CARTER-TYPE SEPARABILITY IN ROTATING SPACETIMES

From a Radial–Angular Compatibility Condition to a Universal Master Equation

I · Separability Ansatz & RACC

II · Schwarzian Rigidity

III · Master Equation

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arXiv:2603.04047 · arXiv:2603.08408 · Kim, "A master equation for Carter-separable spacetimes" (2026)

ROADMAP

What We'll Cover in 30 Minutes

I

Separability Ansatz & RACC

arXiv:2603.04047

A Carter-type metric ansatz and the radial–angular compatibility condition that builds rotating spacetimes coupled to general matter.

II

Schwarzian Rigidity of Kerr

JHEP · arXiv:2603.08408

Local equilibrium in a Carter frame forces a projective, constant-Schwarzian structure — the local origin of Kerr's hidden symmetry.

III

The Master Equation

Kim (2026)

The remaining diagonal Einstein equations collapse to one sourced equation, unifying Kerr–Carter, Plebański–Demiański, and matter deformations.

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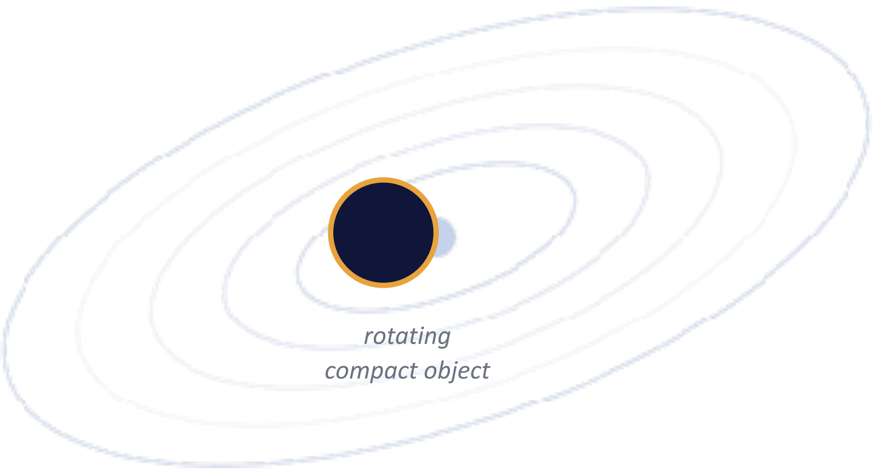
Synthesis & Outlook

How the three results fit into a single research program, and what remains open.

MOTIVATION

Rotation Is Everywhere — and Hard to Model

- Black holes, neutron stars, and rotating remnants: angular momentum is the generic case, not the exception
- Rotation brings purely relativistic gravitomagnetic effects (frame dragging) with no Newtonian counterpart
- **Two observational revolutions demand ever more general rotating solutions:**



Gravitational Waves

LIGO–Virgo–KAGRA: binary black-hole mergers

Black-Hole Imaging

Event Horizon Telescope: shadows of supermassive black holes

Kerr Is Uniquely Rigid — But Why?

- Kerr (1963): the astrophysical rotating black hole
  - Separable geodesics & wave equations (Carter, 1968)
  - Petrov type D, Killing–Yano tensor, hidden symmetry
- Newman–Janis algorithm generates Kerr-type metrics, but reaches only a narrow slice of all rotating solutions
- Usual explanations for Kerr's rigidity are global: uniqueness theorems, asymptotic flatness, horizon regularity

Is Kerr's separable structure already fixed — locally — by the Einstein equations, before any global assumption is imposed?

