One-loop Partition Functions from Quasinormal Modes and Trace Formulas

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Based on my ongoing work with Cynthia Keeler and Carlos Perez-Pardavila

Motivation: Quantum Gravity Effects

- Quantum gravity effects are typically associated with the Planck scale.

$$\ell_p = \sqrt{\frac{\hbar G}{c^3}} \approx 1.616 \times 10^{-35} \text{ m}$$

- Two main questions

  1. Regimes where QG corrections to classical observables become dominant
  2. Physical process where QG effects are observable

Motivation: Enhancement of QG effects

- Horizons exhibit a quantum uncertainty set by the geometric mean of the UV and IR scales. [Marolf (2003), Verlinde-Zurek (2019)]

$$\delta L \sim \sqrt{\ell_p \cdot L} \qquad \text{(in 4D spacetime)}$$

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- The statistical description of black holes breaks down at low temperature due to large quantum fluctuations.

[Preskill-Schwarz-Shapere-Trivedi-Wilczek (1991), Iliesiu-Turiaci (2022)]

$$S(T) \supset \log T^{3/2}$$

Motivation: Gravitational Path Integral

- Gravitational path integral and semiclassical expansion on a fixed background: [Gibbons–Hawking, (1977)]

$$Z = \int Dg\, D\phi\, e^{-S(g,\phi)} \simeq \int D\phi\, e^{-S(g_c,\phi)}$$

- One-loop is the leading quantum correction; encodes spectral data (determinant of the kinetic operator $\mathcal{O}$)

$$Z \simeq \underbrace{(\det \mathcal{O})^{-1/2}}_{Z^{\text{1-loop}}} e^{-S_{\text{cl}}} \quad \text{(for scalar field)}$$

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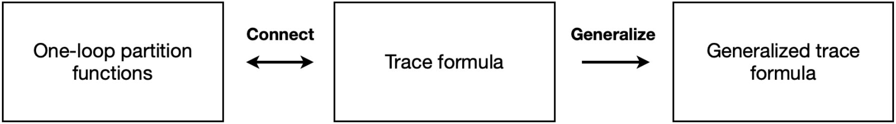
- Questions:
  1. How to compute one-loop partition functions beyond highly symmetric cases?
  2. Understand the large quantum fluctuations in the near-extremal BHs from the four-dimensional Lorentzian perspective.

Our Approach



Goal

Connection between one-loop partition function & trace formula
$\rightarrow$ Toward one-loop partition functions on more general backgrounds

Plan of My Talk

1. Review of heat kernels and quasinormal modes in BTZ black holes
2. BTZ one-loop partition functions from trace formulas
3. Results on de Sitter space and product manifolds

One-loop partition function: heat kernel method

- One-loop partition function $\sim$ One-loop determinant

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- One-loop determinant via the trace of heat kernel operator $K = e^{-t\mathcal{O}}$

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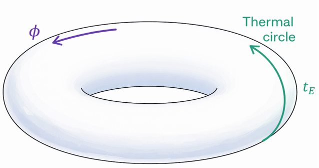
- Solve the differential equation in position basis:

$$(\partial_t + \mathcal{O}_x) \, K(t, x, y) = 0$$

Heat kernels for massive scalar fields in $\mathbb{H}^3/\mathbb{Z}$ [Giombi-Maloney-Yin (2008)]

- Euclidean BTZ: quotient space $\mathbb{H}^3/\mathbb{Z}$

$$(t_E, \phi) \sim (t_E + \beta, \phi + \theta)\,, \qquad (t_E, \phi) \sim (t_E, \phi + 2\pi)$$



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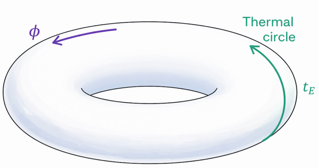

