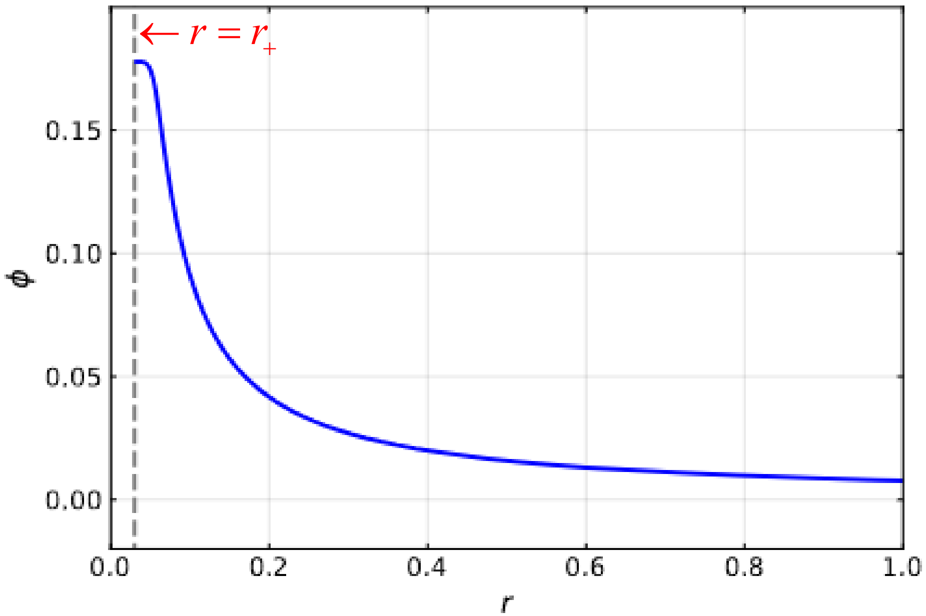
→→ secondary scalar charges $\boxed{Q_s(M)}$

De-Cheng Zou, Xu Yang, Meng-Yun Lai, Hyat Huang, Bo Liu, Jutta Kunz, YSMyung, Rui-Hong Yue, arXiv:2404.19521v3

Nonlinear scalar hairs for polynomial coupling functions

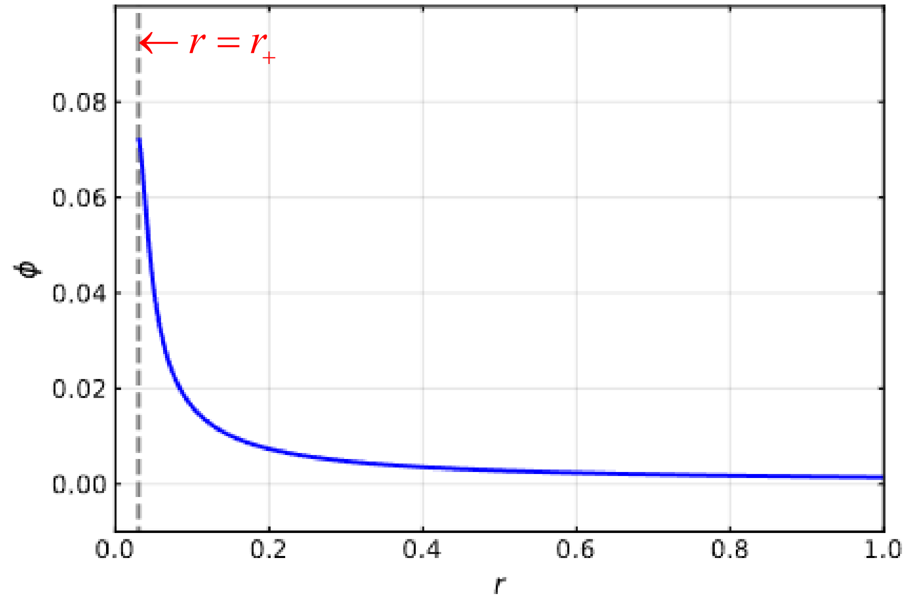
$$\varsigma_1(\phi) = \alpha\phi^4 - \beta\phi^8$$

$\Downarrow$



$$\varsigma_2(\phi) = \alpha\phi^4 - \beta\phi^6$$

$\Downarrow$



3. Nonlinear scalarization in the EMS theory

- EMS theory with $f(\phi)=1+\alpha\phi^4$ $(e^{\alpha\phi^4})$

$$S_{\rm EMS}=\frac{1}{16\pi}\int d^4x\sqrt{-g}\left[R-2\partial_\mu\phi\partial^\mu\phi-f(\phi)F^2\right]$$

$\rightarrow$ $\boxed{V_{eff}=f(\phi)F^2<0\rightarrow f(\phi)>0\rightarrow \text{Evasion of no-hair theorem}}$

$\rightarrow$ $\boxed{\text{subclass IIB: } f(0)=1, f'(0)=0, f''(0)=0; \text{secondary scalar}}$

- Bald BH $\rightarrow$ RNBH
- Linearized scalar eq: $\bar{\Box}\,\varphi=0$ $\rightarrow$ No tachyonic instability $\rightarrow$ $\boxed{\text{RNBH: stable}}$

$\rightarrow$ Instead, nonlinear instability to trigger nonlinear scalarized black holes

- Solving full eqs $\rightarrow$ two branches of SCBHs

J. L. Blazquez-Salcedo, C. A. R. Herdeiro, J. Kunz, A. M. Pombo and E. Radu, arXiv: 2002.00963
Wei Xiong, Cheng-Yong Zhang, Peng-Cheng Li, arXiv: 2312.11879

Equations of motion

- Einstein Eq : $G_{\mu\nu} = 2\partial_\mu\phi\partial_\nu\phi - (\partial\phi)^2 g_{\mu\nu} + 2f(\phi)T^M{}_{\mu\nu}$
- Maxwell Eq : $\nabla^\mu(f(\phi)F_{\mu\nu}) = 0$
- Scalar Eq : $\boxed{\Box\phi - \frac{1}{4}f'(\phi)F^2 = 0}$

Nonlinear scalarized charged BHs

- Solution ansatz $\rightarrow \left\{ \begin{array}{l} ds^2_{\text{SCBH}} = -N(r)e^{-2\delta(r)}dt^2 + \dfrac{dr^2}{N(r)} + r^2 d\Omega_2^2 \quad \text{with} \quad N(r) \equiv 1 - \dfrac{2m(r)}{r} \\ A = v(r)\, dt, \quad \phi = \phi(r) \end{array} \right\}$
- $\boxed{\text{Eqs and solution forms in near-horizon and asymptotic regions} \Rightarrow \text{Next}}$
- Reduced Area and Reduced Temperature

$$a_H = \frac{A_H}{16\pi M^2} = \frac{r_H^2}{4M^2},\ t_H = 8\pi M T_H = 2M\ N'(r_H)\ e^{-\delta(r_H)}$$

- Linearized scalar potential around NSCBHs

$$V_\phi(r) = \frac{e^{-2\sigma}N}{r^2}\left[1 - N - 2r^2\phi'^2 + \frac{Q^2}{f}\left\{2\phi'^2 - \frac{8\alpha}{rf}\phi^3\phi' - \frac{1}{r^2f^2}\left(\alpha^2\phi^6(\phi^2-10) + 2\alpha\phi^2(\phi^2+3) + 1\right)\right\}\right]$$