PART I

A Carter-Type Ansatz and the Radial–Angular Compatibility Condition

H.-C. Kim & W. Lee · arXiv:2603.04047 [gr-qc]

- A general separable metric ansatz
- The RACC condition: $G_{\hat{r}}{}^{\hat{\theta}} = 0$
- Rotating black holes, angular deficits, wormholes

Stationary Axisymmetric Spacetimes: a Hard PDE Problem

$$ds^2 = g_{m\gamma}(r, \theta)\, dx^m\, dx_\gamma$$

- Metric components depend on two coordinates (r, θ): a highly coupled, nonlinear PDE system
- Direct substitution into the Einstein equations is analytically tractable only in special, highly symmetric cases
- **We need an ansatz that keeps full generality while exposing structure**

> *Strategy: build a Carter-type ansatz with a canonical orthonormal frame, then impose an alignment condition on the Einstein tensor — rather than trying to classify all solutions at once.*

axisymmetric,
stationary
(∂_t , ∂phi Killing)

A Generalized Carter-Type Metric Ansatz

$$ds^2 = -\frac{\Sigma\Delta}{q\,(\Gamma - a^2 p)^2}(dt - ap\,d\phi)^2 + \frac{\Sigma}{q\Delta}dr^2 + \Sigma\,d\theta^2 + \frac{\Sigma \sin^2\theta}{(\Gamma - a^2 p)^2}(a\,dt - \Gamma\,d\phi)^2$$

- **Σ(r,θ)**
  – area function of the symmetry orbits — a genuine dynamical degree of freedom
- **Γ(r), Δ(r)**
  – auxiliary radial function, and a free radial “potential” fixed later by matter
- **p(x), q(x), x ≡ cosθ**
  – angular functions

$$q = 1, \quad p = 1 - \cos^2\theta, \quad \Sigma = r^2 + a^2\cos^2\theta, \quad \Delta = r^2 + a^2 - 2Mr + v(r)$$

Kerr & Kerr–Newman are recovered for the standard angular choice

A Canonical Frame Adapted to the Radial–Angular Split

$$\omega^{\hat{0}} = \sqrt{\frac{\Sigma\Delta}{q}}\,\frac{dt - ap\,d\phi}{\Gamma - a^2 p}$$

$$\omega^{\hat{1}} = \sqrt{\frac{\Sigma}{q\Delta}}\,dr$$

$$\omega^{\hat{2}} = \sqrt{\Sigma}\,d\theta$$

$$\omega^{\hat{3}} = \frac{\sqrt{\Sigma}\sin\theta}{\Gamma - a^2 p}(a\,dt - \Gamma\,d\phi)$$

- $\omega^{\hat{1}} \propto dr$ and $\omega^{\hat{2}} \propto d\theta$ — radial and angular directions are manifestly separated
- $\omega^{\hat{0}}$, $\omega^{\hat{3}}$ span the stationary & axial Killing directions
- Not an arbitrary tetrad — it is the canonical frame tied to the ansatz itself
- For Kerr and Kerr–Newman, the Einstein tensor is already diagonal in this frame

The Radial–Angular Compatibility Condition (RACC)

$$G_{\hat{1}\hat{2}} = R_{\hat{1}\hat{2}} = 0$$

"no mixed radial–angular momentum flux or shear stress in the canonical Carter frame"

- RACC requires the effective stress tensor to be aligned with the r–θ split of the Carter frame
- **It is an alignment condition — not a claim that some tetrad can diagonalize the Einstein tensor pointwise**
- A local SO(2) rotation in the $(\hat{r},\hat{\theta})$ plane could diagonalize a single point, but it mixes dr and dθ and destroys the radial–angular split built into the ansatz
- RACC selects a geometrically natural "aligned sector" containing Kerr, Kerr–Newman — and genuinely new non-vacuum geometries

From RACC to a Solvable System

RACC reduces to a nonlinear differential equation for Γ, p, q, Σ:

$$\left(\frac{a^2\dot{p}}{\Gamma - a^2 p}\right)' = \frac{\Sigma\dot{\Sigma}'}{\Sigma^2} - \frac{1}{3}\frac{\dot{q}}{q}\frac{\Sigma'}{\Sigma} - \frac{2}{3}\frac{\dot{\Sigma}'}{\Sigma}$$

Δ(r) always decouples — left free for the matter / equation-of-state input

Key trick — separate the area function through one combined variable:

$$\Sigma(r,x) = P(x)\,\Sigma(y)\,, \qquad y = \sigma(r) + Q(x)$$

A nonlinear, higher-order PDE — no general solution exists.
This ansatz reduces it to two ODEs plus a Riccati equation.

