Dual gravities from entanglement entropy

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Based on :

1) D. Kim, CP, and J. Yoo, in preparation (Dual of TFIM)

2) J. Huh and CP, arXiv:2603.25270

3) CP, C. Hwang, K. Cho, and S. Kim, Phys. Rev. D (2022)

Motivation

AdS/CFT correspondence

Classical SUGRA
on AdS space-time

$\longleftrightarrow$ 1 to 1

Super-CFT at the AdS boundary
(**in a strong coupling regime**)

Due to the conformal symmetry, the IR theory is trivial.

How about a non-conformal and non-supersymmetric QFT like a condensed matter theory?

IR $\longleftarrow$ UV

non-trivial RG flow

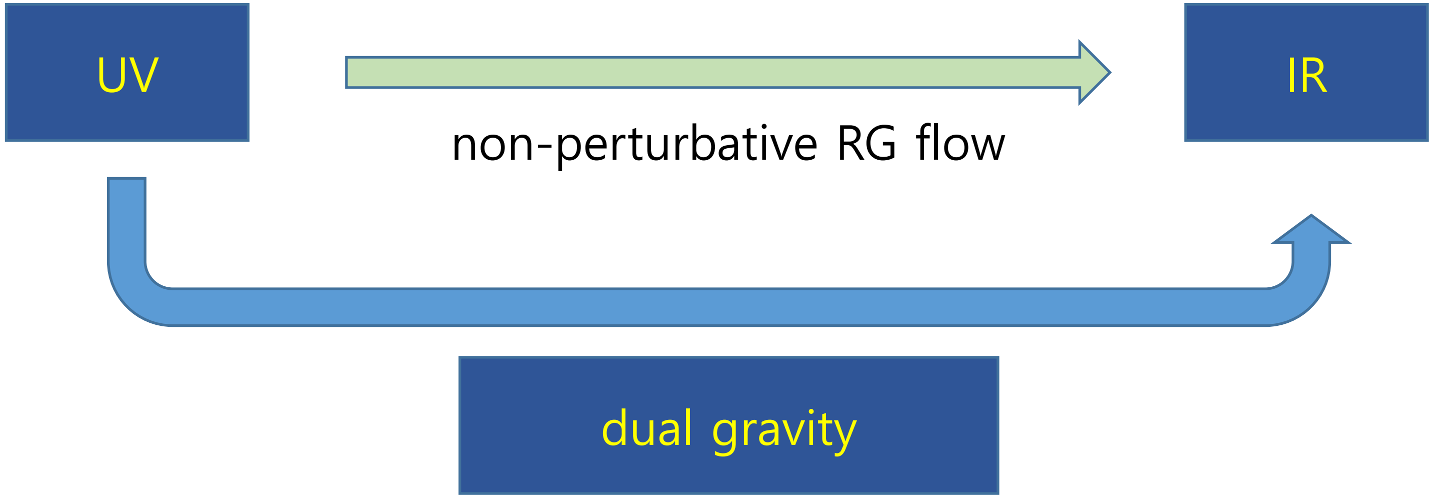
- IR physics
- QCD and CMT

- known asymptotic AdS geometries
(dual QFT?)
- dual geometries?

Can we find the dual geometry of a given QFT data?

If we know the RG flow of QFT, then we can reconstruct its dual gravity.

To investigate IR (macroscopic) physics from the fundamental (microscopic) QFT, we need to figure out a non-perturbative RG flow.

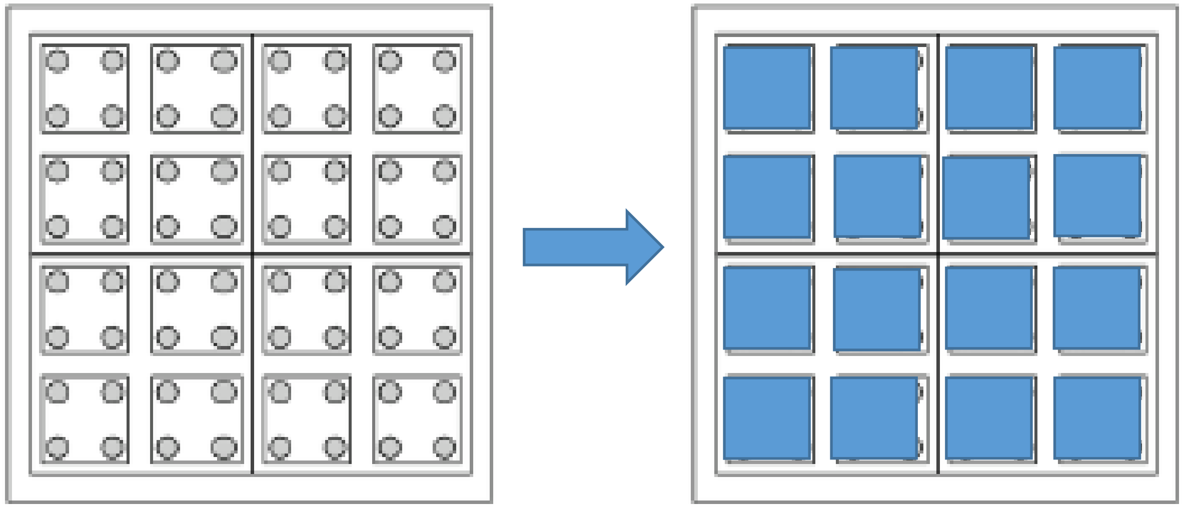


RG flow descriptions

Real-space RG (Migdal-Kadanoff, CMT)

Block spin renormalization

RG flow of Entanglement entropy



General properties of the entanglement entropy

The leading term of the entanglement entropy is provided by the short distance interaction between two subsystems near the boundary. In the continuum limit, this term causes a UV divergence and its coefficient is proportional to the area of the entangling surface ∂A (*UV cutoff sensitive, regularization scheme dependent*).

$$S_E \sim \frac{Area(\partial A)}{\epsilon^{d-1}} + \text{ subleading finite terms}$$

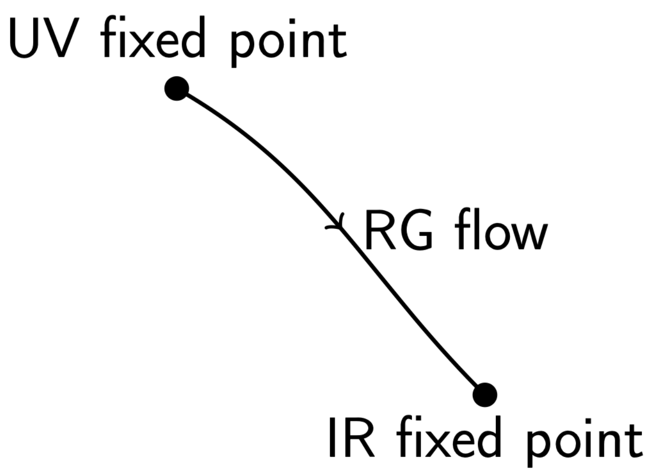
c-theorem by Zamoldchikov

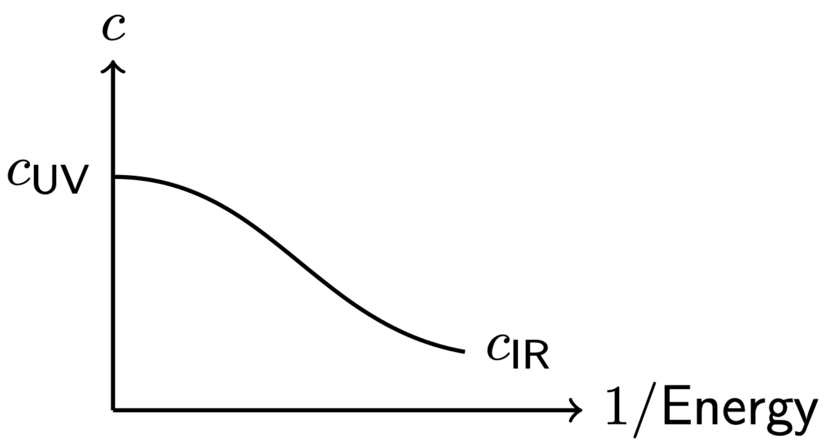
When a 2-dim. CFT is deformed by a relevant operator, it flows to a new IR fixed point.

In this case, the central charge, which describes degrees of freedom of a system, monotonically decreases along the RG flow.

In higher dimensional theory, is there a theorem similar to the C-theorem?

- For d=4, there exists two central charges, a and c. It has been believed that the a-type anomaly satisfies the c-theorem (***a-theorem***).





Ryu-Takayanagi conjecture

One of the most remarkable successes in the AdS/CFT correspondence is the microscopic derivation of the Bekenstein-Hawking entropy for a BPS black hole

$$S_{BH} = \frac{A}{4G}$$

This idea relates the gravitational entropy to the degeneracy of the dual quantum field theory with its microscopic description.

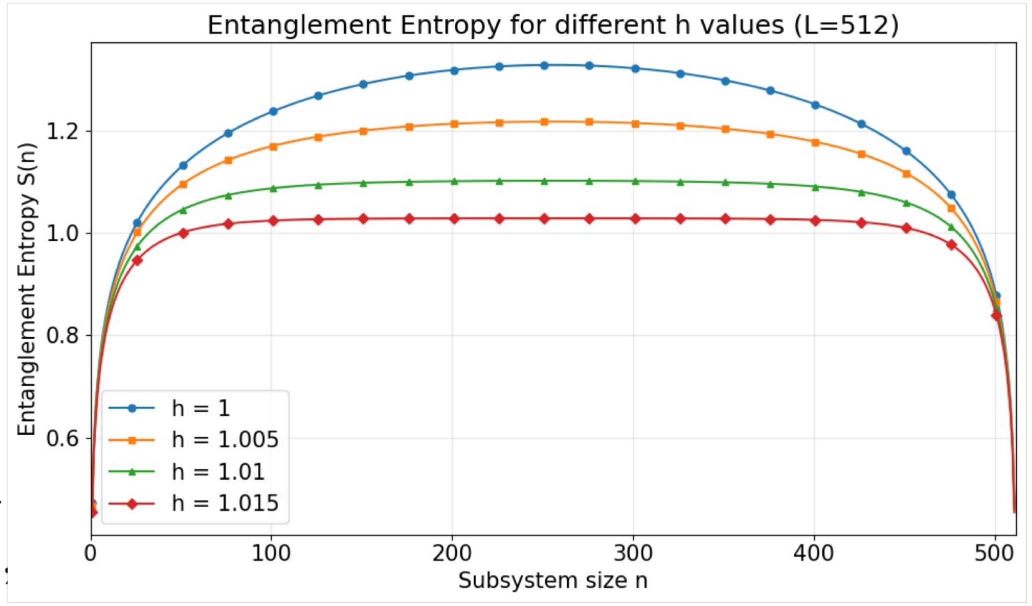
On the other hand, there exists a different kind of entropy called the entanglement entropy in quantum mechanical systems which measures the entanglement between quantum states.

Ryu and Takayanagi proposed the formula following the black hole entropy

$$S = \frac{\text{Area of } \gamma}{4G}$$

The goal of this work is to figure out the entanglement entropy in the strong coupling regime following the AdS/CFT correspondence.