Four coupled differential equations

- $m' = \frac{1}{2} r^2 N \phi'^2 + \frac{Q^2}{2r^2 f(\phi)}$
- $\delta'(r) + r\phi'^2 = 0,$
- $v'(r) = -\frac{Q e^{-\delta}}{r^2 f(\phi)},$
- $(e^{-\delta} r^2 N \phi')' + \frac{e^{-\delta} f'(\phi) Q^2}{2r^2 f^2(\phi)} = 0$

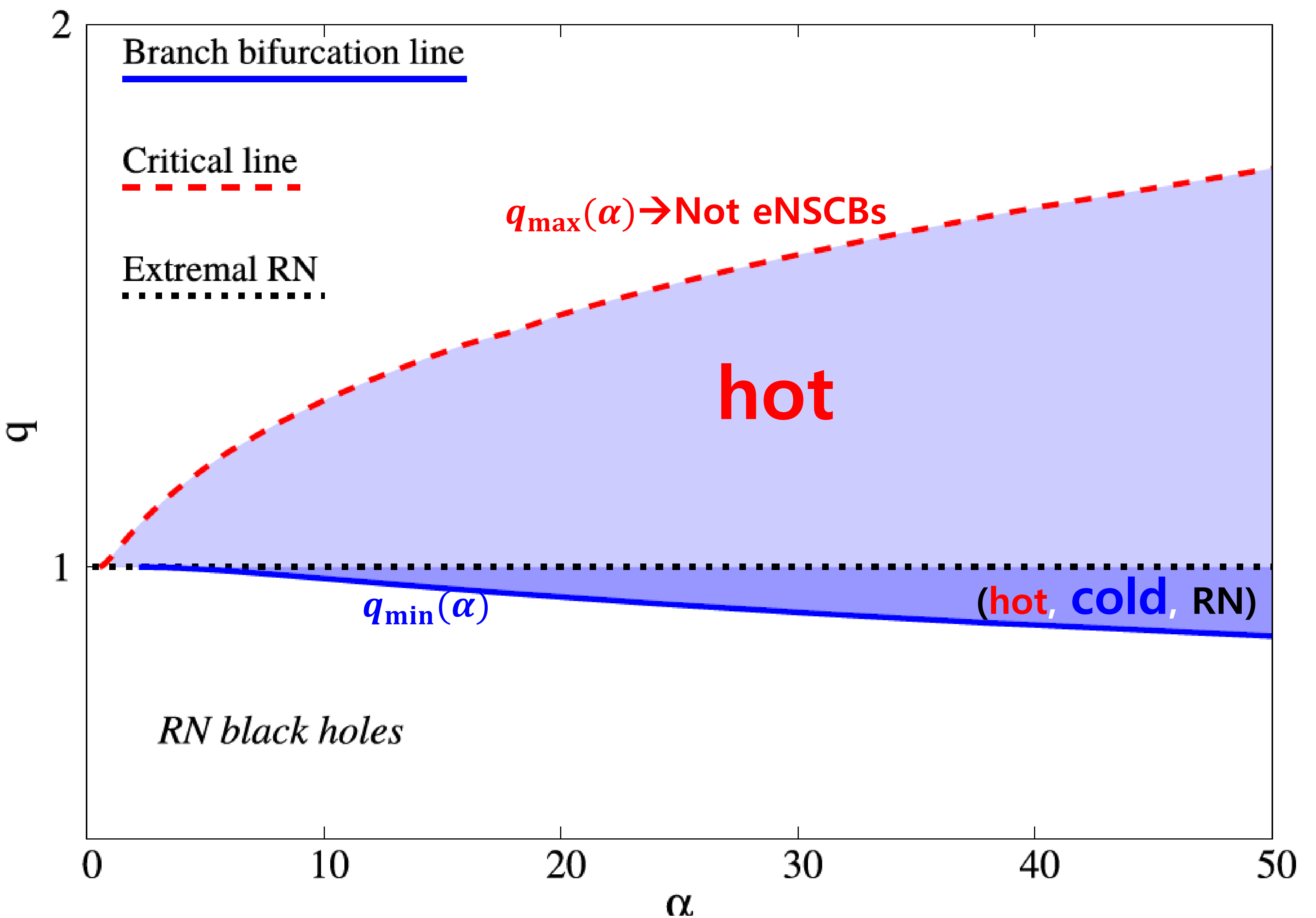
How to solve these equations?

- Near-horizon form: $\begin{cases} m(r) = \dfrac{r_H}{2} + m_1(r - r_H) + \dots, & \delta(r) = \delta_0 + \delta_1(r - r_H) + \dots \\ \phi(r) = \phi_0 + \phi_1(r - r_H) + \dots, & V(r) = v_1(r - r_H) + \dots, \end{cases}$

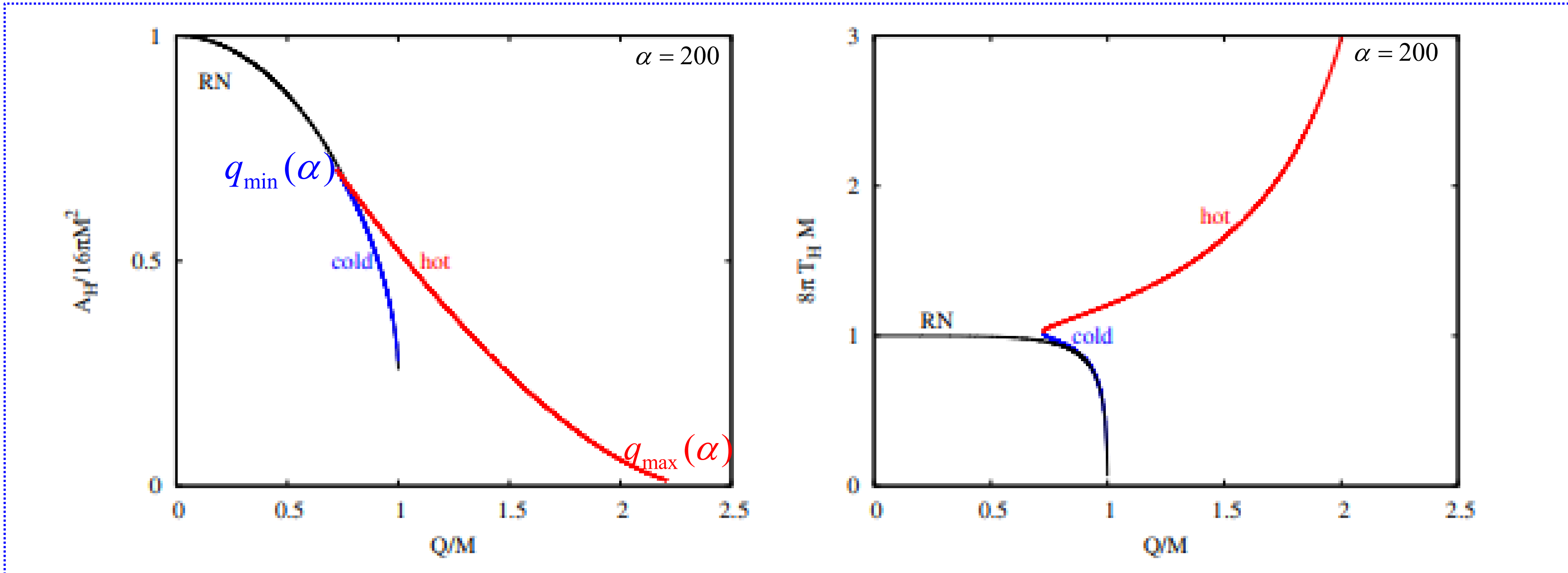
  with $m_1 = \dfrac{1}{2r_H^2}\left[\dfrac{Q^2}{f(\phi_0)}\right]$, $\quad \phi_1 = \dfrac{f'(\phi_0)}{2r_H}\dfrac{1}{\left[Q^2 - r_H^2 f(\phi_0)\right]}$, $\quad \delta_1 = -r_H\phi_1^2$, $\quad v_1 = \dfrac{e^{-\delta_0}Q}{r_H^2 f(\phi_0)}$

