Quadratic effective energy-momentum tensor during Inflation in various gauge conditions

Inyong Cho (SeoulTech)

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- In preparation: Uniform-Density Gauge, Uniform-Expansion Gauge
- Past works: Longitudinal/Spatially-flat/Comoving Gauges

  co-work with Jinn-Ouk Gong (Ewha Womans U.) & Seung Hun Oh (Tech U. of Korea)
  - PRD 106, 4, 084027 (2022): Inflation,
  - PRD 102, 043531 (2020): in Friedmann Universe

Outline

- Introduction to cosmological perturbations
- 2^{nd} order perturbations and back reaction
  :- Quadratic effective Energy-Momentum Tensor (QEMT/2EMT)
- Gauge Problem of QEMT
- Motivation: Gauge Invariance – Gauge Fixed – Observable
- Results (QEMT) for scalar mode in 5 gauge conditions

1. **Longitudinal**
2. **Spatially-flat**
3. **Comoving**
4. **Uniform-Density**
5. **Uniform-Expansion**

- Conclusions

Introduction to cosmological perturbations

Metric: Scalar-Vector-Tensor Decomposition

$$ds^2 = a^2(\eta)\left[-(1+2A)d\eta^2 - 2B_i\, d\eta\, dx^i + (\delta_{ij} + 2C_{ij})dx^i dx^j\right]$$

$$\boxed{A = \alpha, \qquad B_i = \partial_i\beta + S_i,} \qquad \boxed{C_{ij} = -\psi\delta_{ij} + \partial_i\partial_j E + \frac{1}{2}(\partial_i F_j + \partial_j F_i) + h_{ij}.}$$

Gauge Transformations

$$\boxed{x^\alpha \to \tilde{x}^\alpha = x^\alpha + \xi^\alpha} \text{ : infinitesimal coord. transformation}$$

$$\boxed{\xi^\alpha = (\xi^0, \xi^i) = (\xi^0, \xi^i_\perp + \zeta^{,i})} \qquad \xi^i_{\perp,i} = 0, \qquad \xi_{\perp i} = \xi^i_\perp$$

T L

Scalar perturbation

$$\tilde{\alpha} = \alpha - \mathcal{H}T - T',$$

$$\tilde{\beta} = \beta + L' - T,$$

$$\tilde{\psi} = \psi + \mathcal{H}T,$$

$$\tilde{E} = E - L,$$

Gauge Invariant Variables
Bardeen Variables (1980)

$$\boxed{\Phi = \alpha - (Q' + \mathcal{H}Q),}$$

$$\boxed{\Psi = \psi + \mathcal{H}Q}$$

$$\boxed{Q \equiv \beta + E'.}$$

Gauge Fixing

- **Gauge freedom:** most important in scalar perturbation
- Free to choose ξ^0 and ζ → 2 conditions

  ξ^0: **T**, ζ: **L**

e.g.) Longitudinal Gauge

$\boxed{\beta = E = 0 \quad \Rightarrow \quad \xi^0 = \zeta = 0}$: gauge conditions

$\boxed{\widetilde{\alpha} = \alpha = \Phi, \qquad \widetilde{\psi} = \psi = \Psi.}$: the other 2 scalars → gauge invariant

$$\boxed{ds^2 = a^2(\eta)\left[-(1+2\Phi)\,d\eta^2 + (1-2\Psi)\gamma_{ij}\,dx^i dx^j\right]}$$

: so called, "Conformal Newtonian" gauge

Gauge Invariance
vs.
Gauge Fixing

1. Gauge invariance

> is unchanged under transformations between different diffeomorphic representatives of the same physical geometry.

2. Gauge fixing

> fixes the temporal slicing and spatial threading, thereby selecting one definite representative from each diffeomorphism orbit.

1st order Einstein's equation | Brandenberger & Mukhanov (1992, Phys. Rep.)

$\boxed{G^{\mu}{}_{\nu} = 8\pi G T^{\mu}{}_{\nu}}$: Einstein's equation

$\boxed{G^{\mu}{}_{\nu} = {}^{(0)}G^{\mu}{}_{\nu} + \delta G^{\mu}{}_{\nu} + \cdots ,}$

$\boxed{\delta G^{\mu}{}_{\nu} = 8\pi G\, \delta T^{\mu}{}_{\nu}}$: 1st order Einstein's equation