Transverse field Ising model (TFIM)



$$H = -J\sum_i \sigma_i^x\,\sigma_{i+1}^x - h\sum_i \sigma_i^z\,.$$

After Jordan-Wigner and Bogoliubov transformations

The energy spectrum is given by

$$\epsilon_k = 2\sqrt{(h - J\cos\bar{k})^2 + J^2\sin^2\bar{k}},$$

For a small k limit

$$\epsilon_k \;\approx\; 2\sqrt{|h-J|^2 + hJ\bar{k}^2} = \sqrt{m_\psi^2 + v^2k^2},$$

This is the dispersion relation of a free massive fermion theory

$$H = \pm i\,v\,\bar{\psi}\,\partial_x\psi + m_\psi\,\bar{\psi}\psi.$$

which is exactly solvable. The entanglement entropy of this system becomes

$$S_E = -\sum_{j=1}^{\ell}\left[\frac{1+\nu_j}{2}\log_2\left(\frac{1+\nu_j}{2}\right) + \frac{1-\nu_j}{2}\log_2\left(\frac{1-\nu_j}{2}\right)\right]$$

Dual gravity reconstruction of closed TFIM

$$H = -J\sum_i \sigma_i^x\,\sigma_{i+1}^x - h\sum_i \sigma_i^z .$$

$$H_{CFT} = -J\left(\sum_i \sigma_i^x\,\sigma_{i+1}^x + \sum \sigma_i^z\right)$$

$$S = \frac{1}{2\kappa^2}\int d^3x\sqrt{-G}\left(\mathcal{R} - 2\Lambda \qquad\qquad\qquad\qquad \right)$$

CFT

AdS

$\Downarrow$

$\Downarrow$

Mass (relevant) deformation

Geometry deformation

$$H_{def} = -(h-J)\sum_i \sigma_i^z .\ \left(\ \sigma_i^z \sim \bar{\psi}\psi\ \right)$$

In the UV region, $\Delta = 1.$

The bulk field ϕ is the dual of $O = \bar{\psi}\psi.$

$$V(\phi) \approx \frac{1}{2}m_\phi^2\phi^2 + \cdots, \quad \text{with} \quad m_\phi^2 = -1$$

In the IR regime, plateau appears

Deformation of the inner geometry

$$S = \frac{1}{2\kappa^2}\int d^3x\sqrt{-G}\left(\mathcal{R} - 2\Lambda \boxed{-\frac{1}{2}\partial_M\phi\,\partial^M\phi - \frac{V(\phi)}{R^2}}\right)$$

Dual gravity theory

$$S = \frac{1}{2\kappa^2}\int d^3x\sqrt{-G}\left(\mathcal{R} - 2\Lambda \boxed{- \frac{1}{2}\partial_M\phi\,\partial^M\phi - \frac{V(\phi)}{R^2}}\right)$$

Metric ansatz for the dual of the closed TFIM (asymptotic global AdS)

$$ds^2 = -g(r)^2dt^2 + h(r)^2r^2d\theta^2 + \frac{dr^2}{f(r)^2},$$

we set $h(r) = 1$. Then, the unknown functions are $f(r)$ $g(r)$, $\phi(r)$, and $V(r)$

General equations are

$$\begin{aligned} 0 &= g'' + \frac{f'}{f}\left(g' - \frac{g}{r}\right), \\ \phi'^2 &= \frac{2}{r}\left(\frac{g'}{g} - \frac{f'}{f}\right), \\ V(r) &= 2 - \frac{R^2f^2}{r}\left(\frac{f'}{f} + \frac{g'}{g}\right), \end{aligned}$$

(1) From the TFIM entanglement entropy, we can determine $f(r)$

(2) If we can determine one of them, the others are automatically fixed due to equations of motion.

Analytic calculation at the critical point (CFT)

The RT formula results in

$$S_E = \frac{1}{4G}\int_{-\Delta\phi/2}^{\Delta\phi/2} d\phi \sqrt{r^2 + \frac{1}{f^2}\left(\frac{dr}{d\phi}\right)^2},$$

Using the conservation law, we determine the subsystem size and entanglement entropy in terms of a turning point

$$\begin{aligned}
\Delta\phi(r_t) &= \frac{2\pi}{L}n = \int_{r_t}^{\infty} dr\, \frac{2r_t}{rf\sqrt{r^2 - r_t^2}},\\
S_E(r_t) &= \frac{c}{3R}\int_{r_t}^{\infty} dr\, \frac{r}{f\sqrt{r^2 - r_t^2}},
\end{aligned}$$

These two integrals automatically satisfy

$$\frac{dS_E(n)}{dn} = \frac{\pi\, r_t(n)}{2GN}.$$

From the known TFIM entanglement entropy $S_E(\Delta\phi)$, we can determine f

$$\frac{1}{f} = -\frac{2r^2}{N}\int_r^{\infty} dr_t\, \frac{1}{\sqrt{r_t^2 - r^2}}\frac{d}{dr_t}\left(\frac{n(r_t)}{r_t}\right)$$

Abel transformation

At the critical point (CFT), the exact entanglement entropy is known as

$$\bar{S}_E(n) = \frac{c}{3} \log \left[\frac{\gamma N}{\pi} \sin \left(\frac{\pi n}{N} \right) \right]$$

From this result, we find the subsystem size as a function of the turning point

$$n(r_t) = \frac{N}{\pi} \text{arccot} \left(\frac{r_t}{R} \right)$$

Applying the Abel transformation, the metric function results in

$$f(r) = \sqrt{\frac{r^2}{R^2} + 1}.$$

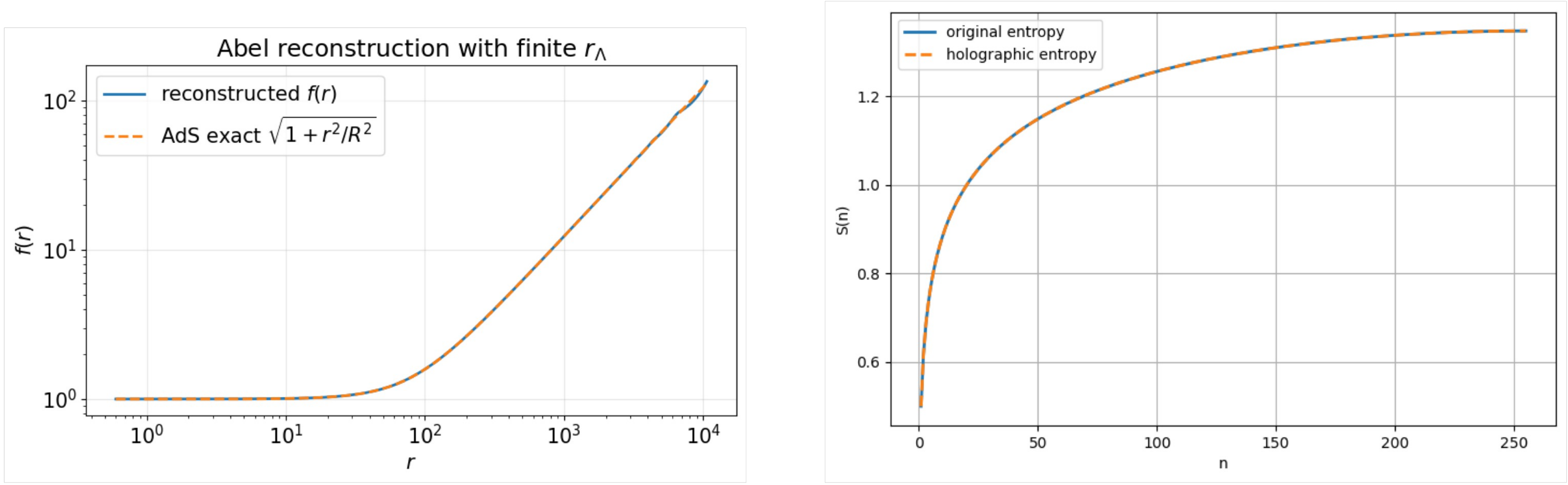
At the critical point, since there is no deformation, we finally obtain $g(r) = f(r)$

This results in the global AdS space, as expected.

Again, we also reconstruct the dual gravity numerically

$$\frac{1}{f(r)} = C - \frac{2r^2}{N} \int_{y_{min}}^{y_{max}} dy \left(\frac{n'(r_t)}{r_t^2} - \frac{n(r_t)}{r_t^3} \right)$$

where we set $y = \sqrt{r_t^2 - r^2}$ and $y_{max} = \sqrt{r_\Lambda^2 - r^2}$.



(a) $f(r)$

(b) Entropy from the reconstructed geometry

Applying this numerical method, we try to reconstruct the dual gravity for the off-critical cases, where there is no known analytic entanglement entropy.

Holographic dual of the mass-deformed (off-critical) TFIM

In the UV region, we expect

$$V(\phi) \approx \frac{1}{2} m_\phi^2 \phi^2 + \cdots ,$$

where $m_\phi^2 = -1$ to describe the mass deformation with $\Delta = 1$

Analytic perturbative results are

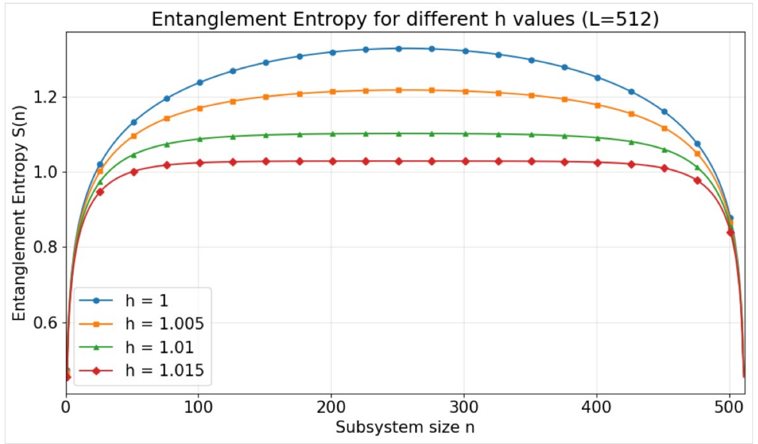
$$\begin{aligned} f(r) &= \frac{r}{R} + \frac{1}{2}\frac{R}{r} - \frac{2-\phi_0^4}{16}\frac{R^3}{r^3} + \cdots , \\ g(r) &= \frac{r}{R} + \frac{2-\phi_0^2}{4}\frac{R}{r} - \frac{2-\phi_0^2}{16}\frac{R^3}{r^3} + \cdots , \\ \phi(r) &= \frac{\phi_0 R}{r}\left(1 - \frac{(2-\phi_0^2)}{8}\frac{R^2}{r^2} + \cdots\right) . \end{aligned}$$