- For quotient geometries, the heat kernel involves the image sum

$$K_{\mathbb{H}^3/\mathbb{Z}}(t, x, x') = \sum_{\gamma \in \mathbb{Z}} K_{\mathbb{H}^3}\left(t, x, \gamma x'\right)$$

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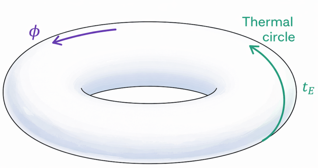

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- One-loop partition function (image sum k, modular parameter $\tau = \frac{1}{2\pi}(\theta + i\beta)$)

$$\boxed{\log Z^{\text{1-loop}} = (\text{divergent term}) + \sum_{k=1}^{\infty} \frac{e^{-2\pi k \tau_2 \sqrt{1+m^2}}}{4k\,|\sin(\pi k \tau)|^2}}$$

One-loop partition function via quasinormal modes

[Denef-Hartnoll-Sachdev (2009)]

- If $Z^{\text{1-loop}}(\Delta)$ is meromorphic $\implies$ $Z^{\text{1-loop}}$: product over poles/zeros:

$$Z^{\text{1-loop}}(\Delta) \;=\; e^{\mathrm{Pol}(\Delta)}\, \frac{\prod(\Delta-\Delta_0)^{d_0}}{\prod(\Delta-\Delta_p)^{d_p}}$$

- Poles occur when the operator develops Euclidean zero-modes:

$$\mathcal{O}(\Delta_p)\,\phi = 0$$

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- DHS formula: Poles occur when quasinormal frequency $\omega_*(\Delta)$ coincides with Matsubara frequency ω_n (for scalar fields)

$$Z^{\text{1-loop}}(\Delta) \;=\; e^{\mathrm{Pol}(\Delta)}\prod_{n,*}(\omega_n-\omega_*(\Delta))^{-1}$$

Connecting QNM to heat kernel: BTZ example

[Keeler-Martin-Svesko (2018)]

- DHS product over QNM frequencies (angular momentum ℓ and radial quantum number p):

$$\omega_* = \pm\ell - 2\pi i T_{L,R}(2p + \Delta), \qquad p \in \mathbb{Z}_{\geq 0},\ \ell \in \mathbb{Z}$$

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- Expand the log-term $\log(1 - x) = -\sum_{k \geq 1} x^k/k$.
- DHS formula reproduces the heat kernel expression exactly: the integer k becomes the image-sum index in the heat kernel method.

$$\boxed{\log Z^{\text{1-loop}} \ \triangleq\ \sum_{k=1}^{\infty} \frac{e^{-2\pi k \tau_2 \sqrt{1+m^2}}}{4k\,|\sin(\pi k \tau)|^2}}$$

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Takeaway

Connection between the Euclidean path integral and the Lorentzian path integral

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1. Review of heat kernels and quasinormal modes in BTZ black holes
2. BTZ one-loop partition functions from trace formulas
3. Results on de Sitter space and product manifolds

Trace formula: general idea [Bogomolny (2003)]

- Trace formula

$$\text{Quantum spectral sum} \quad \leftrightarrow \quad \text{Classical orbit sum}$$

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Quantum spectral sum $\leftrightarrow$ Classical orbit sum

- Basically, calculating the density of states

$$d(E) = -\frac{1}{\pi}\int \mathrm{Im}\, G_E(\mathbf{x},\mathbf{x})\mathrm{d}\mathbf{x}$$

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$$d(E) = -\frac{1}{\pi}\int \mathrm{Im}\, G_E(\mathbf{x},\mathbf{x})\mathrm{d}\mathbf{x}$$

- As a concrete example, we will see

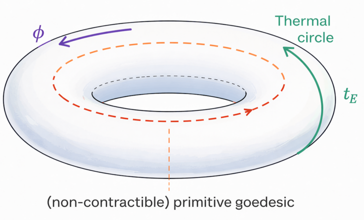
$$\text{Selberg trace formula for } \mathbb{H}/\mathbb{Z} \quad \xrightarrow[\text{generalization}]{} \quad \text{Gutzwiller trace formula}$$