Classifying Solutions: a Γ Equation and a Riccati Equation

$\Gamma(r)$ — up to six solution branches

- $\dot{N} = 0$: $\Gamma(r)$, $\Sigma(r)$ independently free, or algebraically linked
- Polynomial closure (g_k truncated): $\Gamma \to$ Möbius-type rational function of $\sigma(r)$
- Quartic closure with $p \neq$ const: generically over-determined — no solution
- $p =$ const, $\dot{Q} \neq 0$: a distinct degenerate branch requiring separate care

$\Sigma(y)$ — a Riccati equation

- $2\mathring{\Sigma} - \varsigma^2 =$ cubic/quartic source in y — no general solution, but broad exact families:
- Rational: $\Sigma \propto y^2$ or $y/(y_0 y^2+1)^2$
- Hyperbolic / exponential: tanh, cosh closed forms
- Bessel-function branch: I_k, K_k of imaginary argument
- A generation technique builds new solutions from any known one (Riccati → linear ODE trick)

A Generalized Kerr–Newman–NUT Black Hole / Wormhole

1 Impose RACC + Carter-frame diagonality ($G_{\hat{0}\hat{3}} = 0$)

2 Impose angular isotropy ($G_{\hat{2}\hat{2}} = G_{\hat{3}\hat{3}} \Leftrightarrow w_1 = p\_r/\rho = -1$)

3 Solve: $\sigma(r) = r^2$, $Q(x)$ fixed by $P(x)$

$$ds^2 = -\frac{\Delta}{y}\left(dt - \frac{a^2 - Q}{a}d\phi\right)^2 + \frac{y}{\Delta}dr^2 + P\,y\,d\theta^2 + \frac{P\sin^2\theta}{y}\left(a\,dt - (r^2 + a^2)d\phi\right)^2$$

$$y = r^2 + Q(x)$$

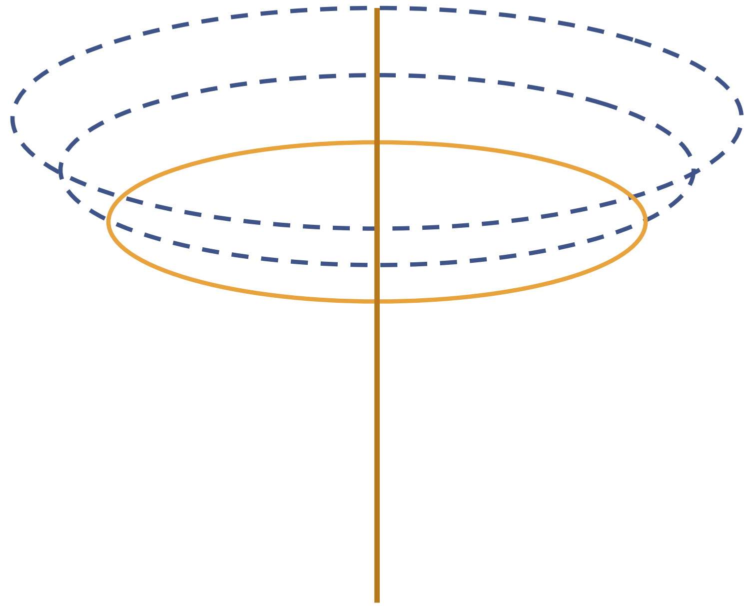
- **$\alpha \neq 1$**
  – angular deficit — a cosmic-string-like tension along the poles
- **$q_0 \neq 0$**
  – NUT charge
- **$\Delta(r)$ free**
  – encodes the radial matter content

Rotating Black Holes with a Global-Monopole-Like Deficit

- $\Delta(r) = \Delta_2(r^2 - 2Mr + \Delta_0)$: quadratic radial function from the angular-isotropy closure
- Large-r area of a constant-r sphere grows as $A(r) = 4\pi(w_2/\Delta_2)\, r^2 + O(1)$ — a genuine solid-angle deficit:

$$\delta\Omega = 4\pi\left(1 - \frac{w_2}{\Delta_2}\right)$$

- Sourced by anisotropic matter with $p_r = -\rho$ ($w_1 = -1$) and $p_\theta = p_\phi \neq -\rho$ ($w_2 = w_3 \neq 1$)
- Physically: a cosmic-string-like tension threading the rotation axis of a spinning black hole



angular deficit
along the poles

A Cosmological-Constant Sector — and Rotating Wormholes

Λ-like matter ($w_1=w_2=w_3=-1$)

