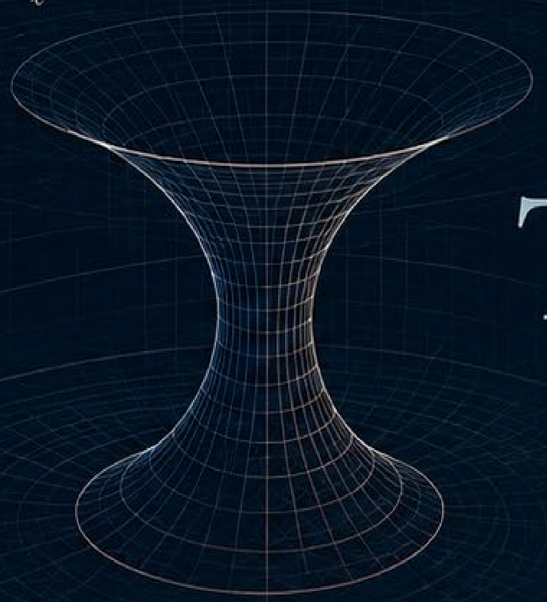
$$S = \frac{1}{16\pi G}\int_M (R - 2\Lambda)\sqrt{-g}\, d^3x$$

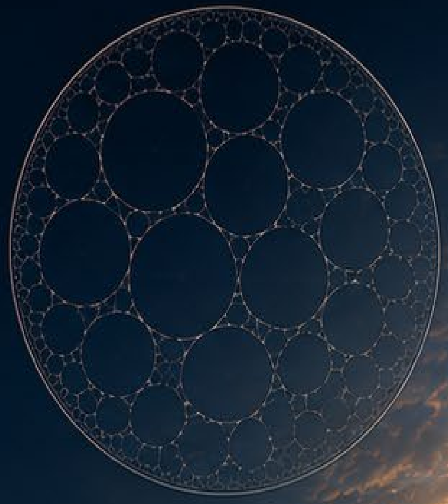
$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = 0$$

$$\Lambda = -\frac{1}{\ell^2}$$

AdS_3



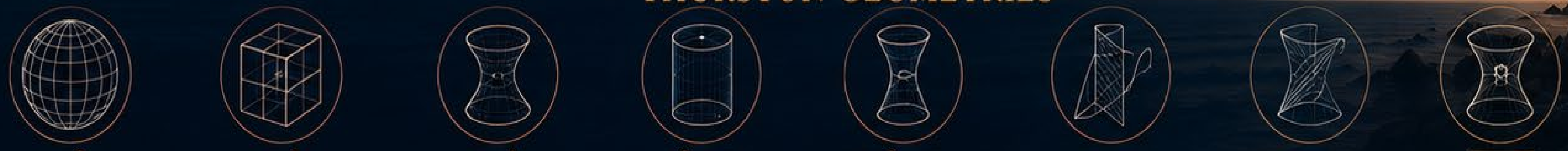
Three-dimensional Gravity and Thurston Geometry



CQUeST 2026 (July 20~24)

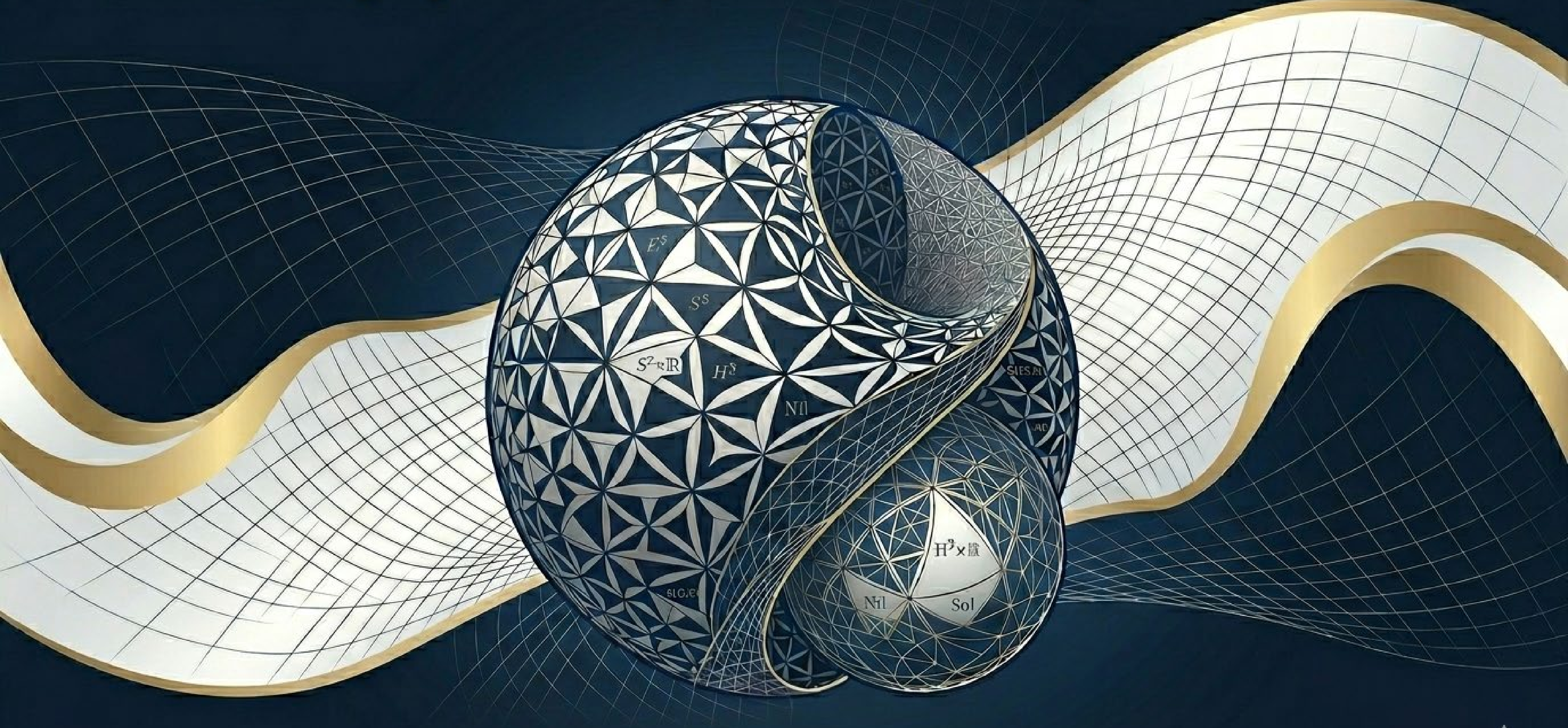
Hyun Seok Yang (GIST)