Selberg trace formula for $\mathbb{H}_3/\mathbb{Z}$ [Bogomolny (2003), Perry-Williams (2003)]

- Selberg trace formula for a test function h and its Fourier transf g:

$$\boxed{\sum_j h(\lambda_j) = \underbrace{(\text{volume term})}_{k=0} + \sum_{\gamma\in\text{p.p.o.}} \sum_{k=1}^{\infty} \frac{\ell_p}{2|\sin \pi k\tau|^2} g\left(k\ell_p\right)}$$

1. spectral sum = geometric sum over primitive periodic orbits (and their winding numbers k)
2. primitive periodic orbit (p.p.o.): a closed orbit with minimal length that is not a repetition of of a shorter orbit.
3. ($\checkmark$) If we choose $h(\lambda) = e^{-\lambda t}$, the spectral sum becomes the heat kernel trace $\mathrm{Tr}\, e^{-t\mathcal{O}}$.



Gutzwiller trace formula [Bogomolny (2003)]

- Density of states from semi-classical Green's function:

$$\boxed{d(E) = (volume) + \frac{1}{\pi\hbar} \sum_{\gamma \in \text{p.p.o}} \sum_{k=1}^{\infty} \frac{T_\gamma}{\sqrt{|\det(M_\gamma^k - I)|}} e^{k\left(\frac{S_\gamma}{\hbar} - \frac{\pi}{2}\mu_\gamma\right)} + \cdots}$$

1. In Hamilton-Jacobi, $S_\gamma = \oint_\gamma \mathbf{p} \cdot d\mathbf{q}$: classical action along γ.
2. Monodromy matrix M_γ encodes the linearized deviation along a periodic orbit,

$$\delta x(T_\gamma) = M_\gamma \, \delta x(0)$$

3. (✓) If we consider $\mathcal{M} = \mathbb{H}^3/\mathbb{Z}$, it reproduces the Selberg trace formula.

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Main Points

Gutzwiller trace formula is applicable to a generic manifold with isolated primitive orbits (where $\det(M_\gamma^k - I) \neq 0$).

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de Sitter space is different!

- $d+1$-dimensional Euclidean de Sitter space dS_{d+1} is a sphere S^{d+1}.

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- $d + 1$-dimensional Euclidean de Sitter space dS_{d+1} is a sphere S^{d+1}.
- Euclidean de Sitter space (sphere) has continuous/infinite families of closed geodesics.
- $\det(M_\gamma - 1) = 0 \implies$ Gutzwiller trace formula is not useful.

From Poisson resummation to dS heat kernel trace [Marklof (2008)]

- Instead, the spectral sum is encoded in the Poisson resummation (sphere trace formula). For S^2,

$$\boxed{\sum_{r=0}^{\infty} \underbrace{(2r+1)}_{D_r^{(d+1)}|_{d=1}} h\left(r+\frac{1}{2}\right) = (\mathrm{vol.}) + \sum_{n\neq 0}(-1)^n \int_{-\infty}^{\infty} |\rho|\, h(\rho)\, e^{2\pi i n\rho}\, d\rho}$$

1. Poisson resummation connects the sum over eigenmodes on S^2 to a sum over closed geodesics on the sphere.
2. (✓) If we choose $h(\rho) = e^{-\tau(\rho^2+m^2-\frac{1}{4})}$, the spectral sum becomes the heat kernel trace $\mathrm{Tr}\, e^{-t\mathcal{O}}$.

$$\mathrm{Tr}\, K = \sum_{r=0}^{\infty}(2r+1)\, e^{-\tau\left(\left(r+\frac{1}{2}\right)^2+m^2-\frac{1}{4}\right)}$$

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3. We can generalize the relation to the case of $S^p \times S^q$. (Next, for $AdS_2 \times S^p$?) [In progress with C. Keeler and C. Perez-Pardavila]