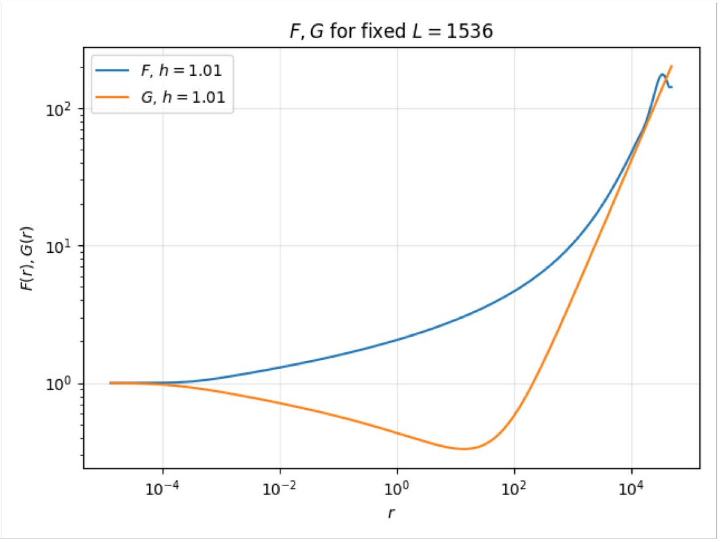
which leads to the following entanglement entropy $(\ell/L \to 0$ and $m_\psi \ell \to 0)$

$$\begin{aligned} \bar{S}_E &= \frac{c}{3}\left[\log\left(\frac{2r_\Lambda}{R}\frac{\pi\ell}{L}\right) - \frac{\pi^2}{6}\left(\frac{\ell}{L}\right)^2 \left\{1 + \frac{\pi^2}{20}\left(\frac{\ell}{L}\right)^2 + \mathcal{O}\left(\frac{\ell^4}{L^4}\right)\right\}\right. \\ &\left. \quad - \frac{\pi^4}{120}\left(\frac{\ell}{L}\right)^4 (m_\psi \bar{a})^4 \left\{1 + \mathcal{O}\left(\frac{\ell^2}{L^2}\right)\right\} + \cdots\right] . \end{aligned}$$

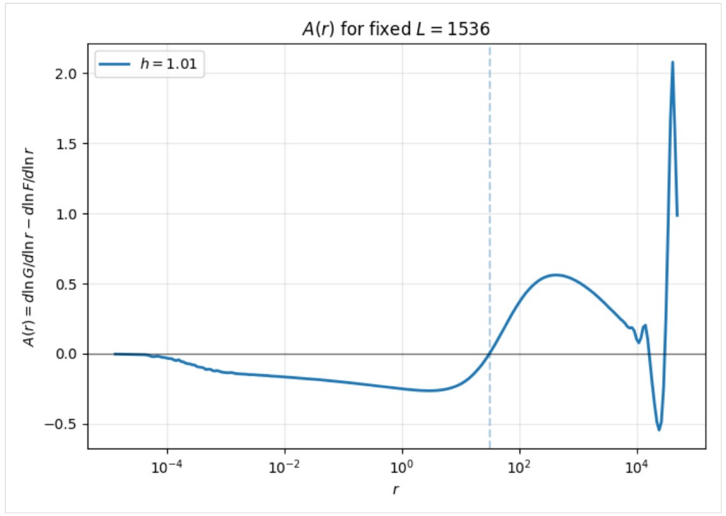
Dual gravity solution

TFIM entanglement entropy data results

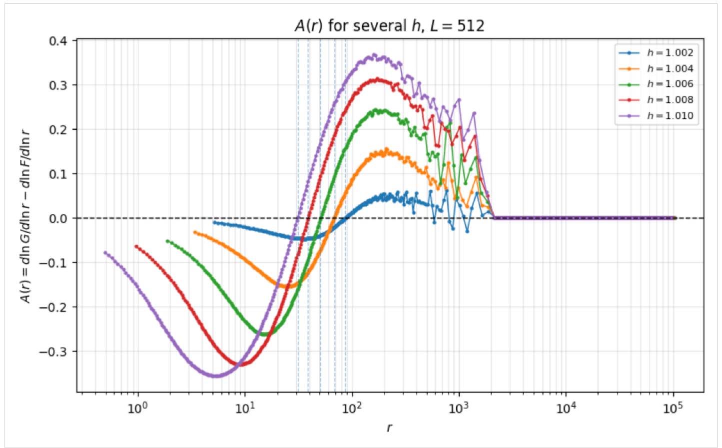




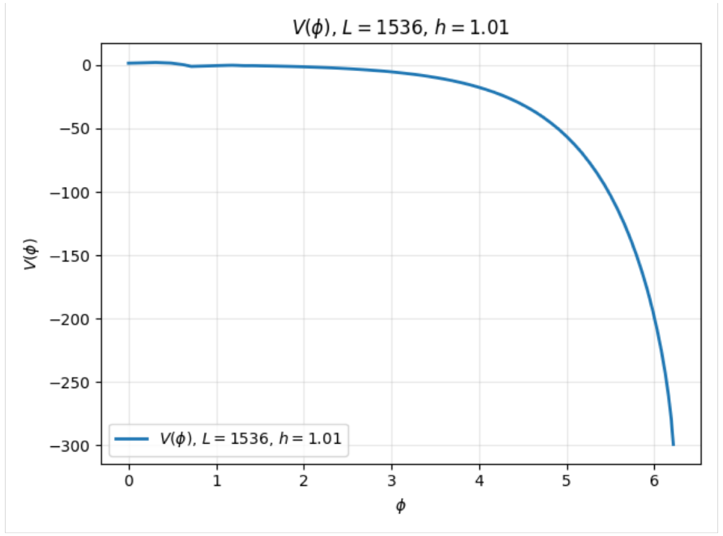
(a) $f(r)$ and $g(r)$



(b) $\phi'^2(r)$



(a) $r\,\phi'^2(r)/2$



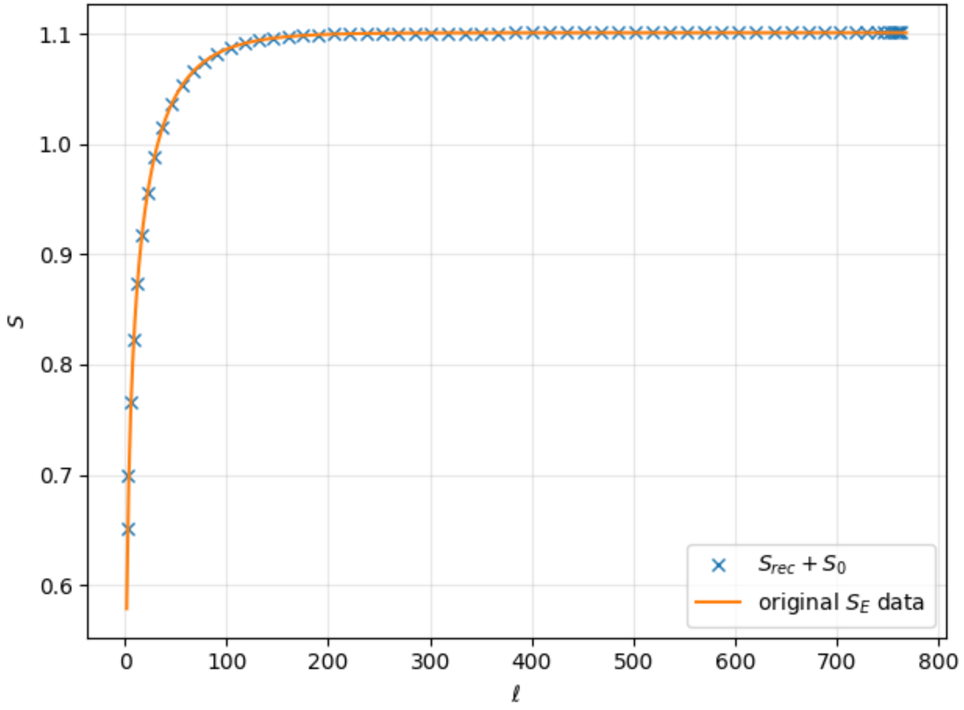
(b) $V(\phi)$

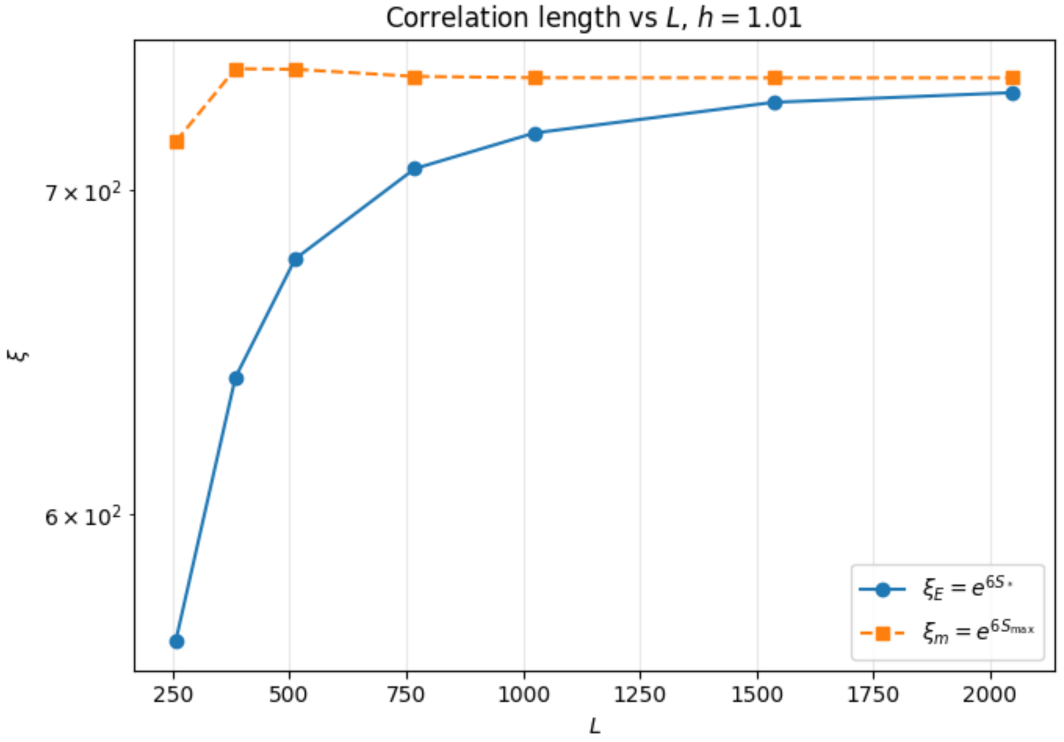
In the infinite-size limit ($L \to \infty$),