THURSTON GEOMETRIES



S^3	E^3	H^3	$S^2 \times \mathbb{R}$	$H^2 \times \mathbb{R}$	Nil	Sol	$\widetilde{SL_2\mathbb{R}}$
(ELLIPTIC)	(EUCLIDEAN)	(HYPERBOLIC)	(SPHERICAL)	(HYPERBOLIC)	(NILPOTENT)	(SOLVABLE)	($SL_2\mathbb{R}$)

Based on an ongoing work with my two students (Jin-Young Park and Yuheun Sun)



Dynamics of Gravity

Let us count the number of independent components of Riemann curvature tensors in $D = d + 1$ dimensions:

Riem = Weyl $\oplus$ Ricci, Ricci = Traceless Ricci $\oplus$ Ricci scalar.

$$\#(\text{Riem}) = \frac{D^2(D^2-1)}{12}, \quad \#(\text{Ricci}) = \frac{D(D+1)}{2}, \quad \#(\text{Weyl}) = \frac{D(D+1)(D+2)(D-3)}{12};$$

In $D = 4$, $\#(\text{Riem}) = 20$, $\#(\text{Ricci}) = 10$, $\#(\text{Weyl}) = 10$.

The only dimension with $\#(\text{Ricci}) = \#(\text{Weyl})$.

The only dimension with semi-simple Lorentz group $\mathrm{SO}(4) = \mathrm{SU}(2)_+ \times \mathrm{SU}(2)_-/\mathbb{Z}_2$.

Weyl tensor components: $R_{efab} = \pm\frac{1}{2}\varepsilon_{ab}{}^{cd} R_{efcd}$ (Gravitational Instantons)

Ricci tensor components: $R_{ab} - \frac{1}{2}\delta_{ab}(R - 2\Lambda) = \frac{8\pi G}{c^4} T_{ab}$ (Einstein Equations)

More generally, Einstein manifolds satisfying $R_{ab} = \Lambda\delta_{ab}$ are gravitational instantons $\oplus$ anti-instantons $\cong SU(2)_+$ Yang-Mills instantons $\oplus$ $SU(2)_-$ Yang-Mills anti-instantons.

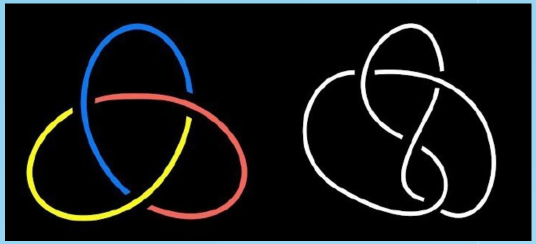
Dynamics of Gravity

$$\#(\mathrm{Riem}) = \frac{D^2(D^2-1)}{12}, \qquad \#(\mathrm{Ricci}) = \frac{D(D+1)}{2}, \qquad \#(\mathrm{Weyl}) = \frac{D(D+1)(D+2)(D-3)}{12};$$

In $D = 3$, $\#(\mathrm{Riem}) = 6$, $\#(\mathrm{Ricci}) = 6$, $\#(\mathrm{Weyl}) = 0$.

The only dimension with $\#(\mathrm{Weyl}) = 0 \iff \#(\mathrm{Riem}) = \#(\mathrm{Ricci})$.

The only dimension with gauge fields $\cong$ scalar fields ($* F = d\phi$, Knot invariants).



The full curvature tensor in three dimensions is totally determined by the Ricci tensor

$$R_{\mu\nu\rho\sigma} = g_{\mu\rho}R_{\nu\sigma} + g_{\nu\sigma}R_{\mu\rho} - g_{\nu\rho}R_{\mu\sigma} - g_{\mu\sigma}R_{\nu\rho} - \frac{1}{2}R(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho}).$$

Any solution of Einstein equations has constant curvature;

$$R_{\mu\nu\rho\sigma} = \Lambda(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho}).$$

Physically, this means that on three-dimensional Einstein spacetimes there are no local propagating degrees of freedom: The theory is topological.

D = 2

- $\#(\text{Riem}) = \frac{D^2(D^2-1)}{12}$, $\#(\text{Ricci}) = \frac{D(D+1)}{2}$, $\#(\text{Weyl}) = \frac{D(D+1)(D+2)(D-3)}{12}$;

- We cannot use the above counting for $D = 2$ because we have used the first Bianchi identity

$$R_{a[bcd]} = 0,$$

which becomes trivial in $D = 2$. Indeed, $\#(\text{Riem}) = 1$ and $\#(\text{Ricci}) = 1$.

- Furthermore, the Gauss-Bonnet theorem asserts that

$$\chi(M) = \frac{1}{2\pi}\int_M d^2x\,\sqrt{g}\,R = 2 - 2g,$$

where $\chi(M)$ is the Euler characteristic of a two-dimensional surface M with genus g.