NOT gauge-invariant, individually

Define Gauge-Invariant Quantities:

$$\delta G_0^{(\mathrm{gi})0} = \delta G_0^0 + \boxed{\left({}^{(0)}G_0^0\right)'(\beta - E')}, \qquad \delta G_i^{(\mathrm{gi})0} = \delta G_i^0 + \left({}^{(0)}G_0^0 - \frac{1}{3}{}^{(0)}G_k^k\right)(\beta - E')_{|i},$$

$$\delta G_j^{(\mathrm{gi})i} = \delta G_j^i + \left({}^{(0)}G_j^i\right)'(\beta - E').$$

$$\delta T_0^{(\mathrm{gi})0} = \delta T_0^0 + \boxed{\left({}^{(0)}T_0^0\right)'(\beta - E')}, \qquad \delta T_i^{(\mathrm{gi})0} = \delta T_i^0 + \left({}^{(0)}T_0^0 - \frac{1}{3}{}^{(0)}T_k^k\right)(\beta - E')_{|i},$$

$$\delta T_j^{(\mathrm{gi})i} = \delta T_j^i + \left({}^{(0)}T_j^i\right)'(\beta - E').$$

$\boxed{\delta G^{\mu}{}_{\nu} = 8\pi G\, \delta T^{\mu}{}_{\nu}} \Rightarrow \boxed{\delta G_{\nu}^{(\mathrm{gi})\mu} = 8\pi G\, \delta T_{\nu}^{(\mathrm{gi})\mu}}$: can be written in gauge-invariant form

Explicit form of G.I. Eq.'s for Inflaton

$$\boxed{\delta G^{(\mathrm{gi})\mu}_{\ \ \nu} = 8\pi G\, \delta T^{(\mathrm{gi})\mu}_{\ \ \nu}}$$

: 1st order equation in gauge-invariant form

$$-3\mathcal{H}\left(\mathcal{H}\Phi+\Psi'\right)+\nabla^2\Psi+3\mathcal{K}\Psi = 4\pi G a^2 \delta T^{(\mathrm{gi})0}_{0},$$
$$\left(\mathcal{H}\Phi+\Psi'\right)_{|i} = 4\pi G a^2 \delta T^{(\mathrm{gi})0}_{i},$$
$$\left[\left(2\mathcal{H}'+\mathcal{H}^2\right)\Phi+\mathcal{H}\Phi'+\Psi''+2\mathcal{H}\Psi'-\mathcal{K}\Psi+\frac{1}{2}\nabla^2 D\right]\delta^i_j-\frac{1}{2}\gamma^{ik}D_{|kj} = -4\pi G a^2 \delta T^{(\mathrm{gi})i}_{j}.$$

Einstein Equation

$$D \equiv \Phi - \Psi.$$

:- derived with NO gauge fixing
:- all written in gauge-invariant (Bardeen) variables
:- valid in arbitrary coord. system

$$\delta\phi^{(\mathrm{gi})} = \delta\phi + \phi_0'\left(\beta - E'\right)$$

$$\delta T^{(\mathrm{gi})0}_{0} = a^{-2}\left[-\phi_0'^{\,2}\Phi+\phi_0'\left(\delta\phi^{(\mathrm{gi})}\right)'+V_{,\phi}\,a^2\delta\phi^{(\mathrm{gi})}\right],$$
$$\delta T^{(\mathrm{gi})0}_{i} = a^{-2}\phi_0'\delta\phi^{(\mathrm{gi})}_{,i},$$
$$\delta T^{(\mathrm{gi})i}_{j} = a^{-2}\left[\phi_0'^{\,2}\Phi-\phi_0'\left(\delta\phi^{(\mathrm{gi})}\right)'+V_{,\phi}\,a^2\delta\phi^{(\mathrm{gi})}\right]\delta^i_j.$$

$$\delta\phi^{(\mathrm{gi})''}+2\mathcal{H}\,\delta\phi^{(\mathrm{gi})'}-\nabla^2\delta\phi^{(\mathrm{gi})}+V_{,\phi\phi}a^2\delta\phi^{(\mathrm{gi})}-4\phi_0'\Phi'+2V_{,\phi}a^2\Phi = 0.$$

Scalar Field Equation

2^{nd} order Perturbations and Back Reaction

Picture

0^{th} order: $G^{(0)}_{\mu\nu} = T^{(0)}_{\mu\nu}$ ($8\pi G=1$)

1^{st} order: $G^{(1)}_{\mu\nu} = T^{(1)}_{\mu\nu}$: solutions

plug in

2^{nd} order: $G^{(2)}_{\mu\nu} = T^{(2)}_{\mu\nu}$

$$\boxed{G^{(1)}_{\mu\nu}[g^{(2)}] + G^{(2)}_{\mu\nu}[g^{(1)}]}$$

Linear in $g^{(2)}$ Quadratic in $g^{(1)}$

quadratic terms only

$$\boxed{G^{(1)}_{\mu\nu}[g^{(2)}] = \cancel{T^{(1)}_{\mu\nu}[\psi^{(2)}]} + T^{(2)}_{\mu\nu}[g^{(1)},\psi^{(1)}] - G^{(2)}_{\mu\nu}[g^{(1)}] \equiv T^{(2,\mathrm{eff})}_{\mu\nu}}$$