- Quartic radial function $\Delta(r) = r^4/\ell^2 + \Delta_2 r^2 + \Delta_1 r + \Delta_0$
- Recovers the non-accelerating Carter canonical form
- Distinct from — and more general than — the standard Kerr–de Sitter metric

Rotating wormholes

- Relax angular isotropy ($w_2 \neq w_3$) via a throat parameter $b^2 \neq 0$
- Static limit → Simpson–Visser-type traversable wormhole
- Rotating case → generalization of the “Alice” wormhole
- Weak energy condition is necessarily violated near the throat

PART II

Emergence of a Universal Schwarzian Structure in Kerr Geometry

H.-C. Kim · JHEP · arXiv:2603.08408 [gr-qc]

- Local equilibrium ⇒ projective rigidity
- $\{\Gamma,R\} = \{p,z\} = C$
- Kerr = a fixed Schwarzian orbit

A Minimal Local Equilibrium Condition

- Same family of stationary-axisymmetric ansatz, in a locally non-rotating orthonormal frame
- Impose: $G_{\hat{a}}{}^{\hat{b}} = 0$ for $a \neq b$ — "no mixed energy / momentum flux"
- For this ansatz, only two mixed components are ever nontrivial: $G_{\hat{0}\hat{3}}$ and $G_{\hat{1}\hat{2}}$
- **No assumption of vacuum, separability, Petrov type, or Killing–Yano structure — purely local**

INPUT

One local condition:

no mixed stress in the Carter frame

→

RESULT

A rigid projective (Schwarzian)

structure for the whole geometry

Characteristic Variables and the First Closure

Radial & angular "clocks" adapted to the mixed equations:

$$\partial_R \equiv \Gamma'(r)\,\partial_r$$

$$\partial_z \equiv \frac{\dot{p}(x)}{q(x)^2}\,\partial_x$$

Integrating $G_{\hat{0}\hat{3}} = 0$ immediately fixes the conformal factor Σ, up to one arbitrary function of $y = R - z$:

$$\Sigma(R,z) = \Sigma_0\,\frac{(\Gamma - a^2 p)\,\bar{\Sigma}(y)}{\sqrt{|\Gamma_R\,(p_z/q^2)|}}\,, \qquad y = R - z$$

The Second Closure: a Riccati Equation

$$\frac{dH}{dy} - \frac{1}{2}H^2 - \frac{1}{2}F(R,z)\,H - \frac{1}{4}G(R,z) = 0$$

- $H(y) \equiv d/dy \log \bar{\Sigma}(y)$: $G_{\hat{1}\hat{2}} = 0$ reduces to Riccati form with coefficients $F(R,z)$, $G(R,z)$
- **Consistency — H depends on y alone — forces F and G to also depend on y alone**

  – a strong new constraint on the metric functions $\Gamma(R)$ and $p(z)$

Closure ⇒ Equal Schwarzian Derivatives

$$\{A, B\} \equiv \frac{A_{BBB}}{A_B} - \frac{3}{2}\left(\frac{A_{BB}}{A_B}\right)^2$$

the Schwarzian derivative —
an invariant of Möbius (projective) transformations

- $F = \partial_y \log S(R,z)$, with $S \equiv (\Gamma - a^2 p)^2/(\Gamma_R\, p_z)$
- “F depends on y only” forces S to factorize: $S(R,z) = \tilde{F}(y)\, \tilde{G}(\tilde{y})$
- **Factorization forces the Schwarzian derivatives of Γ(R) and p(z) to be equal — and constant:**

$$\{\Gamma, R\} = \{p, z\} = C$$

Three Universal Branches, One Alignment

Möbius

C = 0

rational, fractional-linear Γ(R)

Exponential

C < 0

hyperbolic Γ(R) — contains Kerr

Trigonometric

C > 0

circular / periodic Γ(R)