- Jackiw–Teitelboim (JT) gravity (Claudio Teitelboim changed his name to Claudio Bunster in 2005):

$$S_{JT}(M) = -\frac{1}{16\pi G}\int_M d^2x\,\sqrt{g}\,\phi(R+2) - \frac{1}{8\pi G}\int_{\partial M} d^2u\,\sqrt{h}\,\phi(K-1).$$

3-dimensional Gravity

- Three-dimensional gravity has no local propagating degrees of freedom so that the theory is basically topological.
- The surprise is that there exists a precise equivalence of gravity and gauge theory in three dimensions.
- The three-dimensional gauge theory should also be a topological to ensure no local propagating degrees of freedom.
- The well-known topological gauge theory in three dimensions is the Chern-Simons gauge theory:

$$S_{CS}[A] = \frac{k}{4\pi}\int_M \mathrm{Tr}\left[A \wedge dA + \frac{2}{3}A \wedge A \wedge A\right],$$

where k is a constant called level which is related to the gravitational constant G; for instance, $k = \frac{\ell}{4G}$ for AdS_3 with $\Lambda = -\frac{1}{\ell^2}$.

- The gauge group G of the gauge theory depends on a topology of a three-dimensional manifold M and the rank of G is 6 or smaller than 6 in order to accommodate 3 dreibeins $E^a = E^a_\mu dx^\mu$ and 3 spin connections $\Omega^a = \frac{1}{2}\varepsilon^{abc}\,\omega_{bc} = \Omega^a_\mu dx^\mu$.

Chern-Simons Gauge Theory of AdS_3 Gravity

- Three–dimensional anti–de Sitter space AdS_3 can be defined as a quadric (hyperboloid) embedded in a flat 4-dimensional space with two time directions.

Take $\mathbb{R}^{2,2}$ with coordinates $X^A = (X^{-1}, X^0, X^1, X^2)$ with metric

$$ds^2 = -\,dX_{-1}^2 - dX_0^2 + dX_1^2 + dX_2^2 = \eta_{AB} dX^A dX^B,$$

where $\eta_{AB} = \mathrm{diag}(-1, -1, 1, 1)$.

The anti–de Sitter space AdS_3 is the 3-dimensional hyperboloid

$$-X_{-1}^2 - X_0^2 + X_1^2 + X_2^3 = -\ell^2$$

embedded in flat $\mathbb{R}^{2,2}$.

Therefore the underlying algebra in AdS_3 must be completely determined by the Lorentz algebra of $\mathbb{R}^{2,2}$, i.e. $SO(2,2)$:

$$[J_{AB}, J_{CD}] = -\,(\eta_{AC} J_{BD} - \eta_{AD} J_{BC} - \eta_{BC} J_{AD} + \eta_{BD} J_{AC}),$$

where J_{AB} with $A, B = -1, 0,1,2$ are $so(2,2)$ Lorentz generators.

Chern-Simons Gauge Theory of AdS_3 Gravity

- The most general Killing form in the defining representation is

$$\mathrm{Tr}\,(J_{AB}J_{CD}) = -\mu\,(\eta_{AC}\eta_{BD} - \eta_{AD}\eta_{BC}) - \nu\,\varepsilon_{ABCD}.$$

The ν-term is allowed for four-dimensional Lorentz algebras, especially, $SO(4)$ or $SO(2,2)$, due to the existence of the self-dual structure defined by

$$\tilde{J}_{AB} \equiv (*J)_{AB} = \frac{1}{2}\varepsilon_{AB}{}^{CD}J_{CD}, \qquad (*^2 J)_{AB} = J_{AB}.$$

Therefore there exist projection operators acting on the $so(4)$ or $so(2,2)$ Lie algebra vector space defined by $P_\pm \equiv \frac{1}{2}(1 \pm *)$ satisfying the properties

$$P_\pm^2 = P_\pm, \quad P_+ + P_- = 1, \quad P_\pm P_\mp = 0.$$

In terms of components, the projection operators can be written as

$$P_{(\pm)AB}{}^{CD} = \frac{1}{4}(\delta_A^C\delta_B^D - \delta_A^D\delta_B^C \pm \varepsilon_{AB}{}^{CD}).$$

Chern-Simons Gauge Theory of AdS_3 Gravity

- Define (so-called) chiral generators $J_{AB}^{(\pm)} \equiv P_\pm J_{AB}$.

Then one can see that they separately satisfy the Lorentz algebra, i.e.,

$$\left[J_{AB}^{(\pm)}, J_{cD}^{(\pm)}\right] = -\left(\eta_{AC}J_{BD}^{(\pm)} - \eta_{AD}J_{BC}^{(\pm)} - \eta_{BC}J_{AD}^{(\pm)} + \eta_{BD}J_{AC}^{(\pm)}\right),$$

$$\left[J_{AB}^{(\pm)}, J_{cD}^{(\mp)}\right] = 0.$$

The chiral algebras verify the group isomorphism $SO(2,2) = SL(2,\mathbb{R})_+ \times SL(2,\mathbb{R})_-$.

Then one can show that the bilinear pairing reduces

$$\mathrm{Tr}\left(J_{AB}^{(\pm)} J_{cD}^{(\pm)}\right) = -2\,(\mu \mp \nu) P_{(\pm)ABCD}, \qquad \mathrm{Tr}\left(J_{AB}^{(\pm)} J_{cD}^{(\mp)}\right) = 0.$$

Now we split the Lorentz generators $J_{AB} = \left(J_{-1,a} \equiv P_a,\ \ J_{ab} \equiv \varepsilon_{ab}{}^{c}\, J_c\right)$,

where the $\mathbb{R}^{2,2}$ index $A = (-1, a)$ is accordingly split.

Here we choose the orientation as $\varepsilon_{(-1)012} = \varepsilon_{012} = 1$.