Summary and Outlook

- We are developing a systematic way to compute one-loop partition functions in more generic manifolds by using trace formulas.
- We classified two geometric types, whose properties lead to different spectral structures in the corresponding one-loop partition functions.
  1. AdS-type: isolated primitive geodesics
  2. dS-type: infinite families of closed geodesics
- Outlook:
  1. Product spaces:
     - $S^2 \times S^2$ (for Nariai black holes in de Sitter space)
     - $AdS_2 \times S^2$ (for near-extremal RN black holes)
  2. Gauge fields and graviton
  3. Implications for QG enhancement in 4D near-extremal black holes?

Thank you for listening

Questions or Comments?

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Appendix: Group theoretic structure in the heat kernel method

[Keeler-Martin-Svesko (2018)]

- One-loop partition function from the heat kernel method

$$Z^{\text{1-loop}}(\tau, \bar{\tau}) = e^{\mathrm{Pol}(\Delta)} \prod_{\tilde{\ell}, \tilde{\ell}'=0}^{\infty} \frac{1}{1 - q^{\tilde{\ell}+\Delta/2}\, \bar{q}^{\,\tilde{\ell}'+\Delta/2}}$$

where $q = e^{2\pi i \tau}$ and $\tilde{\ell} = p + |\ell|,\ \tilde{\ell}' = p$.

- This is a character of the Virasoro algebra; the product structure reflects $SL(2, \mathbb{Z})$ of the BTZ geometry.

Appendix: Quasinormal mode method

- Compute quasinormal frequencies

$$\omega_* = \omega_*(\Delta; p, \ell, \dots).$$

  - Example (BTZ scalar QNM):

$$\omega_* = -\ell - 2\pi i T_R(2p+\Delta), \quad \ell - 2\pi i T_L(2p+\Delta).$$

- Impose thermal regularity:

$$\omega_n = 2\pi i n T \quad \text{(or rotating generalization)}.$$

- Matching condition

$$\omega_*(\Delta) = \omega_n$$

  determines the pole structure of $Z^{(1)}(\Delta)$.
- Reconstruct the determinant using Weierstrass factorization:

$$Z^{(1)}(\Delta) = e^{\mathrm{Pol}(\Delta)} \prod_{\text{poles}} (\Delta - \Delta_p)^{-d_p}.$$

Appendix: Massive scalar fields on $\mathrm{H}^3/\mathbb{Z}$

- QNM frequencies:

$$\omega_* = \pm\ell - 2\pi i T_{L,R}(2p + \Delta).$$

- Matching with Matsubara modes produces a product representation:

$$Z^{\text{1-loop}}(\tau,\bar\tau) = e^{\mathrm{Pol}(\Delta)} \prod_{\tilde\ell,\tilde\ell'} \frac{1}{1 - q^{\tilde\ell + \Delta/2}\bar q^{\tilde\ell' + \Delta/2}}.$$

- Equivalent to the heat kernel result for $\mathrm{H}^3/\mathbb{Z}$.

Appendix: Selberg trace formula and zeta function

- Trace formula (schematic):

$$\sum_j h(\lambda_j) = (\text{continuous/tunneling term}) + \sum_{\gamma \in P} \sum_{n \geq 1} \frac{\ell_\gamma g(n\ell_\gamma)}{2\sinh(n\ell_\gamma/2)}.$$

1. Selberg zeta (Euler product form):

$$Z_\Gamma(s) = \prod_{p.p.o.} \prod_{m=0}^{\infty} \left(1 - e^{-(m+s)\ell_p}\right).$$

2. The derivative $Z'(s)/Z(s)$ is associated with the trace formula with a particular choice of the test function (resolvent trace formula).