the entanglement entropy of a massive theory has the following form

$$S_E(r_*) = \frac{c}{3} \log \frac{\xi_*}{\bar{a}} + S_0,$$

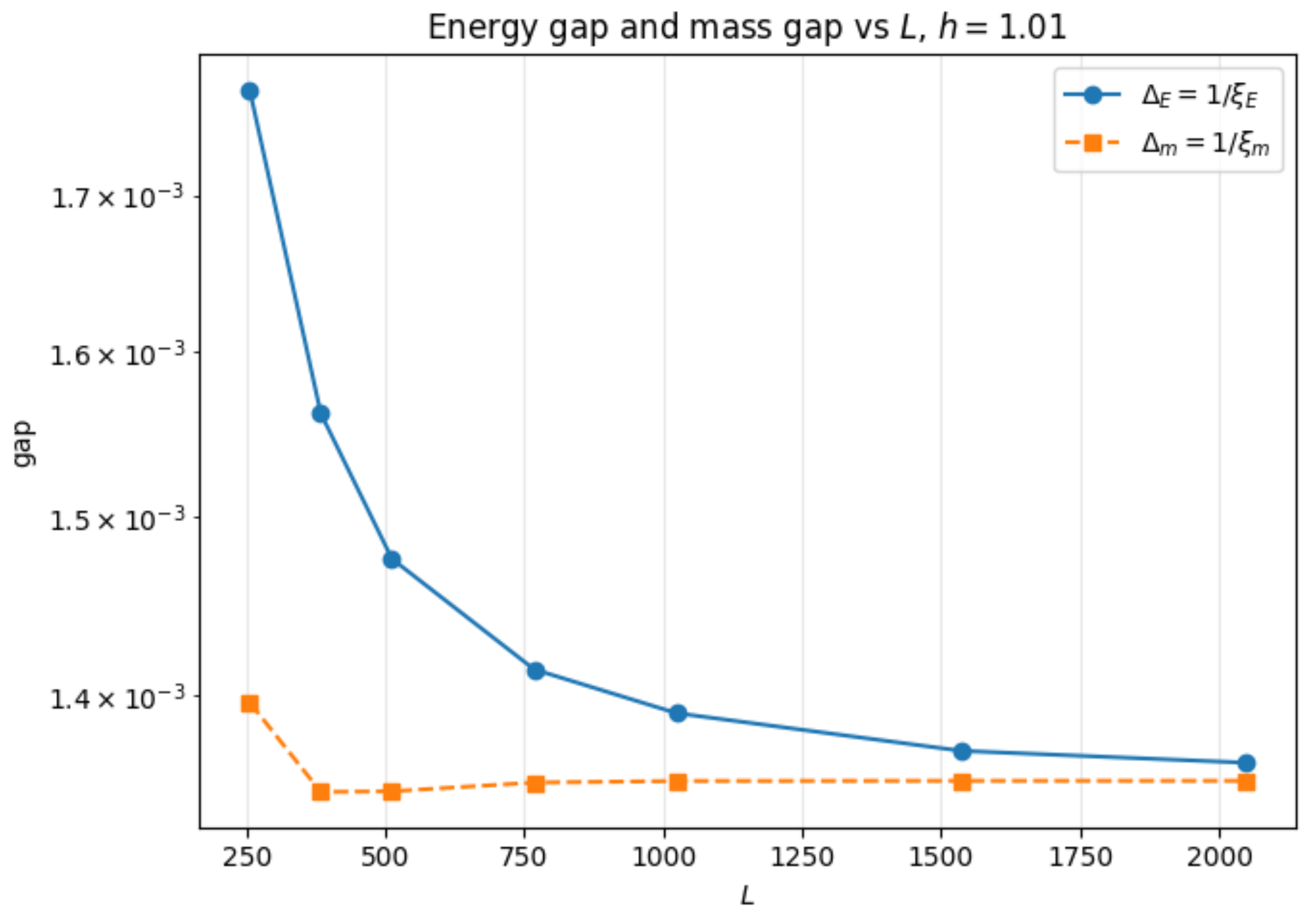
where ξ_* is a correlation length, inversely proportional to a mass gap.





The reconstructed dual gravity can explain this feature correctly $E_{\text{gap}}(m_\psi, L) \simeq \sqrt{m_\psi^2 + v^2 k_{\text{min}}^2}$ with $k_{\text{min}} \sim \frac{2\pi}{L}$.

Mass gap of closed TFIM



Conclusion

- We studied how to reconstruct the dual gravity theory from the known entanglement entropy data.

- So fat, this is possible only for a single scalar deformation.

- Multi-scalar deformation, Vector deformation …

- Holographic dual of other condensed matter systems and models

Thank you!

RG flow descriptions

(1) Momentum-space RG (integrate out higher frequency modes)

(i) 1PI (one particle irreducible) RG (perturbative, QFT for high energy physics)

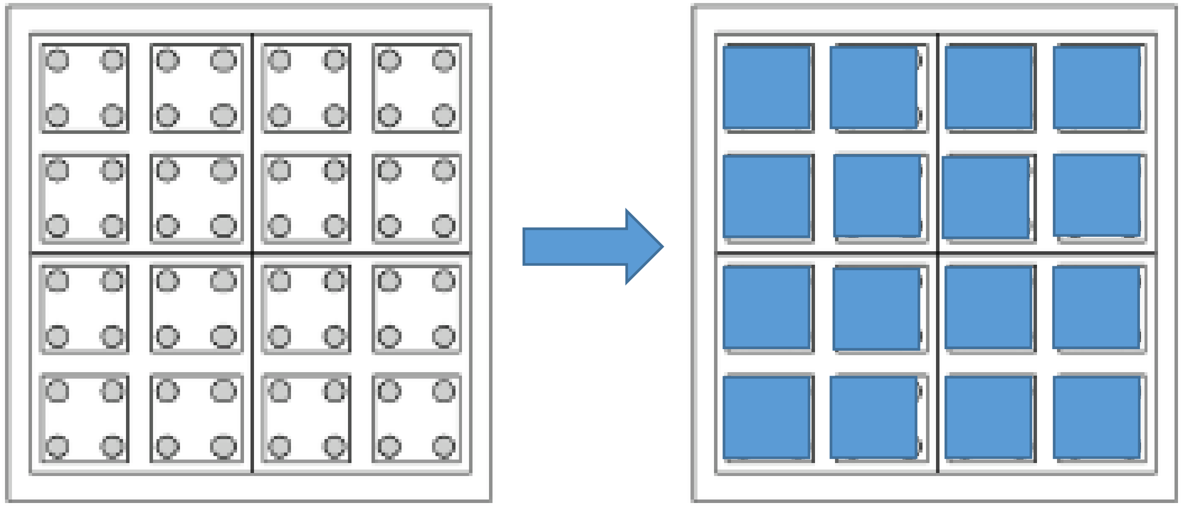
(ii) Wilsonian Exact RG equation with a hard cutoff (non-perturbative) → holographic renormalization

(ii) Polchinski exact RG equation with a soft cutoff (non-perturbative)

(2) Real-space RG (Migdal-Kadanoff, CMT)

Block spin renormalization

→ RG flow of Entanglement entropy



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The goal of this work is to figure out the entanglement entropy in the strong coupling regime following the AdS/CFT correspondence.

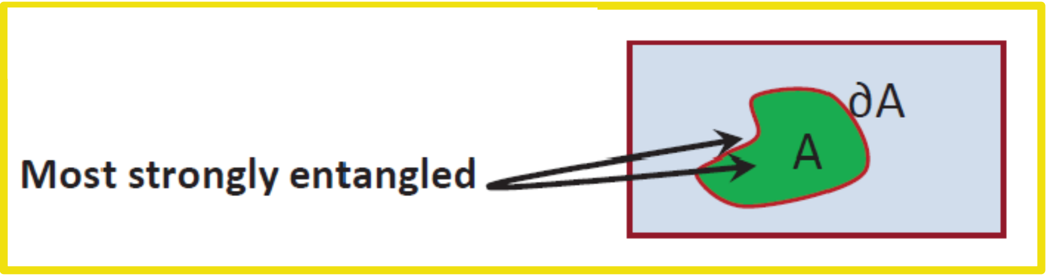
Aspects of the holographic entanglement entropy

General properties of the entanglement entropy

1) Area law of the entanglement entropy

The leading term of the entanglement entropy is provided by the short distance interaction between two subsystems near the boundary. In the continuum limit, this term causes a UV divergence and its coefficient is proportional to the area of the entangling surface ∂A (*UV cutoff sensitive, regularization scheme dependent*).

$$S_E \sim \frac{Area(\partial A)}{\epsilon^{d-1}} + \text{ subleading finite terms}$$



2) Subleading finite terms

There exists the terms not relying on a UV cutoff, which can provide an important physical information associated with the long range correlations.

In general, the entanglement entropy crucially depends on the shape and size of the entangling surface.

(i) for d=odd

$$A = \Omega_{d-2}\left[\frac{1}{d-2}\left(\frac{l}{\epsilon}\right)^{d-2} + F + \mathcal{O}\left(\frac{\epsilon}{l}\right)\right]$$

- No logarithmic term

- There exists a constant term, F, which is identified with a free energy of the 3-dimesional dual CFT for d=3.

- For d=3,

  F is the exact same as the free energy of 3-dim. CFT which has been checked by the comparison with the localization result.

(ii) for d=even

$$A = \Omega_{d-2}\left[\frac{1}{d-2}\left(\frac{l}{\epsilon}\right)^{d-2} + a' \ \log\left(\frac{l}{\epsilon}\right) + \mathcal{O}(1)\right]$$

with

$$a' = (-)^{d/2-1}\frac{(d-3)!!}{(d-2)!!}$$

- There exists a universal logarithmic term. Its coefficient is universal in that it is independent of the regularization scheme.

- The coefficient of the logarithmic term is independent of the entangling surface area, which is related to the a-type anomaly.

- Weyl anomaly of 4-dim. CFT,

$$\langle T^{\alpha}_{\alpha}\rangle = -\frac{c}{8\pi}W_{\mu\nu\rho\sigma}W^{\mu\nu\rho\sigma} + \frac{a}{8\pi}\tilde{R}_{\mu\nu\rho\sigma}\tilde{R}^{\mu\nu\rho\sigma}$$

with

$$\begin{aligned} W_{\mu\nu\rho\sigma}W^{\mu\nu\rho\sigma} &= R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 2R_{\mu\nu}R^{\mu\nu} + \frac{1}{3}R^2, \\ \tilde{R}_{\mu\nu\rho\sigma}\tilde{R}^{\mu\nu\rho\sigma} &= R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 4R_{\mu\nu}R^{\mu\nu} + R^2. \end{aligned}$$

As a consequence, *the logarithmic term is related to the anomaly and crucially depends on the dimension and shape of the entangling surface.*

c-theorem by Zamoldchikov

When a 2-dim. CFT is deformed by a relevant operator, it flows to a new IR fixed point.

In this case, the central charge, which describes degrees of freedom of a system, monotonically decreases along the RG flow.

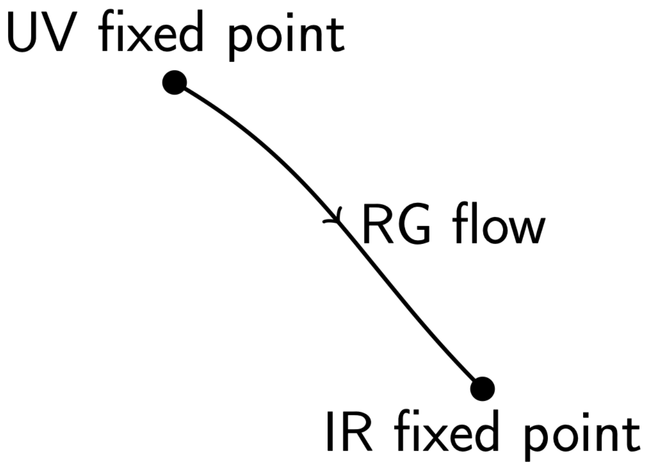
In higher dimensional theory, is there a theorem similar to the C-theorem?