- Asymptotic form: $\begin{cases} m(r) = M - \dfrac{Q^2 + Q_s^2}{2r} + \cdots, & \phi(r) = \dfrac{Q_s}{r} + \cdots, \\ \delta(r) = \dfrac{Q_s^2}{2r^2} + \dots, & V(r) = \Phi_e - \dfrac{Q}{r} + \dots, \end{cases}$

Domain of existence of (hot, cold) branches of NSCBHs



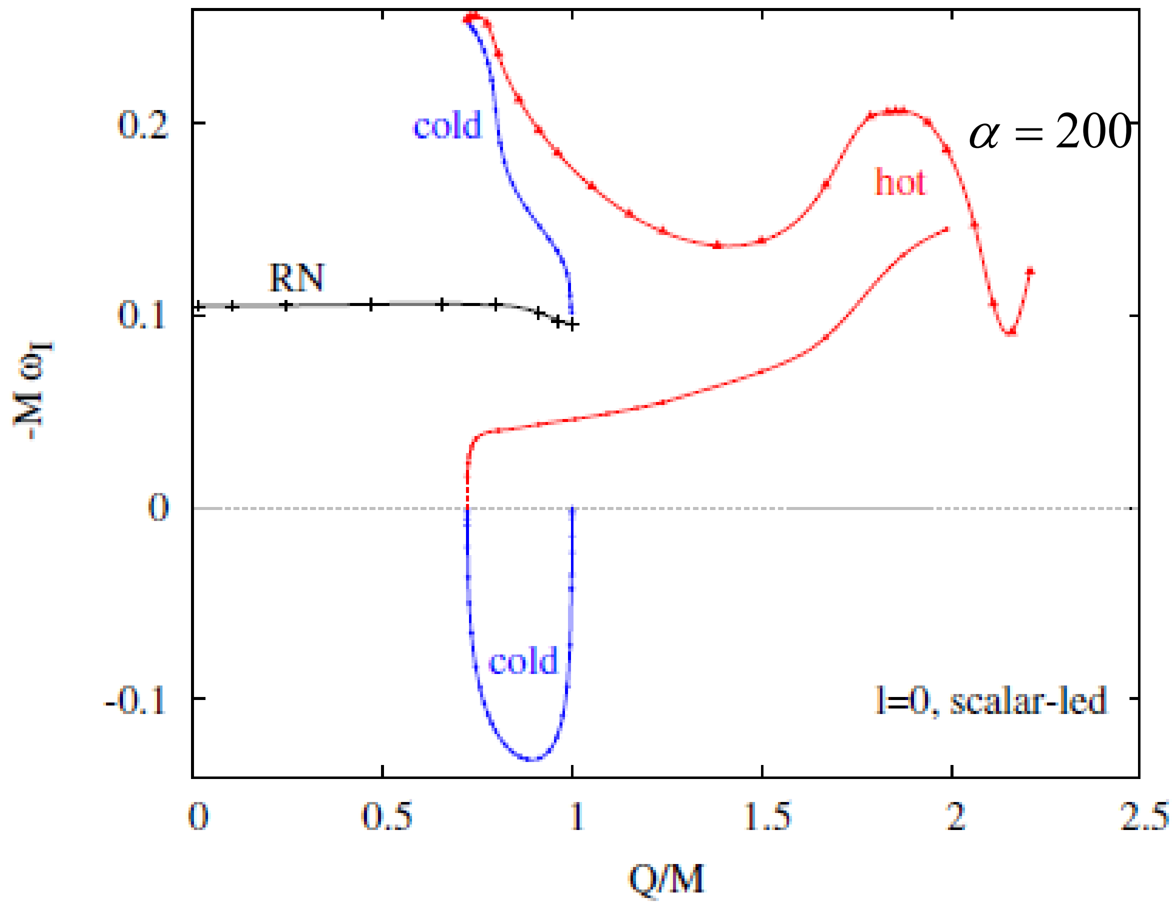
Reduced Area and Reduced Temperature



- Cold branch: it starts at $q = 1$ (extremal RNBH) and ends at $q_{\min}(\alpha)$
- Hot branch : it starts at $q_{\min}(\alpha)$ while disconnecting to RNBH, and ends at $q_{\max}(\alpha)$

- $a_H^{RN} = \frac{1}{4}\left(1+\sqrt{1-q^2}\right)^2 \rightarrow \frac{1}{4} \leq a_H^{RN} \leq 1, \quad t_H^{RN} = \frac{4\sqrt{1-q^2}}{\left(1+\sqrt{1-q^2}\right)^2} \rightarrow 0 \leq t_H^{RN} \leq 1$

QNM frequency of *s*-mode scalar



RN, hot →stable
cold →unstable

Jose Luis Blázquez-Salcedo, Carlos A. R. Herdeiro, Sarah Kahlen, Jutta Kunz, Alexandre M. Pombo, Eugen Radu, arXiv: 2008.11744

4. Scalarized extremal BH solutions

in Einstein-Maxwell-conformally coupled scalar (EMCS) theory

- $S_{\rm EMCS} = \frac{1}{16\pi G}\int d^4x\sqrt{-g}\left[R - F_{\mu\nu}F^{\mu\nu} - \beta\left(\phi^2 R + 6\partial_\mu\phi\partial^\mu\phi\right)\right]$

------------------------------------ $\Rightarrow$ Conformal symmetry

- Einstein Eq.: $G_{\mu\nu} = 2T^{\rm M}_{\mu\nu} + T^{\phi}_{\mu\nu}$,

$$\begin{cases} T^{\rm M}_{\mu\nu} = F_{\mu\rho}F_{\nu}{}^{\rho} - \frac{F^2}{4}g_{\mu\nu}, \\ T^{\phi}_{\mu\nu} = \beta\left[\phi^2 G_{\mu\nu} + g_{\mu\nu}\nabla^2(\phi^2) - \nabla_\mu\nabla_\nu(\phi^2) + 6\nabla_\mu\phi\nabla_\nu\phi - 3(\nabla\phi)^2 g_{\mu\nu}\right] \end{cases}$$