Chern-Simons Gauge Theory of AdS_3 Gravity

- Then the $SO(2,2)$ Lorentz algebra is also split as

$$[P_a, P_b] = \varepsilon_{ab}{}^{c}\, J_c, \qquad [J_a, P_b] = \varepsilon_{ab}{}^{c}\, P_c, \qquad [J_a, J_b] = \varepsilon_{ab}{}^{c}\, J_c$$

and the Killing form becomes

$$\mathrm{Tr}(P_a P_b) = \mu\eta_{ab}, \qquad \mathrm{Tr}(J_a J_b) = \mu\eta_{ab}, \qquad \mathrm{Tr}(P_a J_b) = \nu\eta_{ab}.$$

The chiral generators are similarly split as

$$J_a^{(\pm)} \equiv -\frac{1}{2}\varepsilon_a{}^{bc} J_{bc}^{(\pm)} = \frac{1}{2}(J_a \mp P_a), \qquad P_a^{(\pm)} \equiv J_{-1a}^{(\pm)} = \mp J_a^{(\pm)},$$

and satisfy the Lie algebras

$$[J_a^{(\pm)}, J_b^{(\pm)}] = \varepsilon_{ab}{}^{c} J_c^{(\pm)}, \qquad [J_a^{(\pm)}, J_b^{(\mp)}] = 0.$$

The nondegenerate pairing for the chiral generators reduces to

$$\mathrm{Tr}\left(J_a^{(\pm)}\, J_b^{(\pm)}\right) = \frac{1}{2}(\mu \mp \nu)\eta_{ab}, \qquad \mathrm{Tr}\left(J_a^{(\pm)}\, J_b^{(\mp)}\right) = 0.$$

Chern-Simons Gauge Theory of AdS_3 Gravity

- Define an $SO(2,2)$ connection

$$A = \frac{1}{2}\omega^{AB}J_{AB} = \omega^{-1a}J_{-1a} + \frac{1}{2}\omega^{ab}J_{ab} = E^a P_a + \Omega^a J_a,$$

with the identification $\omega^{-1a} = E^a$ and $\Omega^a = \frac{1}{2}\varepsilon^{abc}\,\omega_{bc}$.

The expansion in the basis of chiral generators leads to

$$A = \frac{1}{2}\omega^{AB}\left(J^{(+)}_{AB} + J^{(-)}_{AB}\right) = (\Omega^a - E^a)J^{(+)}_a + (\Omega^a + E^a)J^{(-)}_a$$

$$\equiv A^a_{(+)}J^{(+)}_a + A^a_{(-)}J^{(-)}_a = A^{(+)} + A^{(-)},$$

where $A^a_{(\pm)} = (\Omega^a \mp E^a)$.

In order to recover the cosmolocal constant $\Lambda = -\frac{1}{\ell^2}$, rescale the dreibeins as $E^a = \frac{e^a}{\ell}$ and $P_a \equiv \ell\, p_a$

so that $A = e^a p_a + \Omega^a J_a$.

Then the translation generators obey the algebra $[p_a, p_b] = \frac{1}{\ell^2}\,\varepsilon_{ab}{}^{c}\,J_c$, which commute each other

in the limit $\ell \to \infty$. Thus, in this limit, $SO(2,2) \xrightarrow{\ell\to\infty} ISO(2,1)$.

This shows how to get an asymptotically flat spacetime from the AdS_3 gravity, which may shed light on the flat spacetime holography (ongoing collaboration with Bum-Hoon Lee, Tabasum Rahnuma and Patricio Salgado-Rebolledo).

Chern-Simons Gauge Theory of AdS_3 Gravity

- Let us calculate the two-form curvature

$$F = dA + A \wedge A$$
$$= \left(R^a(\Omega) + \frac{1}{2}\varepsilon^a{}_{bc} E^b \wedge E^c\right) J_a + T^a P_a$$
$$= \mathcal{R}^a J_a + T^a P_a = F^{(+)} + F^{(-)},$$

where $F^{(\pm)} = dA^{(\pm)} + A^{(\pm)} \wedge A^{(\pm)}$ and

$$R^a(\Omega) = d\Omega^a + \frac{1}{2}\varepsilon^a{}_{bc}\, \Omega^b \wedge \Omega^c,$$
$$T^a = dE^a + \varepsilon^a{}_{bc}\, \Omega^b \wedge E^c,$$

are Riemann curvature 2-forms and torsion 2-forms, respectively.

The variation of the Chern-Simons action takes the form after integrating by parts,

$$\delta S_{CS}[A] = \frac{k}{4\pi}\int_M \mathrm{Tr}[2\, \delta A \wedge (dA + A \wedge A)] - \frac{k}{4\pi}\int_{\partial M} \mathrm{Tr}[A \wedge \delta A].$$

Therefore the equation of motion is

$$F = dA + A \wedge A = 0.$$

Thus the solution is locally $A = g^{-1} dg$ where $g \in G$, i.e. a pure gauge.

Chern-Simons Gauge Theory of AdS_3 Gravity

- The curvature-free condition, $F=0$, implies that

$$T^a = dE^a + \varepsilon^a{}_{bc}\,\Omega^b \wedge E^c = 0 \qquad \text{(Torsion-free condition),}$$

$$\mathcal{R}^a\,(\Omega) = R^a(\Omega) + \frac{1}{2}\varepsilon^a{}_{bc}E^b \wedge E^c = 0 \qquad \text{(Einstein's equation).}$$