Appendix: Zeta zeros and spectral data

- Zeta zeros $s_k = \frac{1}{2} + ik_j$ encode Laplacian eigenvalues / resonances.
- Relation:

$$\frac{Z'(s)}{Z(s)} = \sum_{p.p.o.} \ell_p \sum_{n\geq 1} \frac{e^{-n\ell_p(s-1/2)}}{2\sinh(n\ell_p/2)}.$$

- Combined with trace formula, zeros $\leftrightarrow$ spectrum $\leftrightarrow$ QNMs; the zeta provides the meromorphic structure.

Appendix: Derivation of Selberg trace formula

- Quantum problem on hyperbolic surface:

$$(\Delta_{LB} + E_n)\Psi_n = 0,$$

where

$$\Delta_{LB} = y^2(\partial_x^2 + \partial_y^2).$$

- Eigenvalue density from Green function:

$$d(E) = -\frac{1}{\pi}\,\mathrm{Im}\int_D G_E(x,x)\,d\mu.$$

- For quotient geometry $\mathbb{H}/\Gamma$, automorphic Green function:

$$G_E(x,x') = \sum_{g\in\Gamma} G_E^{(0)}(x,gx').$$

- This converts spectral data into a sum over group elements.

Appendix: Group elements and periodic orbits

- Hyperbolic elements of Γ (i.e. $|\mathrm{Tr}\, g| > 2$) correspond to classical periodic orbits.
- Orbit length determined by

$$2\cosh\frac{\ell_p}{2} = |\mathrm{Tr}\, g|.$$

- Density splits into

$$d(E) = \bar{d}(E) + d_{\mathrm{osc}}(E).$$

1. Identity element: mean density
2. Non-trivial classes: oscillatory terms

Appendix: Selberg trace formula (explicit form)

After integrating against a test function $h(k)$:

$$\sum_n h(k_n) = \frac{\mu(D)}{2\pi} \int_{-\infty}^{\infty} k h(k) \tanh(\pi k)\, dk$$

$$+ \sum_{\text{p.p.o.}} \ell_p \sum_{m=1}^{\infty} \frac{1}{2 \sinh(m\ell_p/2)} g(m\ell_p),$$

where

$$g(\ell) = \frac{1}{2\pi} \int_{-\infty}^{\infty} h(k) e^{-ik\ell} dk.$$

1. First term $\rightarrow$ homogeneous part
2. Second term $\rightarrow$ sum over primitive periodic orbits
3. Exact trace formula (not semiclassical)

Appendix: Hamilton-Jacobi formalism

Stationary phase condition:

$$\frac{\partial S_\gamma}{\partial x} = 0 \quad \Rightarrow \quad \text{periodic orbit.}$$

Classical action:

$$S_p(E) = \oint_p \mathbf{p} \cdot d\mathbf{q}.$$

Action derivative and phase:

$$\frac{dS_p}{dE} = T_p \quad \Longrightarrow \quad \frac{S_p(E)}{\hbar} = \frac{1}{\hbar} \int^E T_p(E')\, dE'$$

For classical orbits, monodromy matrix captures the geodesic deivation along a periodic orbit:

$$\delta x(T_p) = M_p \delta x(0).$$

Appendix: Comparing different approaches

Euclidean / Geometric

- Heat kernel and image-sum methods
- Selberg zeta and trace formulas

Lorentzian / Dynamical

- Quasinormal modes (QNM)
- DHS meromorphic-product method

Appendix: Product manifolds

- Heat kernel trace factorizes for product manifolds, e.g, for $S^p \times S^q$

$$\mathrm{Tr}\, K_{S^p \times S^q}(\tau) = e^{-\tau m^2} \cdot \mathrm{Tr} e^{\tau \nabla^2_{S^p}} \cdot \mathrm{Tr} e^{\tau \nabla^2_{S^q}}$$

- Thus, we can construct the sphere partition function from the Poisson resummation as in the case of a single sphere.

- For $AdS_2 \times S^p$, we need to consider the different relationship between the trace formula and one-loop partition function. [In progress with C. Keeler and C. Perez-Pardavila]