- $\nabla^\mu F_{\mu\nu} = 0$
- $\nabla^2\phi - \frac{1}{6}R\phi = 0 \rightarrow \left[R = 0\right] \rightarrow \nabla^2\phi = 0$

→→Two scalarized extremal BHs with constant and secondary scalar charges

- Solution ansatz: $ds^2_{\text{csh}} = \bar{g}_{\mu\nu} dx^\mu dx^\nu = -N(r)e^{-2\delta(r)}dt^2 + \dfrac{dr^2}{N(r)} + r^2 d\Omega_2^2$

with $N(r) = 1 - \dfrac{2m(r)}{r}$, $\phi = \bar{\phi}(r)$, $\bar{A}_t = v(r)$

- Constant scalar hair and its extremal black holes for $\beta = \dfrac{1}{3}$

$$\left\langle \begin{array}{l} \bar{G}_{\mu\nu} = \dfrac{2\bar{T}^{\mathrm{M}}_{\mu\nu}}{1-\beta\bar{\phi}_c^2} = 2\bar{T}_{\mu\nu}, \quad \bar{T}_\nu^{\ \mu} = \dfrac{Q^2+q_s^2}{r^4}\,\mathrm{diag}[-1,-1,1,1] \\ \bar{\phi}_c = q_s\sqrt{\dfrac{3}{q_s^2+Q^2}}, \quad \bar{A}_t = \dfrac{Q}{r} - \dfrac{Q}{r_+}, \quad \delta(r) = 0 \\ N(r) = 1 - \dfrac{2M}{r} + \dfrac{Q^2+q_s^2}{r^2} \xrightarrow[\text{extremal}: M=\sqrt{Q^2+q_s^2}]{} \boxed{\bar{\phi}_c = \dfrac{\sqrt{3(M^2-Q^2)}}{M}}, \; N_e(r) = \left(1-\dfrac{M}{r}\right)^2 \end{array} \right\rangle$$

Marco Astorino, arXiv:1307.4021

cBBMB BH solution

- Residual conformal symmtery in $(\bullet\bullet\bullet) \Rightarrow \left[R=0 \text{ and } \Box\phi=0\right]$
- $ds^2_{\text{cBBMB}} = -\left(1-\frac{M}{r}\right)^2 dt^2 + \frac{dr^2}{\left(1-\frac{M}{r}\right)^2} + r^2 d\Omega_2^2, \quad \bar{A}_t = \frac{Q}{r} - \frac{Q}{r_+}, \quad M = \sqrt{Q^2 + Q_s^2}$
- $\phi(r) = \sqrt{\frac{1}{\beta}} \frac{Q_s}{r-M} \Rightarrow \sqrt{\frac{1}{\beta}} \frac{\sqrt{M^2-Q^2}}{r-M}$: Scalar hair is $\boxed{\text{secondary}}$ and $\boxed{\text{it diverges at } r=M}$
- $T_\mu^{\phi\ \nu} = \frac{2Q_s^2}{r^4} \text{diag}\,[-1,-1,1,1], \quad T_\mu^{\text{M}\ \nu} = \frac{2Q^2}{r^4} \text{diag}\,[-1,-1,1,1]$
- $T_\mu^{\phi+\text{M}\ \nu} = \frac{2M^2}{r^4} \text{diag}\,[-1,-1,1,1] \Rightarrow$ Mass-dependent energy-momentum tensor

Numerical solution to cBBMB BH

$\boxed{\bar{\phi}(r) \to \frac{1}{\bar{\psi}(r)}}$ because $\bar{\phi}(r)$ blows up at the horizon

- For $\beta = \frac{1}{3}$, $ds^2_{\text{scbh}} = -\bar{N}(r)e^{-2\delta(r)}dt^2 + \frac{dr^2}{\bar{N}(r)} + r^2 d\Omega_2^2$, $\boxed{\bar{\phi}(r) \to \frac{1}{\bar{\psi}(r)}}$, $\bar{A}_t = v(r)$

- Full Eqs.:

$$\begin{cases} \bar{\psi}'[r\bar{\psi}\bar{N}' - \bar{N}(\bar{\psi}(r\delta'-2) + 2r\bar{\psi}')] + r\bar{N} + \bar{\psi}\bar{\psi}'' = 0, \\ (1-3\bar{\psi}^2)[1+\bar{N}(2r\delta'-1)] - \frac{r\bar{N}\bar{\psi}'}{\bar{\psi}^2}[2\bar{\psi}(r\delta'-2) + 3r\bar{\psi}'] + 3r^2 e^{2\delta}\bar{\psi}^2 v'^2 + \frac{r^2\bar{N}''}{\bar{\psi}}(r\bar{\psi}' + 3\bar{\psi}^3 - \bar{\psi}) = 0, \\ Q\bar{\psi}^2 + e^{\delta}r^2\bar{\psi}^2 v' = 0\left(v'(r) = \frac{Qe^{-\delta}}{r^2}\right), \\ \delta'(3\bar{\psi}^3 - \bar{\psi} + r\bar{\psi}') + r\bar{\psi}'' = 0 \end{cases}$$

- Near-Horizon: $\boxed{\bar{N}(r) = \sum_{i=2}^{\infty} N_i (r-r_+)^i}$, $\delta(r) = \sum_{i=0}^{\infty}\delta_i (r-r_+)^i$, $\boxed{\bar{\psi}(r) = \sum_{i=1}^{\infty}\psi_i (r-r_+)^i}$, $v(r) = \sum_{i=1}^{\infty} v_i (r-r_+)^i$

- Lowest order: $1 - r_+^2 N_2 = 0$, $\delta_1\psi_1 + 2\psi_2 = 0 \Rightarrow \boxed{N_2 = \frac{1}{r_+^2},\ \psi_2 = -\frac{\delta_1\psi_1}{2}}$

De-Cheng Zou, YSMyung, arXiv:1911.08062

• NH Coefficients: $\begin{cases} N_2=\dfrac{1}{r_+^2}, \quad N_3=-\dfrac{6(r_+^2-Q^2)\psi_1^2+4}{3r_+^3}, \quad N_4=\dfrac{5+15(r_+^2-Q^2)\psi_1^2}{3r_+^4}-\dfrac{3(r_+^2-Q^2)^2\psi_1^4}{r_+^4}, \\ \delta_0, \quad \delta_1=\dfrac{2[1-3(r_+^2-Q^2)\psi_1^2]}{3r_+}, \qquad \delta_2=-\dfrac{1-15(r_+^2-Q^2)\psi_1^2+36(r_+^2-Q^2)^2\psi_1^4}{9r_+^2}, \\ \psi_1, \quad \psi_2=-\dfrac{[1-3(r_+^2-Q^2)\psi_1^2]\psi_1}{3r_+}, \qquad \psi_3=\dfrac{2\psi_1[1-3(r_+^2-Q^2)\psi_1^2]^2}{9r_+^2}, \\ v_1=-\dfrac{e^{-\delta_0}Q}{r_+^2}, \quad v_2=\dfrac{e^{-\delta_0}Q}{r_+^3}\left[\dfrac{4}{3}-(r_+^2-Q^2)\psi_1^2\right], \quad v_3=-\dfrac{e^{-\delta_0}Q}{r_+^4}\left[\dfrac{14}{9}-\dfrac{7}{3}(r_+^2-Q^2)\psi_1^2+2(r_+^2-Q^2)^2\psi_1^4\right] \end{cases}$