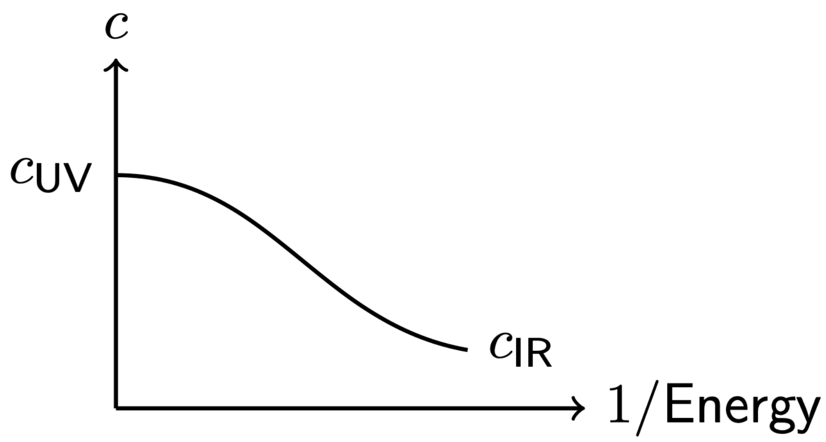
- For d=4, there exists two central charges, a and c. It has been believed that the a-type anomaly satisfies the c-theorem (***a-theorem***).

- For d=3, it has been conjectured that the free energy monotonically decreases along the RG flow (***F-theorem***).

F-theorem in 3-dim. CFT [Jafferis-Klebanov-Pufu-Safdi 2011, Myers-Sinha 2010]

RG flow under a relevant deformation





$$F_{\rm UV}(\mathbb{S}^3) \geq F_{\rm IR}(\mathbb{S}^3) \ , \qquad F = -\log Z(\mathbb{S}^3)$$

HEE with a spherical entangling surface in a 3-dim. CFT

$$S_E = \alpha \frac{l}{\epsilon} - F(\mathbf{S}^3)$$

For the entanglement entropy of thermal systems

we can also derive the similar structure, where the quantum entanglement transfers into a thermal quantity with a small quantum corrections.

For the three-dimensional AdS (BTZ) black hole

$$ds^2 = -\frac{R^2}{z^2} f(z) dt^2 + \frac{R^2}{z^2 f(z)} dz^2 + \frac{R^2}{z^2} dx^2, \qquad \text{with} \qquad f(z) = 1 - \frac{z^2}{z_h^2}$$

thermodynamic quantities are given by

$$\begin{aligned} T_H &= \frac{1}{2\pi} \frac{1}{z_h}, \\ S_{th} &= \frac{1}{4G} \frac{l}{z_h}, \\ E &= \frac{1}{16\pi G} \frac{l}{z_h^2}. \end{aligned}$$

and satisfy the first law of thermodynamics $\quad dE = T_H dS_{th}$

RG flow of the entanglement entropy

Thermodynamics-like law of the entanglement entropy in the UV limit (area law)

$$T_E \; S_E = E \qquad \text{with} \qquad T_E = \frac{6}{\pi l} .$$

- This relation is defined in the UV region with neglecting higher order corrections.

- It reproduces the linearized Einstein equation of the dual geometry.

- It is not valid in the IR region.

In order to go beyond the linearized lever and to describe the RG flow correctly, we need generalized concepts involving all higher order corrections.

We define a generalized thermodynamics-like law and generalized entanglement temperature involving all higher order correction and satisfying in the entire region

$$\bar{T}_E \; \bar{S}_E = \bar{E}$$

In the IR region ($z_0 \approx z_h$),

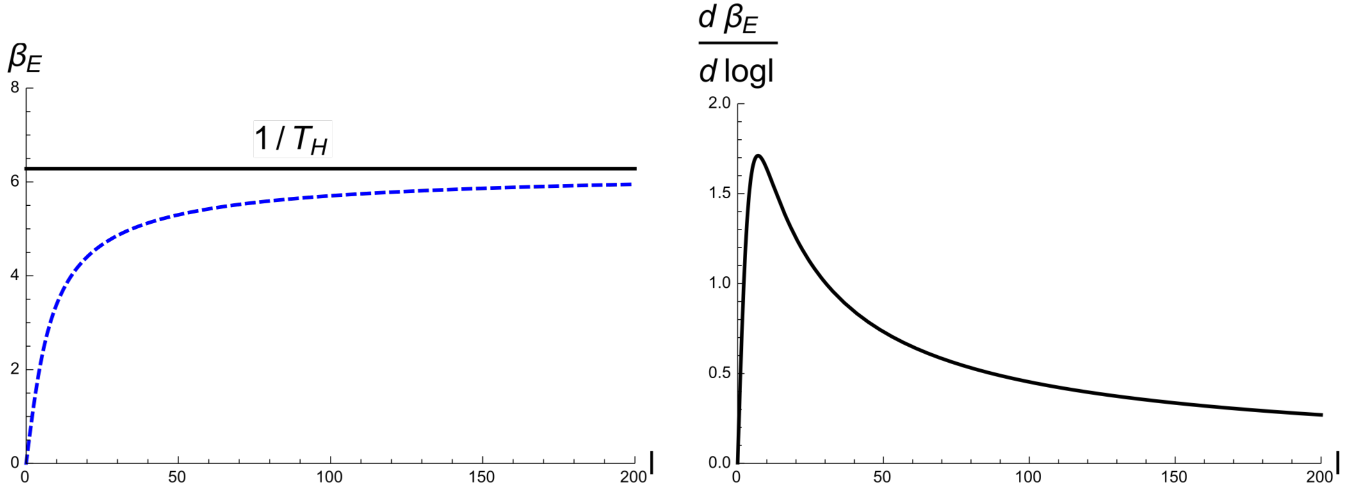
Reexpressing it in terms of the black hole entropy involved in the volume l, we reach to the similar result obtained from the black hole and CFT calculations

$$\bar{S}_E = S_{th} - \frac{1}{2G} \log S_{th} + \mathcal{O}(1)$$

Since $S_{th} \to \infty$ in the IR limit, the IR entanglement entropy reduces to the thermal entropy (volume law) with small quantum corrections. Also, we can see that the generalized entanglement temperature reduces to the real temperature.

$$\beta_E = 2\pi z_h + \frac{4\pi z_h^2}{l} \log\left(\frac{z_h}{l}\right) + \cdots ,$$

$$l \frac{d\beta_E}{dl} = 2\pi z_h \left[\coth \frac{l}{2z_h} - \frac{2z_h}{l} \left\{1 + \log\left(\frac{2z_h}{l} \sinh \frac{l}{2z_h}\right)\right\}\right].$$



(a) (b)

Regardless of the dimensionality and microscopic detail of the dual field theory, the IR entanglement entropy reduces to

$$\bar{S}_E = S_{th} + S_{correction}$$

S_{th} : Universal

$S_{correction}$: depending on the dual theory

For a two-dimensional scale invariant theory

$$S_{correction} \sim -\log S_{th}$$

Intriguingly, the universality of the IR entanglement entropy proposed from the holography

$$\bar{S}_E \approx S_{th}$$

occurs in the real space renormalization group flow of the lattice theory (Ising model). This follows the volume law

$$S_{th} \sim l$$

Reconstruction of the dual gravity from the entanglement entropy I

Abel transformation

When the following integral equation is given

$$F(y) = \int_y^\infty dr \frac{f(r)}{\sqrt{r^2 - y^2}},$$

the solution is determined by the inverse Able inversion

$$f(r) = -\frac{2r}{\pi} \int_r^\infty dy \, \frac{1}{\sqrt{y^2 - r^2}} \frac{dF(y)}{dy} = -\frac{2}{\pi} \frac{d}{dr} \int_r^\infty dy \, \frac{y \, F(y)}{\sqrt{y^2 - r^2}},$$

For example, under the following general metric ansatz

$$ds^2 = \frac{r^2}{R^2} \left(- F(r) dt^2 + dx^2 \right) + \frac{R^2}{r^2} \frac{1}{G(r)} dr^2,$$

the holographic entanglement entropy is governed by

$$S_E = \frac{1}{4G} \int_{-\ell/2}^{\ell/2} dx \sqrt{\frac{r^2}{R^2} + \frac{R^2}{r^2 G(r)} \left(\frac{dr}{dx} \right)^2 }. \quad \Rightarrow \quad \begin{aligned} \frac{\ell(r_t)}{2R^2 r_t} &= \int_{r_t}^\infty dr \frac{1}{r^2 \sqrt{G(r)} \sqrt{r^2 - r_t^2}}, \\ S_E(r_t) &= \frac{R}{2G} \int_{r_t}^\infty dr \frac{1}{\sqrt{G(r)} \sqrt{r^2 - r_t^2}}, \end{aligned}$$

these relations are rewritten as

$$\frac{dS_E}{d\ell} = \frac{dS_E/dr_t}{d\ell/dr_t} = \frac{1}{4G}\frac{r_t(\ell)}{R}. \qquad \frac{1}{\sqrt{G(r)}} = -\frac{r^2}{\pi R^2}\frac{d}{dr}\int_r^\infty dr_t\,\frac{\ell(r_t)}{\sqrt{r_t^2 - r^2}}.$$

Ex 1) When the following entanglement entropy is given

$$S_E = \frac{R}{2G}\log\left(\frac{\ell r_\Lambda}{R^2}\right)$$

we obtain $G(r) = 1.$, which is the metric of an AdS space.

Ex 2) The following entanglement entropy

$$S_E = \frac{c}{3}\log\left[\frac{r_\Lambda}{\pi R^2 T_H}\sinh\left(\pi T_H \ell\right)\right]$$

leads to

$$G(r) = 1 - \frac{r_h^2}{r^2}.$$

This is the metric factor of the BTZ black hole. These examples show that one can reconstruct the dual geometry only from the entanglement entropy data.