Proof) Using the relation $\omega^a{}_b = -\varepsilon^a{}_{bc}\Omega^c$,

$$R^a{}_b(\omega) = d\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b = -\varepsilon^a{}_{bc}d\Omega^c + \varepsilon^a{}_{cd}\,\varepsilon^c{}_{be}\Omega^d \wedge \Omega^e$$

$$= -\varepsilon^a{}_{bc}d\Omega^c + (\delta^a_b\eta_{de} - \delta^a_e\eta_{bd})\,\Omega^d \wedge \Omega^e = -\varepsilon^a{}_{bc}d\Omega^c + \Omega^a \wedge \Omega_b = -\varepsilon^a{}_{bc}\left(d\Omega^c + \frac{1}{2}\varepsilon^c{}_{de}\,\Omega^d \wedge \Omega^e\right)$$

$$\Longrightarrow R^a(\Omega) = \frac{1}{2}\varepsilon^{abc}R_{bc}(\omega) = \frac{1}{2}\varepsilon^{abc}\left(-\varepsilon_{bcd}d\Omega^d + \Omega_b \wedge \Omega_c\right) = d\Omega^a + \frac{1}{2}\varepsilon^{abc}\Omega_b \wedge \Omega_c.$$

In $D=3$, $\quad R_{abcd} = \varepsilon_{ab}{}^e\,\varepsilon_{cd}{}^f\,S_{ef} \Longrightarrow R_{ac} = \eta^{bd}R_{abcd} = \varepsilon_{ab}{}^e\,\varepsilon_c{}^{bf}S_{ef} = -(\eta_{ac}\eta^{ef} - \delta^f_a\delta^e_c)S_{ef}$

$$= S_{ac} - \eta_{ac}S$$

$$\Longrightarrow R = \eta^{ac}R_{ac} = S - 3S = -2S \Longrightarrow S = -\frac{1}{2}R.$$

$$\Longrightarrow S_{ab} = R_{ab} - \frac{1}{2}\eta_{ab}R \quad \Longrightarrow R_{abcd} = \varepsilon_{ab}{}^e\,\varepsilon_{cd}{}^f\left(R_{ef} - \frac{1}{2}\eta_{ef}R\right).$$

$$R_{ab}(\omega) = \frac{1}{2}R_{abcd}E^c \wedge E^d = \frac{1}{2}\varepsilon_{ab}{}^e\,\varepsilon_{cd}{}^f\left(R_{ef} - \frac{1}{2}\eta_{ef}R\right)\varepsilon^{cd}{}_g(*\,E^g)$$

$$= -\varepsilon_{ab}{}^e\left(R_{ef} - \frac{1}{2}\eta_{ef}R\right)(*\,E^f)$$

$$\Longrightarrow \mathcal{R}^a\,(\Omega) = \frac{1}{2}\varepsilon^{abc}R_{bc}(\omega) + \frac{1}{2}\varepsilon^a{}_{bc}E^b \wedge E^c = -\,\frac{1}{2}\varepsilon^{abc}\,\varepsilon_{bc}{}^e\left(R_{ef} - \frac{1}{2}\eta_{ef}R\right)(*\,E^f) + \frac{1}{2}\varepsilon^a{}_{bc}\,\varepsilon^{bc}{}_f(*\,E^f)$$

$$= \left(R^a{}_b - \frac{1}{2}\delta^a_bR - \delta^a_b\right)(*\,E^b) = 0 \;\Longrightarrow\; R^a{}_b - \frac{1}{2}\delta^a_bR - \delta^a_b = 0 \;\Longrightarrow\; R^a{}_b + 2\,\delta^a_b = 0. \qquad QED$$

Chern-Simons Gauge Theory of AdS_3 Gravity

These results can be confirmed by calculating the action explicitly

$$S_{CS}[A] = \frac{k\mu}{4\pi}\int_M (CS(\Omega) + E^a \wedge T_a) + \frac{k\nu}{4\pi}\int_M (2E^a \wedge R_a(\Omega) + \frac{1}{3}\varepsilon_{abc}E^a \wedge E^b \wedge E^c + d(E^a \wedge \Omega_a))$$
$$= \frac{k(\mu-\nu)}{8\pi}\int_M CS(A^{(+)}) + \frac{k(\mu+\nu)}{8\pi}\int_M CS(A^{(-)}),$$

where $CS(\Omega) = \mathrm{Tr}\left(\Omega \wedge d\Omega + \frac{2}{3}\Omega \wedge \Omega \wedge \Omega\right)$ and $CS\left(A^{(\pm)}\right) = \mathrm{Tr}\left(A^{(\pm)} \wedge dA^{(\pm)} + \frac{2}{3}A^{(\pm)} \wedge A^{(\pm)} \wedge A^{(\pm)}\right)$.

(I) $\mu = 0$: $\delta E^a \Longrightarrow$ Einstein equation, $\delta\Omega^a \Longrightarrow$ Torsion-free condition, $T^a = 0$.

(II) $\nu = 0$: $\delta E^a \Longrightarrow$ Torsion−free condition, $T^a = 0$, $\delta\Omega^a \Longrightarrow$ Einstein equation.

(III) $\mu \neq 0$ & $\nu \neq 0$: Topologically massive gravity with a gravitational Chern-Simons coupling μ.

(Its holographic aspects have not been explored.)

3-dimensional gravity = Gauge theory?

- E. Witten, Three-Dimensional Gravity Reconsidered, arXiv:0706.3359
- In Chern-Simons gauge theory, the dreibein may cease to be invertible.
- There is a classical solution $A = e = \omega = 0$. No spacetime is allowed in gauge theory.
- The symmetry of gauge theory (Diffeomorphism $\oplus$ Gauge symmetry) is bigger than gravity.
- Topological sum in quantum gravity $\not\Leftrightarrow$ In Chern-Simons gauge theory?
- All physical informations: topology & boundary conditions match together?
- In gravity, black hole solutions (BTZ) are allowed with a horizon of finite length and Bekenstein-Hawking entropy.
- Is it possible to accommodate a huge deneracy in Chern-Simons gauge theory which is essentially a topological theory.
- Topology & boundary conditions are not determined by an underlying action.