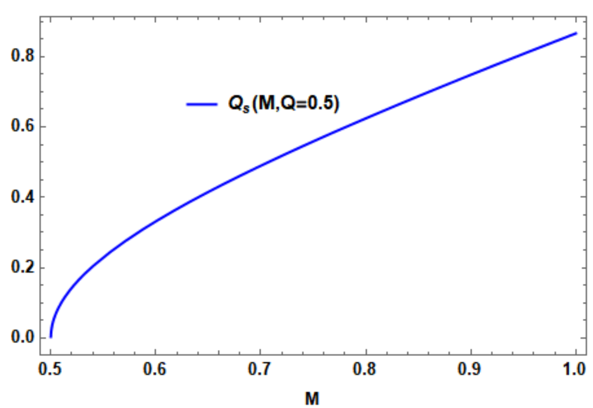
• Asymptotic form with Q_s :

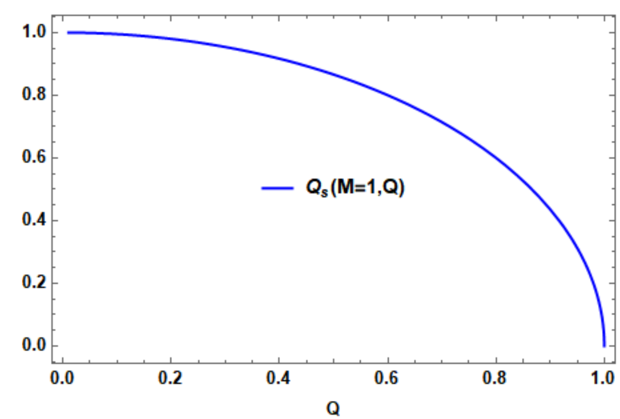
$$\left\{\bar{N}(r)=1-\frac{2M}{r}+\frac{Q^2+Q_s^2}{r^2}+\cdots, \quad \bar{\phi}(r)=\frac{\sqrt{3}Q_s}{r}+\frac{\sqrt{3}Q_s M}{r^2}+\cdots, \quad \delta(r)=\frac{Q_s^2(Q^2+Q_s^2-M^2)}{6r^4}+\cdots, \quad v(r)=\frac{Q}{r}-\frac{Q}{r_+}\right\}$$

• $\boxed{\text{Asymptotic flatness}}\ (\delta_0\approx 0,\ \delta_n=0,\ \delta(r)=0) \Rightarrow r_+\approx M,\ \boxed{M^2\approx Q^2+Q_s^2,}\ \ \psi_1\approx\dfrac{1}{\sqrt{3}Q_s}=\dfrac{1}{\sqrt{3(r_+^2-Q^2)}} \Rightarrow \boxed{Q_s\text{: dependent}}$

$\Downarrow$

$\boxed{Q_s \text{ : independent scalar charge} \xrightarrow[\delta(r)=0]{} \text{dependent (secondary) scalar charge}}$





Imposing Asymptotic flatness condition: Numerical solution→ cBBMB BH

- Near-horizon:

$$\begin{cases} \bar{N}(r) = \dfrac{(r-r_+)^2}{r_+^2} - \dfrac{2(r-r_+)^3}{r_+^3} + \dfrac{3(r-r_+)^4}{r_+^4} + \dots \ \rightarrow \left[\dfrac{(r-r_+)^2}{r^2}\right]_{r=r_+}, \\ \bar{\psi}(r) = \psi_1(r-r_+) = \dfrac{r-r_+}{\sqrt{3}Q_s} = \dfrac{r-r_+}{\sqrt{3(r_+^2-Q^2)}}, \quad \boxed{\delta(r)=0}, \\ v(r) = -\dfrac{Q(r-r_+)}{r_+^2} + \dfrac{Q(r-r_+)^2}{r_+^3} - \dfrac{Q(r-r_+)^3}{r_+^4} + \dots \rightarrow -\dfrac{Q}{r_+}\left[\dfrac{(r-r_+)}{r}\right]_{r=r_+} \end{cases}$$

- Asymptotic form:

$$\begin{Bmatrix} \bar{N}(r) = 1 - \dfrac{2M}{r} + \dfrac{M^2}{r^2} = \left(1-\dfrac{M}{r}\right)^2, \ M = \sqrt{Q^2+Q_s^2} \\ \bar{\phi}(r) = \dfrac{\sqrt{3}Q_s}{r} + \dfrac{\sqrt{3}Q_s \cdot M}{r^2} + \cdots \rightarrow \left(\dfrac{\sqrt{3}Q_s}{r-M}\right)_{r \gg M} = \left(\dfrac{\sqrt{3}\sqrt{M^2-Q^2}}{r-M}\right), \ \boxed{\delta(r)=0}, \\ v(r) = \dfrac{Q}{r} - \dfrac{Q}{r_+} \end{Bmatrix}$$

5-0. Dyonic EMS theory

- $\mathcal{S}_{\text{dEMS}} = \frac{1}{16\pi}\int d^4x\sqrt{-g}\left(R - 2\partial_\mu\phi\partial^\mu\phi - f(\phi)F_{\mu\nu}F^{\mu\nu}\right)$, $\boxed{A = -\frac{Q}{r}dt + P\cos\theta d\varphi}$

- $ds^2 = -e^{-2\delta(r)}N(r)dt^2 + \frac{dr^2}{N(r)} + r^2(d\theta^2 + \sin^2\theta d\varphi^2)$ with $N(r) \equiv 1 - \frac{2m(r)}{r}$

- Full Eqs.: $\left\{\begin{array}{l} m' = \frac{1}{2}r^2N\phi'^2 + \frac{1}{2r^2}\left(\frac{Q^2}{f(\phi)} + f(\phi)P^2\right), \quad \delta' + r\phi'^2 = 0, \\ (e^{-\delta}r^2N\phi')' + \frac{e^{-\delta}}{2r^2 f(\phi)}\boxed{f'(\phi)\left(\frac{Q^2}{f(\phi)} - f(\phi)P^2\right)} = 0 \end{array}\right\}$

=0 $\Rightarrow$ constant scalar hair

D. Astefanesei, C. Herdeiro, A. Pombo, E. Radu, arXiv:1905.08304

5-0. Dyonic EMS theory → extremal BHs with constant scalar hair

- RNBH with $\phi=0$: $f(\phi)=1,\ \delta(r)=0,\ N(r)=1-\dfrac{2M}{r}+\dfrac{Q^2+P^2}{r^2} \xrightarrow[\text{extremal: } M=\sqrt{Q^2+P^2}]{} N_e=\left(1-\dfrac{M}{r}\right)^2$

- BH with $\phi=\phi_c$: $f(\phi_c)=\dfrac{Q}{P},\ \delta(r)=0,\ \tilde{N}(r)=1-\dfrac{2M}{r}+\dfrac{2PQ}{r^2} \xrightarrow[\text{extremal: } M=\sqrt{2PQ}]{} \tilde{N}_e=\left(1-\dfrac{M}{r}\right)^2$