Physically relevant (exponential) branch:

$$\Gamma(R) = \frac{\alpha\, e^{\mu R} + \beta\, e^{-\mu R}}{\gamma\, e^{\mu R} + \delta\, e^{-\mu R}}, \qquad C = -2\mu^2$$

Closure additionally forces a projective alignment between radial and angular sectors:

$$p(z) = \frac{1}{a^2}\,\Gamma(\pm(z - z_0))$$

radial and angular sectors are locked
to the same projective function, up to sign

Kerr Sits on a Fixed Constant-Schwarzian Orbit

$$\Gamma(r) = r^2 + a^2 \quad \Rightarrow \quad \{\Gamma, R\} = -8$$

- $\Gamma(r) = r^2 + a^2 \Rightarrow$ Kerr lives in the exponential ($C<0$) branch
- Trigonometric branch ($C>0$):
  - – $\Sigma(y)$ is a Legendre function of non-integer degree
  - – generically singular / multi-valued at the rotation axis
- **Global axis-regularity + periodicity $\Rightarrow$ trigonometric branch excluded; $C \leq 0$ branches survive**

Kerr is the globally admissible realization of an underlying, locally rigid class.

Hidden Symmetry as a Consequence, not an Assumption

- Within the aligned, separable class: Petrov type D and a (conformal) Killing–Yano tensor follow automatically
- The same Schwarzian structure governs the universal boundary mode of nearly-AdS_2 / JT gravity (Kitaev–Suh; Maldacena–Stanford–Yang)
- Kerr's near-extremal throat sits on the same orbit, $\{\Gamma,R\} = -8$ — a possible geometric link to universal near-extremal ringdown behavior
- Robustness: the same Schwarzian closure survives a broader ansatz with redistributed radial/angular warp factors

PART III

A Master Equation for Carter-Separable Spacetimes and Compatible Sources

H.-C. Kim · arXiv:2606.03758 (2026)

- Off-diagonal equations fix the geometry
- Diagonal equations collapse to ONE equation
- Kerr–Carter, Plebański–Demiański & regular BHs as solutions

Taking the Projective Structure as Given

- Same separable ansatz; the off-diagonal Einstein equations (Part II) already fix Γ(R), p(z), and the Schwarzian branch
- **Focus: the anti-aligned exponential branch**

  – contains the full Lorentzian Kerr–Carter and Plebanski–Demianski real section

$$\Gamma - a^2 p = \frac{\mu}{2}(r^2 + a^2 x_-^2)$$

The familiar $r^2 + a^2\cos^2\vartheta$ structure is now derived — not assumed

Two Automatic Identities Restrict the Matter Content

$$G_{\hat{0}\hat{0}} + G_{\hat{1}\hat{1}} = 0\,, \qquad G_{\hat{2}\hat{2}} - G_{\hat{3}\hat{3}} = 0$$

- These hold identically in the anti-aligned exponential branch — geometric identities, not vacuum equations
- Einstein's equations then force any compatible matter to satisfy:

$$T_{\hat{0}\hat{0}} + T_{\hat{1}\hat{1}} = 0,\ \ T_{\hat{2}\hat{2}} - T_{\hat{3}\hat{3}} = 0 \ \Leftrightarrow\ p_r = -\rho,\ \ p_\theta = p_\phi$$

i.e. radially "vacuum-like" and tangentially isotropic — exactly the ad hoc equation of state widely used in regular rotating black-hole models

One Equation to Rule the Diagonal Sector

$$\mathcal{L}_{CP}[\Delta, Y] = 16\pi\Sigma\,(T_{\hat{0}\hat{0}} + T_{\hat{3}\hat{3}})$$

- **Δ(r)**

  – radial structure function

- **$Y(x) = (1-x^2)Q(x)$**

  – angular structure function

- **Only the single combination $T_{\hat{0}\hat{0}} + T_{\hat{3}\hat{3}}$ sources the entire diagonal system**
- Homogeneous case (S=0): recovers the vacuum–Λ Kerr–Carter / Plebanski–Demianski family