  But everything is there. Then what is physics?
- Holography is not an answer.

Choice of Gauge Group G

- Maximally symmetric spaces: Dimension of $G=6$.
  Other spaces appearing in Thurston geometries: Dimension of $G \le 4$.

- Asymptotically flat: $G = ISO\,(2,1) = SO(2,1) \ltimes \mathbb{R}^3$,
  $G = ISO\,(3) = SO\,(3) \ltimes \mathbb{R}^3$,
  Asymptotically AdS_3: $G = SO(2,2) = SL(2,\mathbb{R})_+ \times SL(2,\mathbb{R})_-$,
  $G = SO(3,1) = SL(2,\mathbb{C})$, $(-X_{-1}^2 + X_0^2 + X_1^2 + X_2^3 = -\ell^2)$, Topology: $\mathbb{R}^3$,
  Asymptotically dS_3: $G = SO(3,1) = SL(2,\mathbb{C})$, $(-X_{-1}^2 + X_0^2 + X_1^2 + X_2^3 = \ell^2)$, Topology: $\mathbb{R} \times \mathbb{S}^2$,
  $G = SO(4) = SU(2)_+ \times SU(2)_-$.

★ The choice of gauge group ⇔ Topology of three manifolds.
The **Thurston geometries** are the eight possible types of "uniform" 3-dimensional geometries

that can appear as the building blocks of 3-manifolds.

- Thurston conjecture states that a three-manifold with a given *topology* has a canonical decomposition into a connected sum of 'simple three-manifolds',

each of which admits one, and only one, of eight homogeneous *geometries*:

$H^3, \mathbb{S}^3, E^3, \mathbb{S}^2 \times \mathbb{S}^1, H^2 \times \mathbb{S}^1$, Sol, Nil and SL(2, R).

The conjecture has been completely proven by G. Perelman.

Model geometries

Now we return to three dimensions. We again have three constant-curvature model geometries:

- The round sphere S^3 (with constant curvature $+1$);
- The Euclidean space $\mathbf{R}^3$ (with constant curvature 0); and
- The hyperbolic space H^3 (with constant curvature -1).

However, one can also now form model geometries from Cartesian or twisted products of lower-dimensional geometries:

- The product $S^2 \times \mathbf{R}$;
- The product $H^2 \times \mathbf{R}$;
- The (universal cover of) $SL_2(\mathbf{R})$ (a twisted bundle over H^2);
- The Heisenberg group (a twisted bundle over $\mathbf{R}^2$); and
- The three-dimensional solvmanifold (a twisted torus bundle over S^1).

These are the eight Thurston geometries.

Thurston Geometries

- **Nil Geometry**

  Associated with the **Heisenberg group** $\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}$. Twisted but not negatively curved.

  Appears in certain torus bundles Distances behave anisotropically (different directions scale differently).

- **Sol Geometry**

  Very unusual "exponentially stretched" geometry.

  A typical metric is $ds^2 = e^{2z}dx^2 + e^{-2z}dy^2 + dz^2$

  Expands in one direction while shrinking in another. Neither isotropic nor constant curvature

  Sol manifolds arise from torus bundles with hyperbolic monodromy.

- $\widetilde{SL_2}(\mathbb{R})$ **Geometry**

  The universal cover of the Lie group $SL_2(\mathbb{R})$. Closely related to hyperbolic geometry

  Appears in Seifert fiber spaces (locally looks like (surface)$\times\mathbb{S}^1$).

  One of the most technically sophisticated geometries.

Isometry of Thurston Geometries

Geometry	Connected Isometry Group	Dimension
E^3	$SO(3) \ltimes \mathbb{R}^3$	6
S^3	$SO(4)$	6
H^3	$SO^+(3,1) \cong PSL(2,\mathbb{C})$	6
$S^2 \times \mathbb{R}$	$SO(3) \times \mathbb{R}$	4
$H^2 \times \mathbb{R}$	$PSL(2,\mathbb{R}) \times \mathbb{R}$	4
Nil	$\mathrm{Nil} \rtimes SO(2)$	4
Sol	Sol	3
$\widetilde{SL(2,\mathbb{R})}$	$\widetilde{SL(2,\mathbb{R})} \times \mathbb{R}$ (locally)	4

Thurston Geometries

- One can construct three-dimensionalChern–Simons gauge theories in which each gauge group is the group of isometries of a given Thurston geometry.
  This is well known for the isotropic geometries $E^3, \mathbb{S}^3$, and H^3.
- Of the Thurston spaces, only $E^3, \mathbb{S}^3$, and H^3 are solutions of pure Einstein gravity.

Geometry	Metric	ℓ^2	Λ
E^3	$ds^2 = dx^2 + dy^2 + dz^2$	No length scale	0
S^3	$ds^2 = \ell^2(dx^2 + \sin^2 x dy^2 + \sin^2 x \sin^2 y dz^2)$	Arbitrary	$\frac{1}{\ell^2} - \sigma\frac{2(3\alpha+\beta)}{\ell^4}$
H^3	$ds^2 = \ell^2(dx^2 + \cosh^2 x dy^2 + \cosh^2 x \cosh^2 y dz^2)$	Arbitrary	$-\frac{1}{\ell^2} - \sigma\frac{2(3\alpha+\beta)}{\ell^4}$
$E^1 \times S^2$	$ds^2 = \ell^2(dx^2 + dy^2 + \sin^2 y dz^2)$	$\frac{1}{2\Lambda}$	$-\frac{\sigma}{4(2\alpha+\beta)}$
$E^1 \times H^2$	$ds^2 = \ell^2(dx^2 + dy^2 + \cosh^2 y dz^2)$	$-\frac{1}{2\Lambda}$	$-\frac{\sigma}{4(2\alpha+\beta)}$
Nil	$ds^2 = \frac{\ell^2}{4}[dx^2 + dy^2 + (xdy - dz)^2]$	$-\frac{1}{2\Lambda}$	$-\frac{\sigma}{8(\alpha+3\beta)}$
$SL(2,\mathbb{R})$	$ds^2 = \ell^2[dr^2 + \sinh^2 r \cosh^2 r d\theta^2 + (d\psi + \sinh^2 r d\theta)^2]$	$-\frac{25\alpha+23\beta}{10\Lambda(\alpha+\beta)}$	$-\sigma\frac{25\alpha+23\beta}{200(\alpha+\beta)^2}$
Sol	$ds^2 = \ell^2(e^{-2z}dx^2 + e^{2z}dy^2 + dz^2)$	$-\frac{1}{2\Lambda}$	$-\frac{\sigma}{8(\alpha+\beta)}$