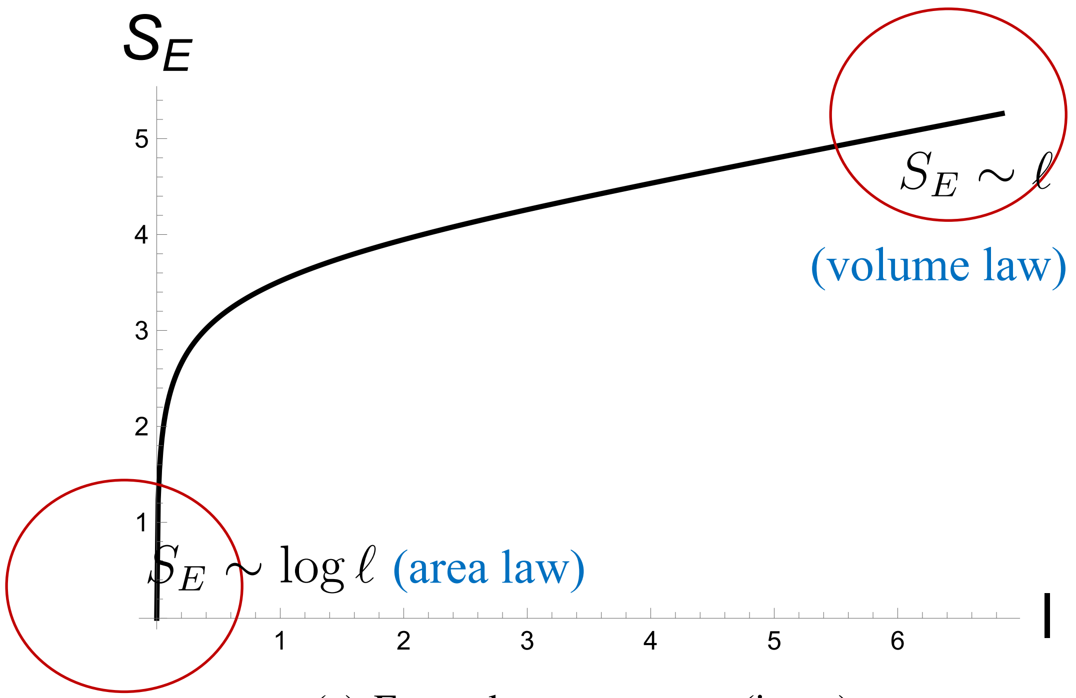
Reconstruction of the dual gravity from the entanglement entropy II

When the entanglement entropy of a 2-dim system consisting of two kinds of matter, radiation and massive particles,

(1) can we reconstruct the dual gravity of this system?

(2) Can we read other physical properties of this system?

Entanglement entropy of a thermal system



(a) Entanglement entropy (input)

1. Logarithmic behavior in the UV limit ($S_E \sim \log \ell$): asymptotic AdS space

2. Volume law in the IR limit ($S_E \sim \ell$): thermal system

3. A general form of the dual gravity (black hole geometry)

$$ds^2 = \frac{R^2}{z^2}\left(-f(z)dt^2 + \frac{1}{f(z)}dz^2 + dx^2\right)$$

Using the above metric ansatz, the corresponding entanglement entropy is determined by

$$S_E = \frac{R^{d-1}\, V_{d-2}}{4G} \int_{-\ell/2}^{\ell/2} dx \frac{\sqrt{z'^2 + f}}{z^{d-1}\sqrt{f}}$$

In this case, the subsystem size and entanglement entropy are characterized by a turning point z_t

$$\ell(z_t) = \int_{\epsilon}^{z_t} dz \frac{2z^{d-1}}{\sqrt{f(z)}\,\sqrt{z_t^{2(d-1)} - z^{2(d-1)}}},$$

$$S_E(z_t) = \frac{R^{d-1}\, V_{d-2}}{2G} \int_{\epsilon}^{z_t} dz\, \frac{z_t^{d-1}}{z^{d-1}\sqrt{f(z)}\,\sqrt{z_t^{2(d-1)} - z^{2(d-1)}}},$$

Solving these integral equations, we find z_t and $f(z_t)$, which determines the black hole geometry.

Parametrizing the turning point as

$$z_i = \epsilon + i\,\delta z \qquad \text{with} \qquad \delta z = \frac{z_h - \epsilon}{N}$$

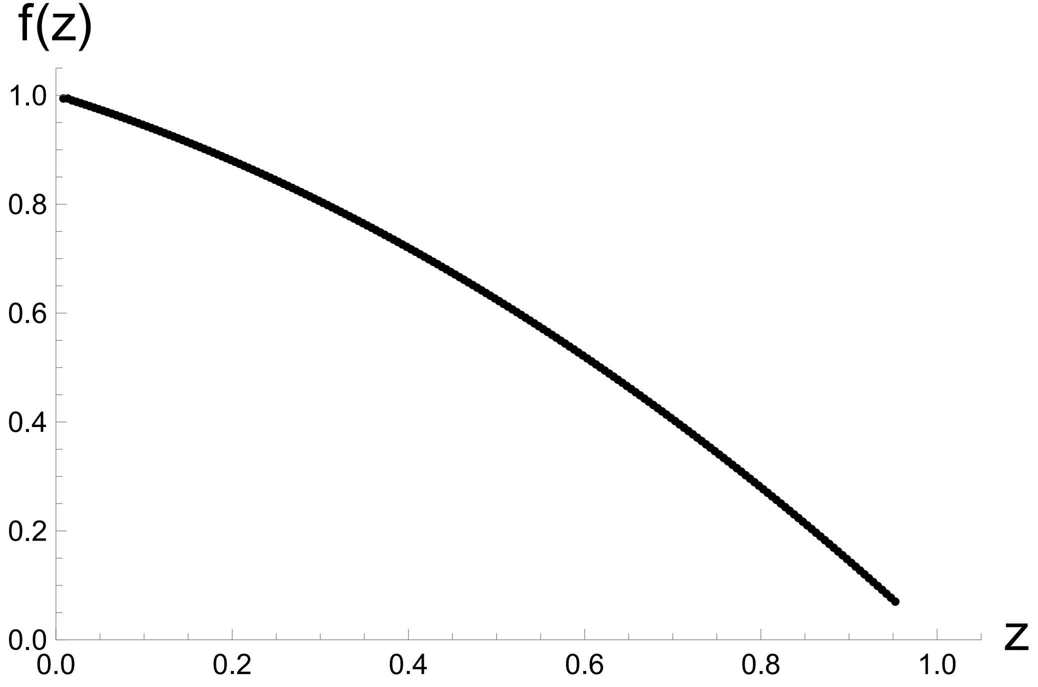
At given z_i, we find $f(z_i)$ satisfying the input entanglement entropy $S_E(\ell)$

$$\ell_n = \sum_{i=1}^{n} \int_{z_{i-1}}^{z_i} dz \frac{2z}{\sqrt{\bar{f}(z_i)}\,\sqrt{z_n^2 - z^2}},$$

$$S_n = \sum_{i=1}^{n} \frac{R}{2G} \int_{z_{i-1}}^{z_i} dz\, \frac{z_n}{z\sqrt{\bar{f}(z_i)}\,\sqrt{z_n^2 - z^2}},$$

As a result, we obtain the blackening factor $\bar{f}(\bar{z}_i)$ at $\bar{z}_i$

$$\bar{f}(\bar{z}_i) = \frac{f(z_{i-1}) + f(z_i)}{2} \quad \text{at} \quad \bar{z}_i = \frac{z_{i-1} + z_i}{2},$$



(b) Blackening factor (output)

After extrapolation, we determine the horizon and Hawking temperature

$$z_h \approx 0.99986$$

$$T_H = -\frac{f'(z_h)}{4\pi} \approx 0.12054$$

To understand other physical properties, we guess the analytic form of the blackening factor

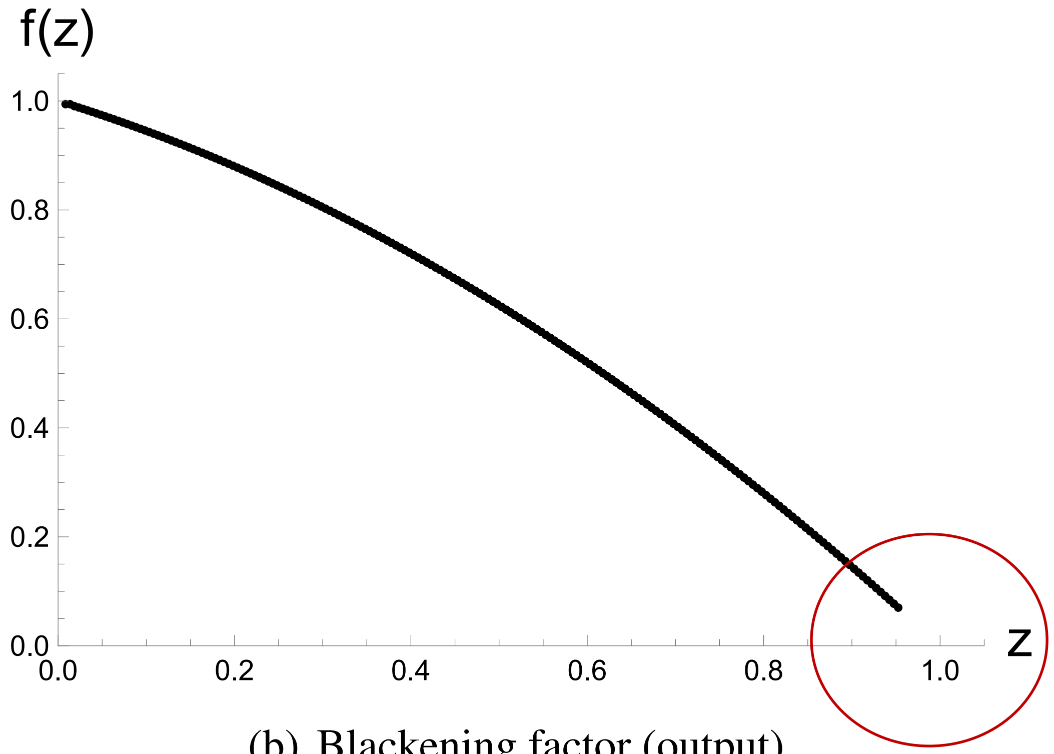
$$f(z) = 1 - \rho_0\, z - M\, z^2,$$

M : number density of massless radiation

ρ_0 : number density of massive particles

Then, the massive particle's density is determined from the above numerical data

$$\rho_0 = \frac{2}{\bar{z}_h} - 4\pi T_H \approx 0.48553$$



(b) Blackening factor (output)

Finally, the following blackening factor is reconstructed from the entanglement entropy data

$$f(z) = 1 - 0.48553\, z - 0.51468\, z^2.$$

From the black hole's thermodynamics, we can derive various thermodynamic quantities of the given system

Internal energy: $U_{th} = \frac{\pi T_H^2}{4} \frac{RV}{G} \approx 0.01141 \frac{RV}{G}$

Pressure: $P_{th} = \frac{2\pi T_H^2 + \rho_0 T_H}{8} \frac{R}{G} \approx 0.01873 \frac{R}{G}.$

Equation of state: $w = 1 + \frac{\rho_0}{2\pi T_H} \approx 1.64107,$

Specific heat: $c_V = \frac{\pi T_H}{2} \frac{RV}{G} \approx 0.18934 \frac{RV}{G} > 0.$