Back-reaction source

$$\boxed{\mathbf{cf)}\ \langle\delta\phi^{(2)}\rangle = 0,\ \langle\delta\rho^{(2)}\rangle = 0}$$

stochastic

QEMT: gauge dependent very possibly

$$G_{\mu\nu} = T_{\mu\nu}$$
$$\Rightarrow \quad G^{(0)}_{\mu\nu} + G^{(1)}_{\mu\nu} + G^{(2)}_{\mu\nu} + \cdots = T^{(0)}_{\mu\nu} + T^{(1)}_{\mu\nu} + T^{(2)}_{\mu\nu} + \cdots .$$

The equality is satisfied order by order, $G^{(n)}_{\mu\nu} = T^{(n)}_{\mu\nu}$.

1st order

$$G^{(1)}_{\mu\nu} = T^{(1)}_{\mu\nu} \longrightarrow \overline{G}^{(1)}_{\mu\nu} = \overline{T}^{(1)}_{\mu\nu}$$

$$\overline{G}^{(1)}_{\mu\nu} = G^{(1)}_{\mu\nu} + \delta G^{(1)}_{\mu\nu}, \qquad \overline{T}^{(1)}_{\mu\nu} = T^{(1)}_{\mu\nu} + \delta T^{(1)}_{\mu\nu}.$$

: terms making the quantity gauge invariant

2nd order

$$G^{(2)}_{\mu\nu} = T^{(2)}_{\mu\nu} \longrightarrow \overline{G}^{(2)}_{\mu\nu} = \overline{T}^{(2)}_{\mu\nu}$$

$$\overline{G}^{(2)}_{\mu\nu} = G^{(2)}_{\mu\nu} + \delta G^{(2)}_{\mu\nu} = G^{(1)}_{\mu\nu}[g^{(2)}] + \underline{G^{(2)}_{\mu\nu}[g^{(1)}]} + \delta G^{(1)}_{\mu\nu}[g^{(2)}] + \delta G^{(2)}_{\mu\nu}[g^{(1)}]$$
$$\overline{T}^{(2)}_{\mu\nu} = T^{(2)}_{\mu\nu} + \delta T^{(2)}_{\mu\nu} = T^{(1)}_{\mu\nu}[g^{(2)}, \psi^{(2)}] + \underline{T^{(2)}_{\mu\nu}[g^{(1)}, \psi^{(1)}]} + \delta T^{(1)}_{\mu\nu}[g^{(2)}, \psi^{(2)}] + \delta T^{(2)}_{\mu\nu}[g^{(1)}, \psi^{(1)}],$$

$$T^{(2,\mathrm{eff})}_{\mu\nu} \equiv T^{(2)}_{\mu\nu}[g^{(1)}, \psi^{(1)}] - G^{(2)}_{\mu\nu}[g^{(1)}].$$

: QEMT responsible for Back-Reaction
→ **hard** to be gauge invariant

QEMT(k)

$$\hat{\tau}_{\mu\nu} \equiv 8\pi G \tau_{\mu\nu}(\mathbf{k}) = 8\pi G \left\langle T^{(2,\mathrm{eff})}_{\mu\nu}(\mathbf{k}) \right\rangle$$

: Fourier Transformed

Interpretation

$$T^{\mu\ (2,\mathrm{eff})}_{\nu} = \begin{pmatrix} -\rho_{\mathrm{eff}} & 0 & 0 & 0 \\ 0 & p_{\mathrm{eff}} & 0 & 0 \\ 0 & 0 & p_{\mathrm{eff}} & 0 \\ 0 & 0 & 0 & p_{\mathrm{eff}} \end{pmatrix}$$

$$\boxed{g^{(2)}_{\mu\nu}}$$

$$\boxed{T^{(2,\mathrm{eff})}_{\mu\nu} \equiv T^{(2)}_{\mu\nu}[g^{(1)}, \psi^{(1)}] - G^{(2)}_{\mu\nu}[g^{(1)}].}$$