Lorentzian Thurston Geometries

- D. Flores-Alfonso , C. S. Lopez-Monsalvo , and M. Maceda, PRL **127**, 061102 (2021); G. Alkaç and D. O. Devecioğlu, PRD **105**, 064023 (2022).

Geometry	Metric	ℓ^2	Λ
Nil	$ds^2=\frac{\ell^2}{4}[dx^2+dy^2-(xdy-dz)^2]$	$\frac{1}{2\Lambda}$	$-\frac{\sigma}{8(\alpha+3\beta)}$
$SL(2,\mathbb{R})$	$ds^2=\frac{\ell^2}{4}[-(d\Psi+\cos\Theta d\Phi)^2+d\Theta^2+\sin^2\Theta d\Phi^2]$	$\frac{25\alpha+23\beta}{10\Lambda(\alpha+\beta)}$	$-\sigma\frac{25\alpha+23\beta}{200(\alpha+\beta)^2}$
New Sol	$ds^2=\ell^2(e^{-2z}dx^2+e^{2z}dy^2-dz^2)$	$\frac{1}{2\Lambda}$	$-\frac{\sigma}{8(\alpha+\beta)}$
Lorentz Sol	$ds^2=\ell^2(2e^{-z}dxdz+e^{2z}dy^2)$	Arbitrary	$\sigma=0$, no effect on the solution
Third Sol	$ds^2=\ell^2(-e^{2z}dy^2-2dxdy+dz^2)$	$-4\sigma\beta$	0
Lorentz-Heisenberg	$ds^2=\frac{\ell^2}{4}(-dx^2+dy^2+(xdy-dz)^2)$	$\frac{1}{2\Lambda}$	$-\frac{\sigma}{8(\alpha+3\beta)}$
AdS	$ds^2=\ell^2[dr^2+\sinh^2 r\cosh^2 r d\theta^2-(d\psi+\sinh^2 r d\theta)^2]$	Arbitrary	$-\frac{1}{\ell^2}-\sigma\frac{2(3\alpha+\beta)}{\ell^4}$

Three-Dimensional New Massive Gravity

- ☞ Since many Thurston geometries are not Einstein spaces, one might wonder if a "master theory" that admits all TGs as solutions can be achieved by a modification of Einstein's equation.
- It was shown that the three-dimensional Thurston geometries are vacuum solutions to the 3D new massive gravity equations of motion. (D. Flores-Alfonso , C. S. Lopez-Monsalvo , and M. Maceda, PRL **127**, 061102 (2021))

$$\Lambda g + G(g) - \frac{1}{2m^2} K(g) = 0.$$

$$S = \frac{1}{\kappa^2} \int d^3x \sqrt{g} \left[R + \frac{1}{m^2} K \right],$$

$$K_{\mu\nu} = 2\Box R_{\mu\nu} - \frac{1}{2}\nabla_\mu \nabla_\nu R - \frac{1}{2}\Box R g_{\mu\nu} + 4R_{\mu\alpha\nu\beta}R^{\alpha\beta} - \frac{3}{2}RR_{\mu\nu} - R_{\alpha\beta}R^{\alpha\beta}g_{\mu\nu} + \frac{3}{8}R^2 g_{\mu\nu}.$$

$$K = R_{\mu\nu}R^{\mu\nu} - \tfrac{3}{8}R^2.$$

Peleman's Gradient Flow

- The logic of a uniformization theorem along the lines of Hamilton's flow is to assume that the manifold admits some metric, and one needs to devise a mechanism to flow the initial metric into a highly symmetric one, which can be obtained by the following parabolic system:

$$\partial_t g_{\mu\nu} = \mathcal{O}(g_{\mu\nu}),$$

where $\mathcal{O}$ is an elliptic operator.

- The parabolicity of the Hamilton flow ensures that the flow will end at fixed points as $t \to \infty$, for which $\mathcal{O}(g_{\mu\nu}) \to 0$. However, since $\mathcal{O}(g_{\mu\nu})$ should transform as a tensor, it has to be constructed from curvature tensors, which makes it impossible to obtain a strictly parabolic system and leads to nonlinearity due to the inverse metric. Many issues regarding these complications should be resolved for a full proof.

Peleman's Gradient Flow

- Perelman made use of the operator

$$\mathcal{O}(g_{\mu\nu}) = -2(R_{\mu\nu} + \nabla_\mu \nabla_\nu f),$$

where f is a scalar field.
It follows from the field equations of the action that

$$S[f, g] = \int d^3x \sqrt{|g|} e^{-f} [R + (\nabla f)^2],$$

and the action is an increasing functional provided that $\partial_t f = \partial_t \ln\sqrt{g}$.

- Perelman finds an operator that vanishes for all Thurston geometries, which satisfies a gradient flow rather than the Hamilton flow.