	Derived value	True value	Error
U_{th}	0.01141	0.01119	1.97 %
P_{th}	0.01873	0.01865	0.43 %
w	1.64107	1.66667	1.54 %
c_V	0.18934	0.18750	0.98 %

with $R = V = G = 1$

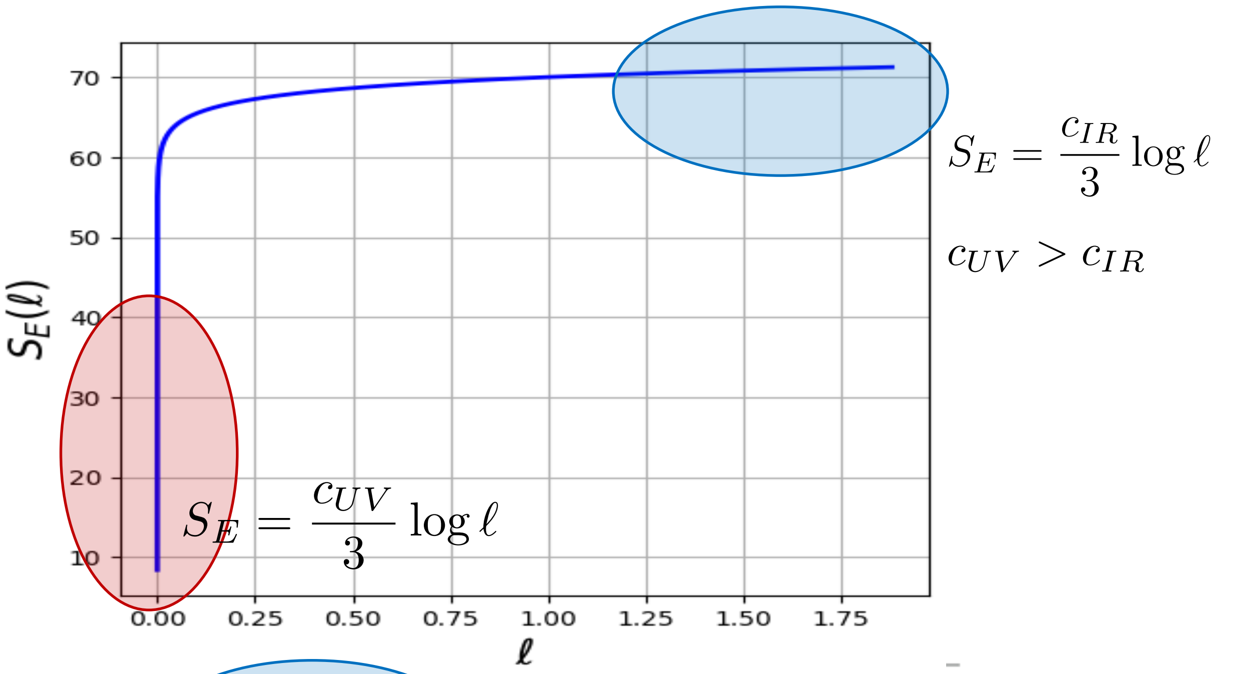
- True values of M and ρ_0 are $M = q_0 = 1/2$ which are utilized to derive the input data.

Reconstruction of the dual gravity theory

Assuming that the following entanglement entropy is given, then what is the dual geometry and gravity theory?

- The given entanglement entropy data shows that a UV CFT deforms and then leads to another IR CFT.

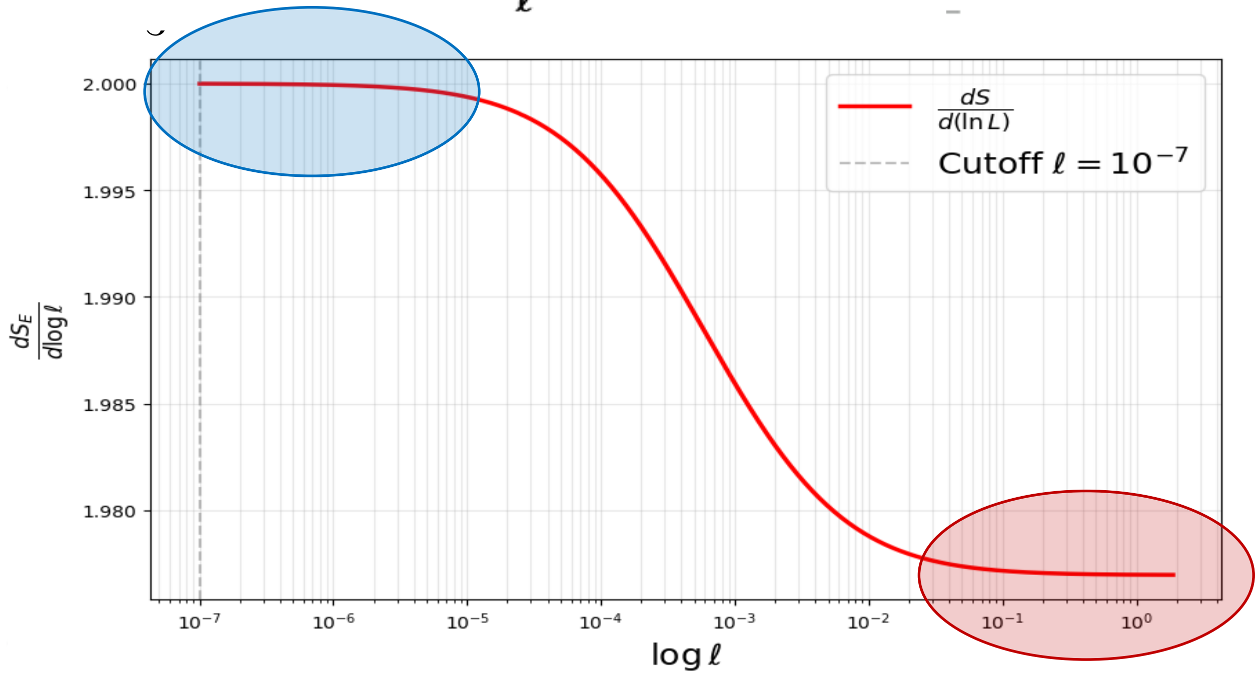
$$S = S_{CFT} + \int d^2x \, \lambda O$$



To reconstruct the dual geometry, we use the following most general metric ansatz, which preserves the boundary Lorentz symmetry,

$$ds^2 = dr^2 + \mu(r)^2 \left(-dt^2 + dx^2\right) ,$$

where $\mu(r) = e^{A(r)}$



What is the dual gravity theory admitting this entanglement entropy?

From the metric ansatz, the holographic entanglement entropy is governed by

$$S_E(r_t) = \frac{1}{4G}\int_{-\ell/2}^{\ell/2} dx\sqrt{r'^2 + e^{2A(r)}} = \frac{1}{2G}\int_{r_t}^{r_\Lambda} dr\sqrt{1 + e^{2A(r)}\left(\frac{dx}{dr}\right)^2},$$

$$\ell(r_t) = \int_{-\ell/2}^{\ell/2} dx = 2\int_{r_t}^{r_\Lambda} dr\,\frac{dx}{dr}.$$

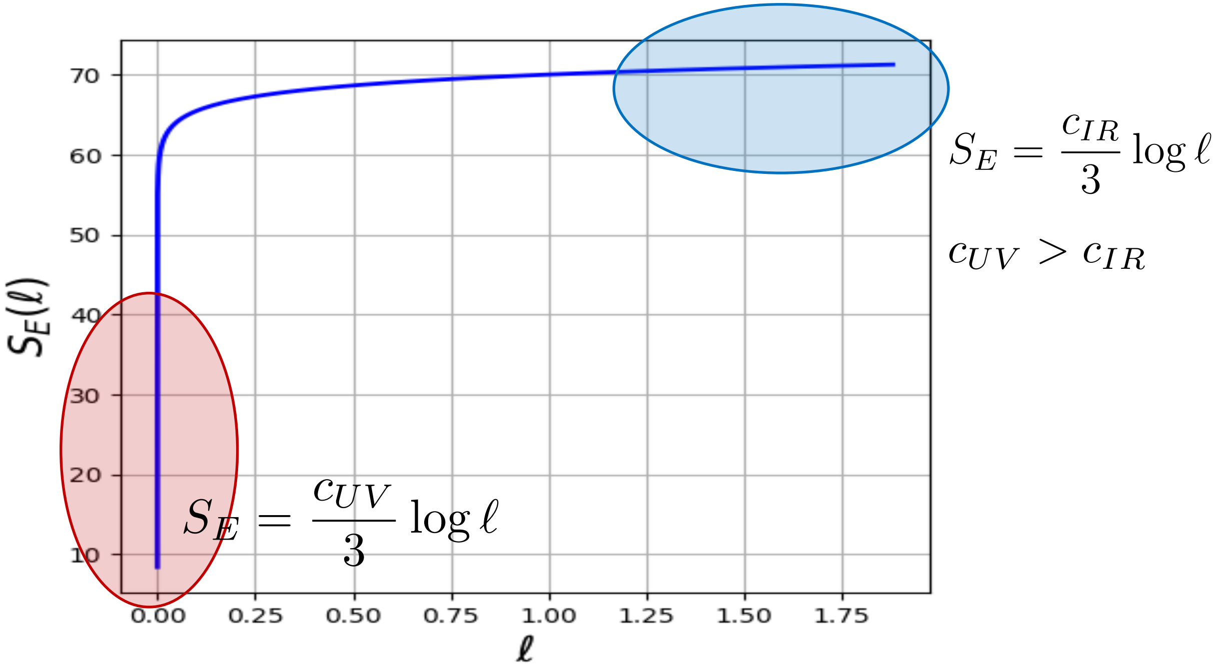
Using the conservation law, we finally obtain the subsystem size and the entanglement entropy as functions of the turning point

$$\ell(r_t) = \int_{r_t}^{r_\Lambda} dr\,\frac{2\,e^{A(r_t)}}{e^{A(r)}\sqrt{e^{2A(r)} - e^{2A(r_t)}}},$$

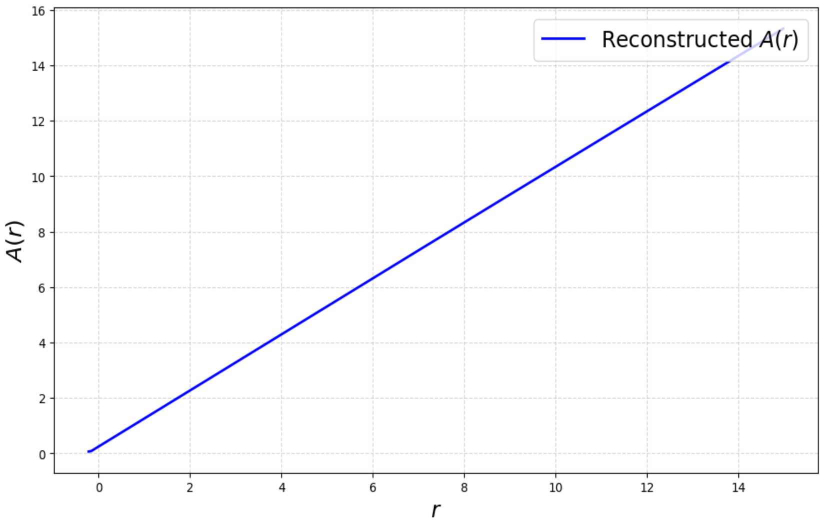
$$S_E(r_t) = \frac{1}{2G}\int_{r_t}^{r_\Lambda} dr\,\frac{e^{A(r)}}{\sqrt{e^{2A(r)} - e^{2A(r_t)}}},$$

We can determine $A(r_t)$ from the previous entanglement entropy data $S_E(\ell)$ by solving these integral equations.