- BH with $\phi=\phi_c$: $f'(\phi)|_{\phi=\phi_c}=0,\ \delta(r)=0,\quad \hat{N}(r)=1-\dfrac{2M}{r}+\dfrac{\frac{Q^2}{f(\phi_c)}+f(\phi_c)P^2}{r^2}$

$$\xrightarrow[\text{extremal: } M=\sqrt{\frac{Q^2}{f(\phi_c)}+f(\phi_c)P^2}]{} \hat{N}_e=\left(1-\frac{M}{r}\right)^2$$

5. EMS theory with two U(1) fields

- $N=4$ supergravity: $S_{\mathrm{N=4}} = \dfrac{1}{16\pi}\int d^4x\sqrt{-g}\left[R - 2\partial_\mu\phi\partial^\mu\phi - e^{-2\phi}F^2 - e^{2\phi}H^2\right],\ F = dA,\ H = dB$

$$\Rightarrow ds^2_{\mathrm{N4BH}} = -\frac{1}{H_1 H_2}dt^2 + H_1 H_2\left(dr^2 + r^2 d\Omega_2^2\right) \text{ with } e^{2\bar{\phi}} = \frac{H_2}{H_1},\quad H_1 = 1 + \frac{\sqrt{2}Q}{r},\quad H_2 = 1 + \frac{\sqrt{2}P}{r}$$

$\rightarrow$ $\boxed{\text{dilatonic BH solution: horizon at } r = 0 \text{ and constant scalar hair } \phi_H = \frac{1}{2}\ln\left[\frac{P}{Q}\right]}$

- EMS with two U(1): $S_{\mathrm{EMSN4}} = \dfrac{1}{16\pi}\int d^4x\sqrt{-g}\left[R - 2\partial_\mu\phi\partial^\mu\phi - \dfrac{F^2}{f(\phi)} - f(\phi)H^2\right]$

- Eqs.:

$$\begin{cases} G_{\mu\nu} = 2\partial_\mu\phi\partial_\nu\phi - (\partial\phi)^2 g_{\mu\nu} + 2T^{U(1)}_{\mu\nu} \\ \quad\quad \text{with } T^{U(1)}_{\mu\nu} = \dfrac{1}{f(\phi)}\left(F_{\mu\rho}F_\nu^{\rho} - \dfrac{F^2}{4}g_{\mu\nu}\right) + f(\phi)\left(H_{\mu\rho}H_\nu^{\rho} - \dfrac{H^2}{4}g_{\mu\nu}\right) \\ \partial_\mu\left(\sqrt{-g}\,\dfrac{F_{\mu\nu}}{f(\phi)}\right) = 0,\ \partial_\mu\left(\sqrt{-g}\,f(\phi)H_{\mu\nu}\right) = 0, \\ \square\phi + \dfrac{1}{4}\dfrac{f'(\phi)}{f(\phi)}\left(-\dfrac{F^2}{f(\phi)} + f(\phi)H^2\right) = 0 \end{cases}$$

Xiao Yan Chew, YSMyung, arXiv:2606.05766

- $ds^2_{\mathrm{SCBH}} = \bar{g}_{\mu\nu}dx^\mu dx^\nu = -N(r)e^{-2\delta(r)}dt^2 + \dfrac{dr^2}{N(r)} + r^2(d\theta^2 + \sin^2\theta d\varphi^2),$

$$N(r) = 1 - \frac{2m(r)}{r}, \quad \bar{\phi} = \phi(r), \quad \bar{A}_t = v_Q(r), \quad \bar{B}_t = v_P(r)$$

- Eqs.: $\left\{ \begin{array}{l} m' = \dfrac{1}{2}r^2 N\phi'^2 + \dfrac{1}{2r^2}\left(f(\phi)Q^2 + \dfrac{P^2}{f(\phi)} \right) \\ \delta' + r\phi'^2 = 0, \ v'_Q(r) = \dfrac{e^{-\delta(r)}Qf(\phi)}{r^2}, \ v'_P(r) = \dfrac{e^{-\delta(r)}P}{r^2 f(\phi)} \\ (e^{-\delta}r^2N\phi')' + \dfrac{e^{-\delta}}{2r^2 f(\phi)} \boxed{f'(\phi)\left(f(\phi)Q^2 - \dfrac{P^2}{f(\phi)} \right)} = 0 \end{array} \right\}$

=0: constant scalar hair

Extremal BHs with constant scalar hair

- RNBH with $\phi=0$: $f(\phi)=1,\ \delta(r)=0,\ N(r)=1-\frac{2M}{r}+\frac{Q^2+P^2}{r^2} \xrightarrow[\text{extremal: } M=\sqrt{Q^2+P^2}]{} N_e=\left(1-\frac{M}{r}\right)^2$

- BH with $\phi=\phi_c$: $f(\phi_c)=\frac{P}{Q},\ \delta(r)=0,\ \tilde{N}(r)=1-\frac{2M}{r}+\frac{2PQ}{r^2} \xrightarrow[\text{extremal: } M=\sqrt{2PQ}]{} \tilde{N}_e=\left(1-\frac{M}{r}\right)^2$

- BH with $\phi=\phi_c$: $f'(\phi)|_{\phi=\phi_c}=0,\ \delta(r)=0,\quad \widehat{N}(r)=1-\frac{2M}{r}+\frac{Q^2 f(\phi_c)+\frac{P^2}{f(\phi_c)}}{r^2}$

$$\xrightarrow[\text{extremal: } M=\sqrt{Q^2 f(\phi_c)+\frac{P^2}{f(\phi_c)}}]{} \widehat{N}_e=\left(1-\frac{M}{r}\right)^2$$

Spontaneous Scalarization approach to finding scalarized extremal black holes

- One constraint : $$\frac{N''(r)}{2} - N\delta''(r) + N'(r)\left(\frac{1}{r} - \frac{3\delta'(r)}{2}\right) + N\delta'(r)\left(\delta'(r) - \frac{1}{r}\right) + N\phi'^2(r)$$
$$= \frac{1}{r^4}\left(f(\phi)Q^2 + \frac{P^2}{f(\phi)}\right) \quad \text{with } N(r) = 1 - \frac{2m(r)}{r}$$

- Near-horizon: $$\begin{cases} N(r) = N_2(r-r_+)^2 + N_3(r-r_+)^3 \cdots, \\ \delta(r) = \delta_0 + \delta_1(r-r_+) + \cdots, \ v_Q(r) = v_{Q1}(r-r_+) + \cdots, \quad v_P(r) = v_{P1}(r-r_+) + \cdots, \\ \phi(r) = \phi_0 + \phi_1(r-r_+) + \cdots \end{cases}$$
$$\text{with } v_{Q1} = \frac{e^{-\delta_0} Q f(\phi_0)}{r_+^2}, \quad v_{P1} = \frac{e^{-\delta_0} P}{r_+^2 f(\phi_0)}$$