Gauge Problem

QEMT for a scalar field: **gauge dependent**

$\boxed{Q = \beta + E'\,.}$ $\boxed{E}$ **: gauge variables**

$$
\begin{aligned}
8\pi G\tau_{00} = \frac{1}{\mathcal{H}' - \mathcal{H}^2}\Big\{ & -\langle(\nabla\Psi')^2\rangle - \langle(\Delta\Psi)^2\rangle - 2(K+\mathcal{H})\langle\nabla\Psi'\cdot\nabla\Psi\rangle - \Big[3(\mathcal{H}'-\mathcal{H}^2) + K^2 + a^2V_{\phi\phi}\Big]\langle(\Psi')^2\rangle \\
& + (9\mathcal{H}' - 10\mathcal{H}^2 - 2L)\langle(\nabla\Psi)^2\rangle + 2\Big[6\mathcal{H}(\mathcal{H}'-\mathcal{H}^2) - KL + 2(\mathcal{H}'-\mathcal{H}^2)a^2\frac{V_\phi}{\phi_0'} - a^2\mathcal{H}V_{\phi\phi}\Big]\langle\Psi\Psi'\rangle \\
& - \Big[L^2 - 4\mathcal{H}(\mathcal{H}'-\mathcal{H}^2)a^2\frac{V_\phi}{\phi_0'} + a^2\mathcal{H}^2V_{\phi\phi}\Big]\langle\Psi^2\rangle\Big\}
\end{aligned}
$$

$$
\boxed{
\begin{aligned}
& + 2\Big\langle\Big\{a^2\frac{V_\phi}{\phi_0'}Q' + \Big[3\mathcal{H}' + (4\mathcal{H}+K)a^2\frac{V_\phi}{\phi_0'} + a^2V_{\phi\phi}\Big]Q - \Delta Q + \Delta E' - 2\mathcal{H}\Delta E\Big\}\Psi'\Big\rangle \\
& + 2\Big\langle\Big\{-\Big(L + 6\mathcal{H}^2 - 2a^2\mathcal{H}\frac{V_\phi}{\phi_0'}\Big)Q' + \Big[2\mathcal{H}(L-3\mathcal{H}') + (L - 2\mathcal{H}' + 4\mathcal{H}^2)a^2\frac{V_\phi}{\phi_0'} + \mathcal{H}a^2V_{\phi\phi}\Big]Q \\
& \qquad - \Delta Q' - (K-\mathcal{H})\Delta Q + 2\mathcal{H}\Delta E' + \Delta^2 E\Big\}\Psi\Big\rangle - (\mathcal{H}' + 2\mathcal{H}^2)\langle Q'^2\rangle - 2\Big[\mathcal{H}(\mathcal{H}' - 4\mathcal{H}^2) + (\mathcal{H}'-\mathcal{H}^2)a^2\frac{V_\phi}{\phi_0'}\Big]\langle Q'Q\rangle \\
& - \Big\{(\mathcal{H}'-\mathcal{H}^2)\Big[4\mathcal{H}^2 + 8\mathcal{H}a^2\frac{V_\phi}{\phi_0'} + a^4\Big(\frac{V_\phi}{\phi_0'}\Big)^2 + a^2V_{\phi\phi}\Big] + 3\mathcal{H}'(\mathcal{H}' - 4\mathcal{H}^2)\Big\}\langle Q^2\rangle \\
& - 2\mathcal{H}\langle\nabla Q'\cdot\nabla Q\rangle + \langle[4\mathcal{H}^2Q + 2\mathcal{H}\Delta E - (\mathcal{H}' + 2\mathcal{H}^2)E']\Delta E'\rangle + 4\mathcal{H}\langle(\mathcal{H}Q' + \mathcal{H}'Q)\Delta E\rangle,
\end{aligned}
}
$$