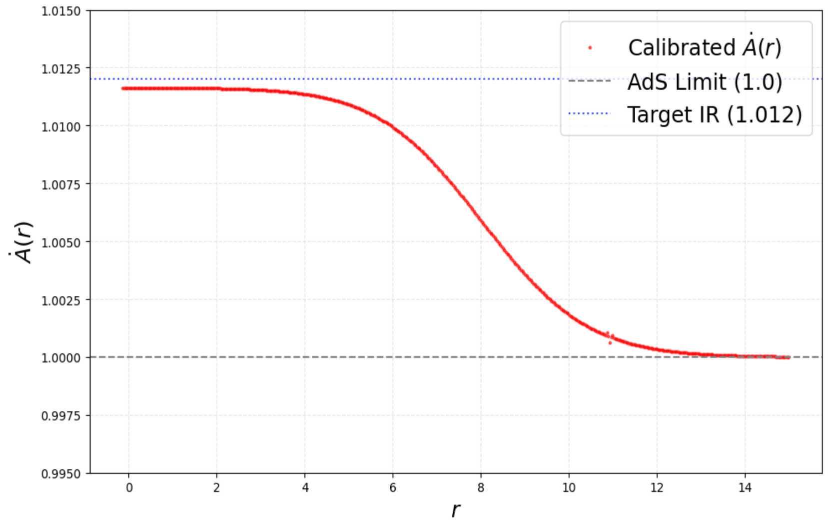
When only the numerical entanglement entropy is known,



we can numerically reconstruct the dual geometry



(a) Reconstructed $A(r)$



(b) $\dot{A} = \frac{dA(r)}{dr}$

Going beyond the dual geometry,

we can further reconstruct the dual gravity theory, which admits the obtained geometry as a geometric solution

To do so, we first guess the general form of the dual gravity theory.

When we identify the scalar operator with the scalar bulk field in the dual gravity, which does not break the boundary Lorentz symmetry, the most general Einstein-scalar gravity theory is

$$\mathcal{S}_{gr} = \frac{1}{2\kappa^2} \int d^3x \sqrt{-G} \left(\mathcal{R} - 2\Lambda_3 - \frac{1}{2} \partial_M \phi \, \partial^M \phi - \frac{V(\phi)}{R^2} \right)$$

Under the previous metric ansatz, the equations of motion are reduced to

$$0 = 4\dot{A}^2 - \dot{\phi}^2 - \frac{4}{R^2} + 2V(\phi),$$

$$0 = 4\ddot{A} + 4\dot{A}^2 + \dot{\phi}^2 - \frac{4}{R^2} + 2V(\phi),$$

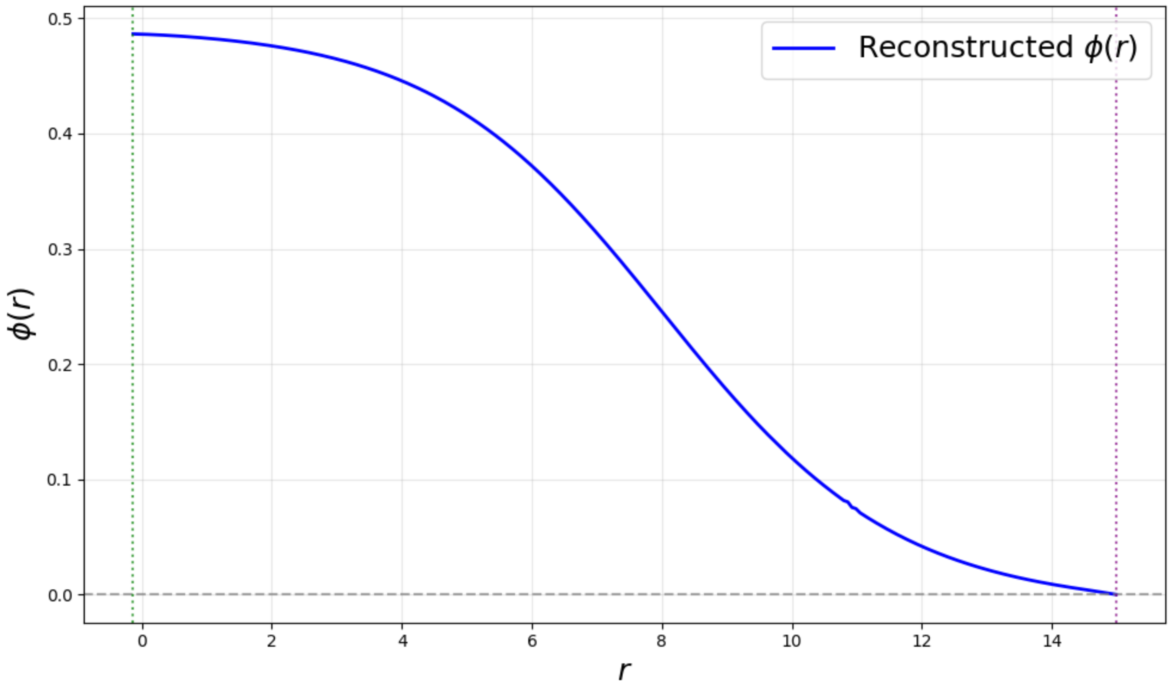
$$0 = \ddot{\phi} + 2\dot{A}\dot{\phi} - \frac{\partial V(\phi)}{\partial \phi},$$

When A(r) is known, we find

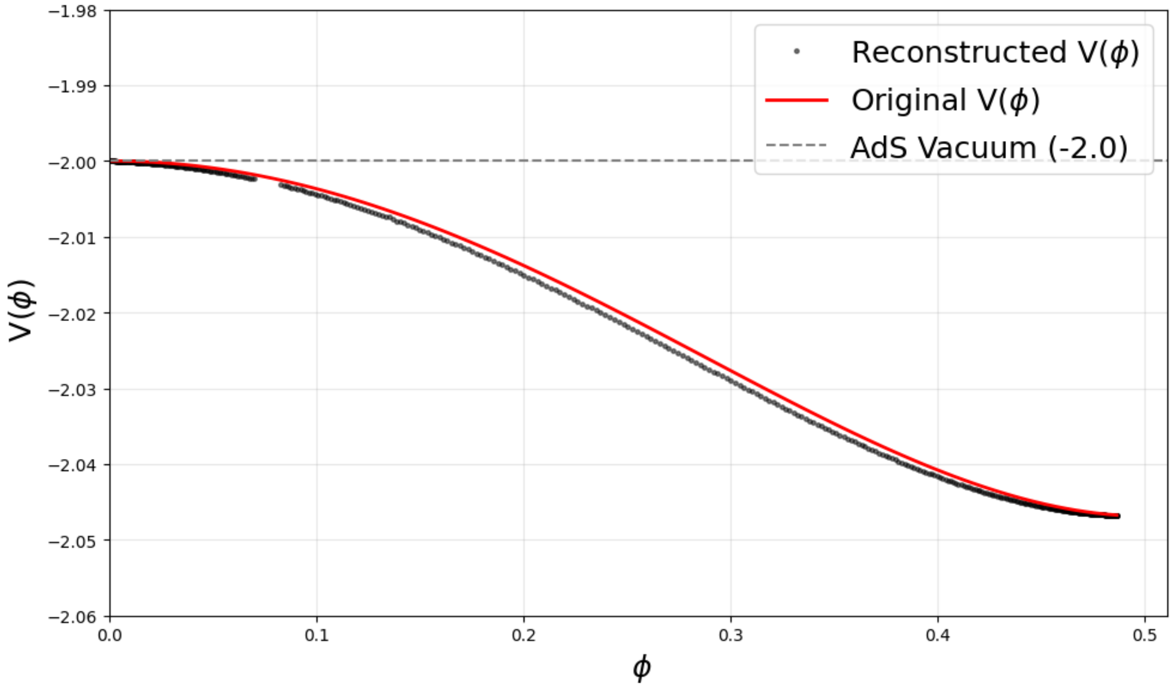
$$\dot{\phi} = \pm\sqrt{-2\ddot{A}}\,.$$

$$V(r) = 2 - R^2 \left(2\dot{A}^2 + \ddot{A} \right).$$

As a result, we obtain the scalar field profile and the scalar potential



(a) $\phi(r)$



(b) $V(\phi)$

The original scalar potential, which was used to derive the numerical entanglement entropy data, is

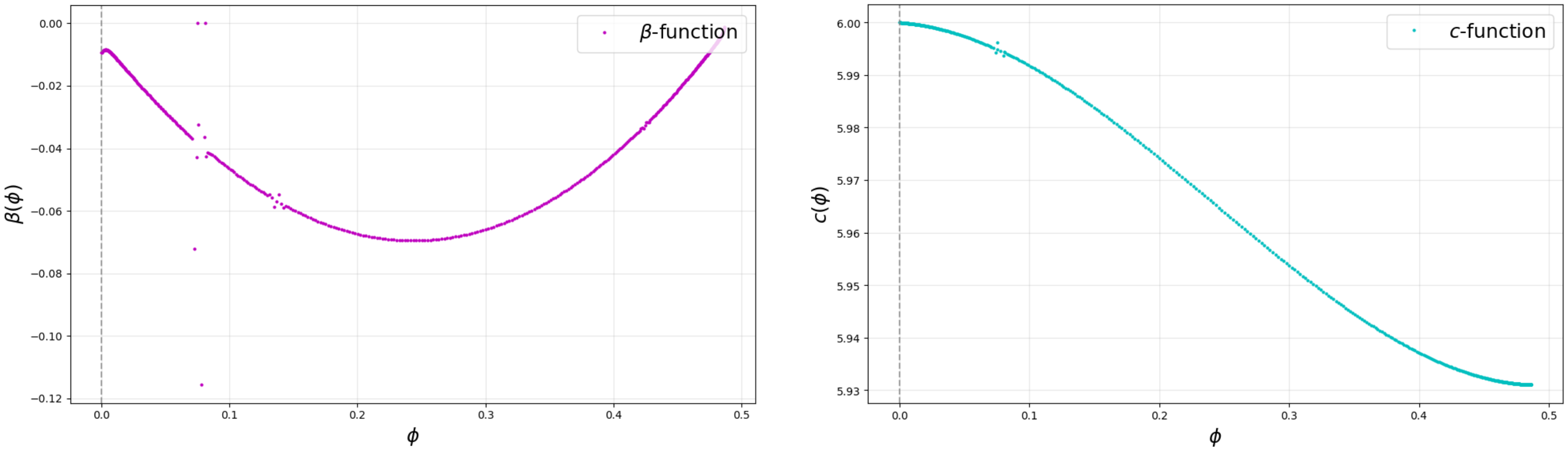
$$V(\phi) = \frac{m_\phi^2}{2}\phi^2 + \frac{\lambda}{4}\phi^4, \qquad \text{for} \quad m_\phi^2 = -3/4 \text{ and } \lambda = 3 \text{ with } R = 1 \text{ and } G = 1/4$$

Using the polynomial that fits the numerical data well,

we can find the form of the scalar potential with a small numerical error

Term	Reconstructed	True Value	Error (%)
$c_2(= m_\phi^2/2)$	-0.372	-0.375	0.73 %
c_3	0.038	0.000	N/A
$c_4(= \lambda/4)$	0.729	0.750	2.74 %

From the reconstructed dual gravity theory, we get more information about the original system



(a) β-function (b) c-function