Homogeneous Solutions: the Carter & Plebański–Demiański Quartics

$$\Delta(r) = \frac{r^4}{\ell^2} + \Delta_2 r^2 + \Delta_1 r + \Delta_0$$

$$Y(x) = \frac{a^2 x_-^4}{\ell^2} - \Delta_2 x_-^2 + \frac{g_1}{a} x_- + \frac{\Delta_0}{a^2}$$

- Non-accelerating representative → Carter's quartic structure functions (mass Δ_1, NUT charge g_1)
- Accelerating representative (conformal factor with A≠0) → the full Plebanski–Demianski quartic, same quartic coefficient in Δ and Y

$$\Sigma(y) = \bar{\Sigma}_0 \frac{a x_- r}{(1 + \bar{A} a x_- r)^2},$$

- **Both are homogeneous solutions of the same master operator L_CP**

A Gauge Choice, Not a Restriction

- A common Möbius transform of $\Gamma(R)$, $p(z)$ leaves $S = (\Gamma - a^2 p)^2/(\Gamma_R\, p_z)$ invariant
- Δ and Y transform with definite projective weights $\Rightarrow$ the master equation keeps its exact form
- **The convenient $\gamma=0$ representative used to derive Kerr–Carter / Plebanski–Demianski is therefore a gauge choice — no generality is lost**

What Matter Can Live in This Sector?

Aligned Maxwell field

$$S_{em} = \frac{4\,q_{em}^2}{r^2 + u^2}\,, \qquad q_{em}^2 = e^2 + g^2$$

- A constant deformation of the structure functions
- Recovers Kerr–Newman(–NUT) and the charged Plebanski–Demianski family

Regular-black-hole sources

$$m(r) = \frac{M r^3}{r^3 + 2ML^2}$$

- A Hayward-type mass function m(r): a localized radial deformation of Δ
- De Sitter-core regular rotating black holes emerge as sourced solutions of the same master equation

Three Papers, One Program

Carter-type separable ansatz — PAPER I

↓

Off-diagonal Einstein equations: $G_{\hat{0}\hat{3}} = 0$, $G_{\hat{1}\hat{2}} = 0$ — PAPERS I – II

↓

RACC + Schwarzian rigidity ⇒ projective structure fixed (Kerr–Carter / Plebanski–Demianski sector) — PAPER II

↓

Diagonal Einstein equations ⇒ Master Equation — PAPER III

↓

Homogeneous
vacuum–Λ: Kerr, Kerr–Newman, NUT, Plebanski–Demianski

Sourced
Maxwell, regular-BH, angular-deficit BH, rotating wormholes

Open Directions

- **Beyond stationarity**

  dynamical generalizations of the Carter-projective construction

- **Stability**

  of angular-deficit black holes and rotating wormholes

- **Cancellation branches**

  non-generic Riccati closures left unclassified

- **Other projective branches**

  aligned-exponential, parabolic, trigonometric — global structure

- **Near-extremal ringdown**

  deeper Schwarzian / JT-gravity connection

- **Beyond GR**

  modified gravity & numerical-relativity initial data

Take-Home Messages

1. A broad Carter-type ansatz plus the RACC systematically builds rotating metrics coupled to general matter — Kerr, Kerr–Newman, NUT, and genuinely new angular-deficit and wormhole geometries.

2. Kerr's separability is not only a global uniqueness fact: local equilibrium (no mixed flux) already forces a rigid Schwarzian / projective structure, and global regularity singles out Kerr.

3. Once that structure is fixed, the entire diagonal Einstein system collapses to one sourced master equation — unifying Kerr–Carter, Plebanski–Demianski, and their charged / regular / anisotropic matter deformations.

Thank You

Questions & discussion welcome

[1] H.-C. Kim & W. Lee, “Stationary axisymmetric systems that allow for a separability structure,” arXiv:2603.04047 [gr-qc].

[2] H.-C. Kim, “Emergence of Universal Schwarzian Structure in Kerr Geometry,” JHEP, arXiv:2603.08408 [gr-qc].

[3] H.-C. Kim, “A master equation for Carter-separable stationary axisymmetric spacetimes and compatible sources,” preprint (2026).

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