Five relations for coefficients → Two extremal solutions

$$\left\{\begin{array}{l} r_+^2 = f(\phi_0)Q^2 + \dfrac{P^2}{f(\phi_0)}, \quad \delta_1 = -r_+\phi_1^2, \quad \left(f'(\phi_0) = 0, \quad f(\phi_0) = \dfrac{P}{Q} \right), \\ \phi_1 = \dfrac{\boxed{\dfrac{f'^2(\phi_0)}{r_+^3}\left(Q^2 - \dfrac{P^2}{f^2(\phi_0)}\right)}}{2r_+^2 N_2 + \dfrac{f''(\phi_0)}{r_+^2}\left(Q^2 - \dfrac{P^2}{f^2(\phi_0)}\right) + \dfrac{f'^2(\phi_0)P^2}{r_+^2 f(\phi_0)}}, \quad N_2 = \dfrac{1}{r_+^4}\left(f(\phi_0)Q^2 + \dfrac{P^2}{f(\phi_0)} \right) \end{array}\right\}$$

- For $f'(\phi_0) \neq 0$, one finds : $f(\phi_0) = \dfrac{P}{Q}, \quad r_+ = \sqrt{2QP}, \quad \phi_1 = \delta_1 = 0, \quad N_2 = \dfrac{1}{r_+^2}$
- For $f'(\phi_0) = 0 \ (f(\phi_0) \neq P/Q) : r_+ = \sqrt{f(\phi_0)Q^2 + \dfrac{P^2}{f(\phi_0)}}, \quad \phi_1 = \delta_1 = 0, \quad N_2 = \dfrac{1}{r_+^2}$

- Asmyptotic form with Q_s:

$$\begin{cases} m(r) = M - \dfrac{f(\phi_\infty)Q^2 + P^2 / f(\phi_\infty) + Q_s^2}{2r} + \cdots, \quad \delta(r) = \dfrac{Q_s^2}{2r^2} + \cdots, \\ v_Q(r) = \Phi_Q - \dfrac{Q}{r} + \cdots,\ v_P(r) = \Phi_P - \dfrac{P}{r} + \cdots,\ \phi(r) = \phi_\infty + \dfrac{Q_s}{r} + \cdots \end{cases}$$

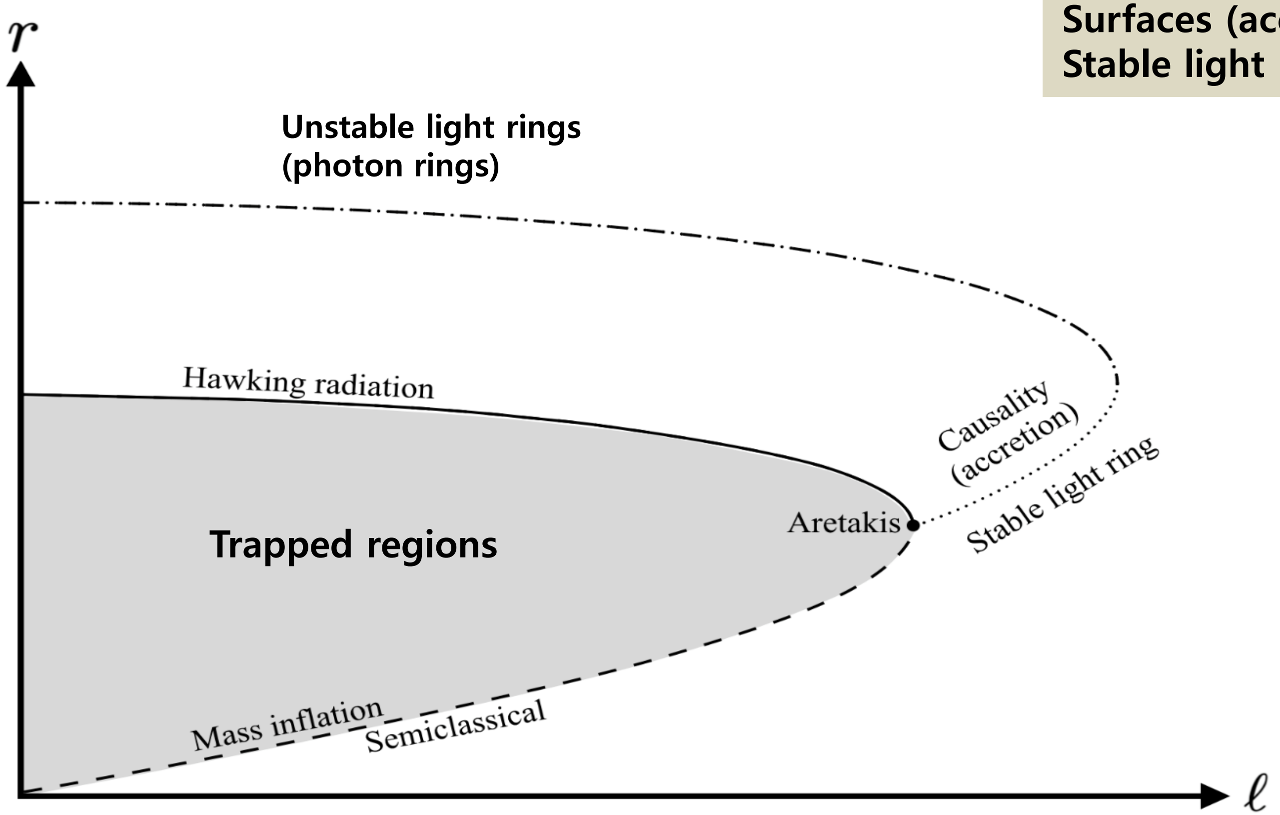
- Asymptotically flat extremal BH $\Rightarrow$ $\boxed{\delta_0 = 0,\ \delta(r) = 0 \to Q_s = 0}$: No scalar charge

- $\begin{Bmatrix} f'(\phi_0) \neq 0 \Rightarrow \phi_\infty = \phi_0 = \phi_c \\ f'(\phi_0) = 0 \Rightarrow \phi_\infty = \phi_0 = \phi_c \end{Bmatrix}$

$\Rightarrow$ Recovering scalarized extremal BHs with constant scalar hair

Family of ss spacetimes between regular BHs and BH mimickers

Event (outer) horizon: Hawking Radiation
Cauchy(inner) horizon: Mass inflation
Extremal horizon: Aretakis instability
Surfaces (accretion disk): accretion instability
Stable light rings: Nonlinear instability