Ricci Flow As A Gradient Flow

- Consider the functional $S[g,f] = \int_M d^3x\sqrt{g}\, e^{-f}(R + |\nabla f|^2)$ for a Riemannian metric $g_{\mu\nu}$ and a function f on a closed manifold M. Its first variation can be expressed as follows:

$$\delta_g S[g,f] = \int_M d^3x e^{-f}\left(\delta_g\left(\sqrt{g}\, R\right) + \delta_g(\sqrt{g}|\nabla f|^2)\right)$$

$$= \int_M d^3x\sqrt{g}e^{-f}\left[\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\right)\delta g^{\mu\nu} + \left(g_{\mu\nu}\nabla^2 - \nabla_\mu\nabla_\nu\right)\delta g^{\mu\nu} + (\nabla_\mu f\nabla_\nu f - \frac{1}{2}g_{\mu\nu}|\nabla f|^2)\delta g^{\mu\nu}\right]$$

$$= \int_M d^3x\sqrt{g}e^{-f}\left[\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\right) + e^f\left(g_{\mu\nu}\nabla^2 - \nabla_\mu\nabla_\nu\right)e^{-f} + (\nabla_\mu f\nabla_\nu f - \frac{1}{2}g_{\mu\nu}|\nabla f|^2)\right]\delta g^{\mu\nu}$$

+surface terms

$$= \int_M d^3x\sqrt{g}e^{-f}\left[R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R - g_{\mu\nu}(\nabla^2 f - |\nabla f|^2) + \nabla_\mu\nabla_\nu f - \nabla_\mu f\nabla_\nu f + \nabla_\mu f\nabla_\nu f - \frac{1}{2}g_{\mu\nu}|\nabla f|^2\right]\delta g^{\mu\nu}$$

+surface terms

$$= \int_M d^3x\sqrt{g}e^{-f}\left[R_{\mu\nu} + \nabla_\mu\nabla_\nu f - \frac{1}{2}g_{\mu\nu}(R + 2\nabla^2 f - |\nabla f|^2)\right]\delta g^{\mu\nu} \text{ +surface terms}$$

$$\Longrightarrow R_{\mu\nu} + \nabla_\mu\nabla_\nu f - \frac{1}{2}g_{\mu\nu}(R + 2\nabla^2 f - |\nabla f|^2) = 0.$$

Ricci Flow As A Gradient Flow

- $\delta_f S[g,f] = \int_M d^3x e^{-f}\ [-(R + |\nabla f|)\delta f + 2\, g^{\mu\nu}\ \nabla_\mu \delta f \nabla_\nu f]$

$$= \int_M d^3x\sqrt{g}e^{-f}\,[-(R + |\nabla f|)\delta f + (-2\nabla^2 f + 2|\nabla f|^2)\delta f + \text{total derivative}]$$

$$= -\int_M d^3x\sqrt{g}e^{-f}\,[(R + 2\nabla^2 f - |\nabla f|^2)\delta f + \text{total derivative}]$$

$$\Rightarrow R + 2\nabla^2 f - |\nabla f|^2 = 0.$$

Combining the two variations $\delta_g S[g,f]$ and $\delta_f S[g,f]$ leads to the total variation:

$$\delta S[g,f] = \int_M d^3x\sqrt{g}e^{-f}\left[\left(R_{\mu\nu} + \nabla_\mu\nabla_\nu f\right)\delta g^{\mu\nu} - \left(\delta f + \frac{1}{2}g_{\mu\nu}\delta g^{\mu\nu}\right)(R + 2\nabla^2 f - |\nabla f|^2)\right] + \text{surface terms}$$

$$= \int_M d^3x\sqrt{g}e^{-f}\left[\left(R_{\mu\nu} + \nabla_\mu\nabla_\nu f\right)\delta g^{\mu\nu} + \left(\delta \ln\sqrt{g} - \delta f\right)(R + 2\nabla^2 f - |\nabla f|^2)\right] + \text{surface terms}$$

$$= \int_M d^3x\sqrt{g}e^{-f}\left[\left(R_{\mu\nu} + \nabla_\mu\nabla_\nu f\right)\delta g^{\mu\nu} + \delta \ln\left(e^{-f}\sqrt{g}\right)(R + 2\nabla^2 f - |\nabla f|^2)\right] + \text{surface terms}$$

Notice that $\delta \ln\left(e^{-f}\sqrt{g}\right)$ vanishes identically iff the measure $dm = e^{-f}\sqrt{g}\ d^3x$ is kept fixed.

Therefore, the symmetric tensor $-\left(R_{\mu\nu} + \nabla_\mu\nabla_\nu f\right)$ is the L^2 gradient of the functional $S[g,f] = \int_M (R + |\nabla f|)dm$ where now f denotes $\ln\frac{dV}{dm}$ with $dV = \sqrt{g}\ d^3x$.

2. **Entropy and Monotonicity**

Perelman introduced an entropy functional that is monotonic under the Ricci flow. This entropy helps in understanding the behavior of the flow and provides a way to control the formation of singularities. The entropy functional is defined as:

$$\mathcal{W}(g, f, \tau) = \int_M \left[\tau(R + |\nabla f|^2) + f - 3\right] (4\pi\tau)^{-3/2} e^{-f} \, dV$$

where R is the scalar curvature, f is a function, and τ is a scale parameter.

3. **Finite Extinction Time**

Perelman showed that the Ricci flow with surgery on a simply connected, closed 3-manifold will become extinct in finite time. This means that the manifold will collapse to a point or a collection of points, indicating that the original manifold was topologically a 3-sphere.

Our Ongoing Work

- How many Thurston geometries can be described by Chern-Simons gauge theory?
- If matter is required, what are the possible Chern-Simons-Matter theories?
- What is the corresponding two-dimensional CFT (WZW & Liouville)?
- What is the physical interpretation of Perelman entropy?
- Many other questions?
- We hope to report our result along these questions soon.

Thank You

for your attention

CQUeST