Raúl Carballo-Rubio, Francesco Di Filippo, et al, arXiv:2501.05505

Aretakis Instability

New Difficulties of Extremal Black Holes

1. Near horizon geometry: **degeneracy** of redshift effect

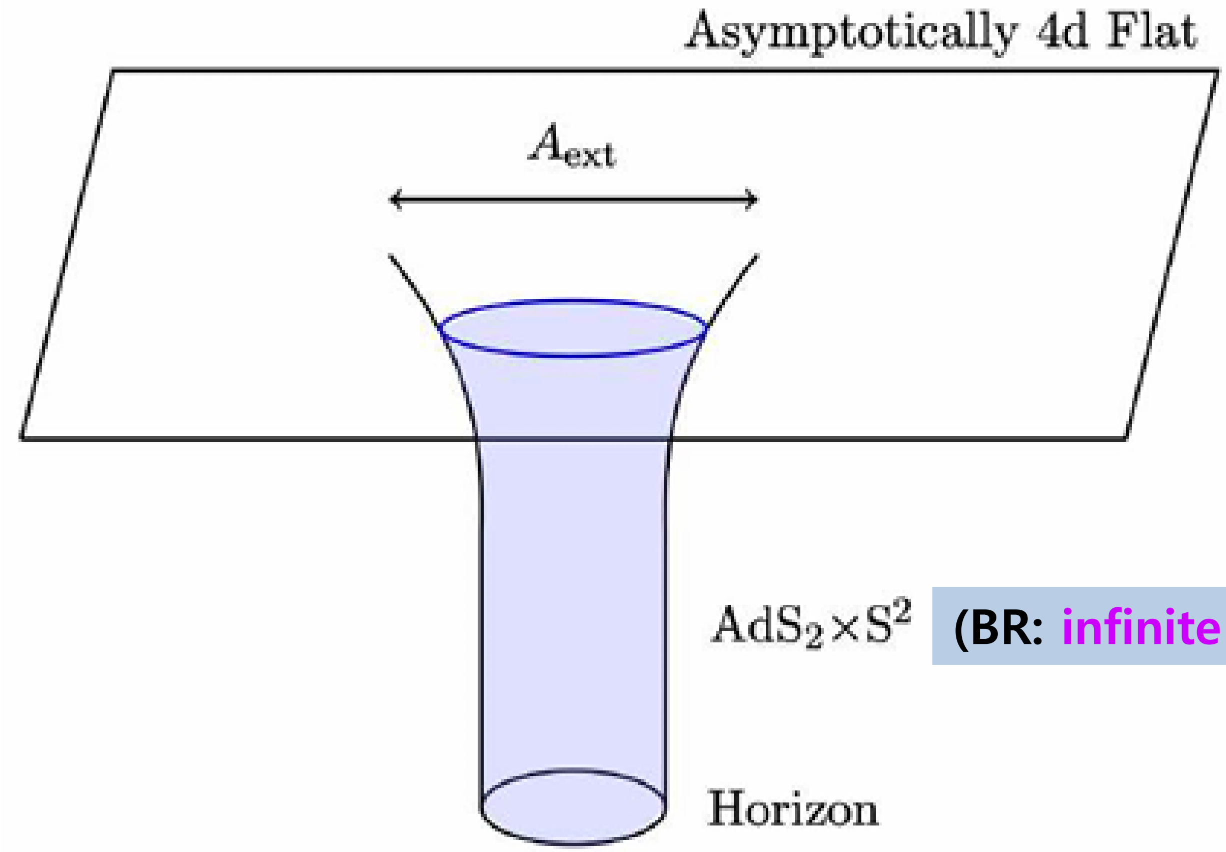
2. Dispersion in intermediate region: trapping effect

3. Near null infinity

4. “Low frequency” obstructions

5. Superradiance

Extremal black hole: Bertotti-Robinson (BR) geometry



(BR: infinite throat)

Entropy function approach to finding scalarized extremal BHs: Entropy→ Only physical quantity

- Bertotti-Robinson metric: $ds^2_{BR} = v_0\left(-r^2dt^2 + \frac{dr^2}{r^2}\right) + v_1(d\theta^2 + \sin^2\theta d\varphi^2)$

  with $A = er\ dt, B = pr\ dt,$ $\boxed{\phi = \phi_0}$: constant scalar on the horizon

- $L = \frac{1}{16\pi}\int d\theta d\varphi\sqrt{-g}\left(R - 2(\nabla\phi)^2 - \frac{F^2}{f(\phi)} - f(\phi)H^2\right) = \frac{1}{2}\left[v_0 - v_1 + \frac{v_1}{v_0}\left(\frac{e^2}{f(\phi_0)} + f(\phi_0)p^2\right)\right]$

- Entropy function: $\varepsilon = 2\pi(eQ + pP - L)$

Entropy function approach to find scalarized extremal BHs

$$\bullet\ \text{Eqs.}:\begin{cases}\dfrac{\partial\varepsilon}{\partial v_0}=0,\ \ \dfrac{\partial\varepsilon}{\partial v_1}=0\Rightarrow -1+\dfrac{v_1}{v_0^2}\left(\dfrac{e^2}{f(\phi_0)}+f(\phi_0)p^2\right)=0\Rightarrow\ \ \left\{v_0=v_1=\dfrac{e^2}{f(\phi_0)}+f(\phi_0)p^2\right\},\\[2ex] \dfrac{\partial\varepsilon}{\partial\phi_0}=0\Rightarrow\left(-\dfrac{e^2}{f^2(\phi_0)}+p^2\right)\dfrac{df}{d\phi_0}=0\\[2ex] \Rightarrow\left\{-\dfrac{e^2}{f^2(\phi_0)}+p^2=0\to f(\phi_0)=\dfrac{e}{p},\ \ \dfrac{df}{d\phi_0}=0\to v_0=v_1=\dfrac{e^2}{f(\phi_0)}+f(\phi_0)p^2\right\},\\[2ex] \dfrac{\partial\varepsilon}{\partial e}=0\Rightarrow\ Q=e\dfrac{v_1}{v_0}\dfrac{1}{f(\phi_0)}\Rightarrow Q=\dfrac{e}{f(\phi_0)}\\[2ex] \dfrac{\partial\varepsilon}{\partial p}=0\Rightarrow\ P=p\dfrac{v_1}{v_0}f(\phi_0)\Rightarrow P=pf(\phi_0)\end{cases}$$

· Scalarized extremal BHs with constant scalar hair

EMS theory with two U(1) fields:

- $f(\phi)=e^{\alpha\phi^3} \Rightarrow f(\phi_0)=\dfrac{P}{Q} \Rightarrow \phi_0=\dfrac{1}{\alpha^{1/3}}\left(\ln\left[\dfrac{P}{Q}\right]\right)^{1/3}$:

  Scalarized extremal BH with $\phi=\phi_0, v_0=v_1=\dfrac{e^2}{f(\phi_0)}+f(\phi_0)p^2=2ep=2QP=r_e^2$

- Entropy for extremal BHs with constant scalar hair $\Rightarrow \varepsilon=\pi v_1=\pi(2QP)$: No change

- $f(\phi)=1-\alpha\phi^2+\beta\phi^4 \Rightarrow \dfrac{df}{d\phi_0}=(2\alpha\phi-4\beta\phi^3)|_{\phi=\phi_0}=0 \Rightarrow \phi_0=\sqrt{\dfrac{\alpha}{2\beta}}, f(\phi_0)=1-\dfrac{\alpha^2}{4\beta}$:

  Scalarized extremal BH with $\phi=\phi_0,\ v_0=v_1=Q^2 f(\phi_0)+\dfrac{P^2}{f(\phi_0)}=r_e^2$

- Entropy for extremal BHs with constant scalar hair

$$\Rightarrow \varepsilon=\pi v_1 \geq \pi(Q^2+P^2)(Q<P,\alpha<2\sqrt{\beta})$$

7. Discussions

How to obtain scalarized extremal BHs?

- cBBMB BH was obtained by imposing on the asymptotic flat condition ($\boxed{\delta(r)=0}$) for its numerical (series) solution. In this case, $Q_s=\sqrt{M^2-Q^2}$ (dependent $\Rightarrow$ secondary)

- Scalarized extremal BHs with constant scalar hair was obtained when imposing $\boxed{\delta(r)=0 \rightarrow Q_s=0}$ on the scalarization process. Also, this was found from Entropy function approach.

- Up to now,
  no scalarized extremal BHs with primary scalar hair.