$$
\begin{aligned}
8\pi G\tau_{0i} =& \, 0, \\
8\pi G\tau_{ij} =& \, \delta_{ij}\Bigg[\frac{1}{\mathcal{H}'-\mathcal{H}^2}\Bigg\{\frac{1}{3}\langle(\nabla\Psi')^2\rangle - \langle(\Delta\Psi)^2\rangle + 2\left(\frac{\mathcal{H}}{3}-K\right)\langle\nabla\Psi'\cdot\nabla\Psi\rangle + \left(\mathcal{H}'-\mathcal{H}^2-K^2+a^2V_{\phi\phi}\right)\langle(\Psi')^2\rangle \\
& + \left[\frac{1}{3}(11\mathcal{H}'-10\mathcal{H}^2)-2L\right]\langle(\nabla\Psi)^2\rangle + 2\left[\mathcal{H}a^2V_{\phi\phi}-KL+2(\mathcal{H}'-\mathcal{H}^2)(K+2\mathcal{H})\right]\langle\Psi\Psi'\rangle \\
& + \left[4(\mathcal{H}'-\mathcal{H}^2)(\mathcal{H}'+2\mathcal{H}^2+L)-L^2+\mathcal{H}^2a^2V_{\phi\phi}\right]\langle\Psi^2\rangle\Bigg\} \\
& + 2\left\langle\left[Q''+3(3\mathcal{H}-K)Q'+\left(\mathcal{H}'+12\mathcal{H}^2+K^2-7\mathcal{H}K-a^2V_{\phi\phi}\right)Q+\frac{1}{3}\Delta E'-\frac{4}{3}(K-\mathcal{H})\Delta E\right]\Psi'\right\rangle \\
& + 2\Bigg\langle\Bigg\{4\mathcal{H}Q''+(6\mathcal{H}'+10\mathcal{H}^2-3L)Q'+\Delta Q'-(K-3\mathcal{H})\Delta Q \\
& - \left[2\mathcal{H}''+a^2\mathcal{H}V_{\phi\phi}-\mathcal{H}K^2+(5\mathcal{H}'+2\mathcal{H}^2)K-11\mathcal{H}'\mathcal{H}-9\mathcal{H}^3\right]Q-\frac{1}{3}\Delta E''-\frac{2}{3}\mathcal{H}\Delta E'-\frac{4}{3}L\Delta E+\frac{2}{3}\Delta^2E\Bigg\}\Psi\Bigg\rangle \\
& + \frac{2}{3}\langle\nabla Q''\cdot\nabla Q\rangle + \frac{2}{3}\langle(\nabla Q')^2\rangle + \frac{4\mathcal{H}}{3}\langle\nabla Q'\cdot\nabla Q\rangle + 2\mathcal{H}\langle Q''Q'\rangle - 2\mathcal{H}'\langle Q''Q\rangle - (\mathcal{H}'-2\mathcal{H}^2)\langle Q'^2\rangle \\
& - 2\left[2\mathcal{H}''+3\mathcal{H}'\mathcal{H}+(\mathcal{H}'-\mathcal{H}^2)(K+\mathcal{H})\right]\langle Q'Q\rangle \\
& - \left\{4\mathcal{H}\mathcal{H}''+3\mathcal{H}'^2+4\mathcal{H}'\mathcal{H}^2+(\mathcal{H}'-\mathcal{H}^2)\left[4\mathcal{H}^2+a^2V_{\phi\phi}+a^4\left(\frac{V_\phi}{\phi_0'}\right)^2\right]\right\}\langle Q^2\rangle \\
& - \frac{2}{3}\langle(2\mathcal{H}Q-3\mathcal{H}E'+\Delta E)\Delta E''\rangle - \frac{1}{3}\left\langle\left[8\mathcal{H}Q'+8(\mathcal{H}'+\mathcal{H}^2)Q-3(\mathcal{H}'+2\mathcal{H}^2)E'+2\Delta E'+4\mathcal{H}\Delta E\right]\Delta E'\right\rangle \\
& - \frac{4}{3}\left\langle\left\{\mathcal{H}Q''+2(\mathcal{H}'+\mathcal{H}^2)Q'+2\left[3\mathcal{H}\mathcal{H}'-\mathcal{H}^3-(\mathcal{H}'-\mathcal{H}^2)K\right]Q\right\}\Delta E\right\rangle\Bigg].
\end{aligned}
$$

Motivation

Gauge invariant $\neq$ Gauge fixed $\neq$ Observable

Three Different Notions

Concept	Meaning	What it solves	What it does not guarantee
Gauge invariant	unchanged under gauge transformation	removes coordinate ambiguity	**not automatically measurable**
Gauge-fixed	defined after complete gauge choice	selects one representative in a chosen gauge	**not automatically physical**
Observable	defined by measurement procedure	**connects theory to experiment/observation data**	**requires operational input**

Central statement

> Gauge invariance is not the same as observability.
> Gauge fixing is not the same as observability.

Step 1: Gauge-Invariant Quantities

A quantity can be free of coordinate ambiguity without being directly measured.

$$\tilde{Q} = Q$$

- ✓ Unchanged under an infinitesimal gauge transformation.
- ✓ Not a coordinate artifact.
- ✓ A natural candidate for physical interpretation.

Examples

- Bardeen potentials: Φ, Ψ
- Comoving curvature perturbation: $\mathcal{R}$
- Mukhanov–Sasaki variable

But

> A gauge-invariant quantity is **not automatically a directly observable quantity**.

Example

- Φ and Ψ are gauge invariant,
- but a telescope does **not** directly measure Φ or Ψ alone.

STEP 2

Step 2: Gauge-Fixed Quantities

- A gauge-fixed quantity is evaluated **after imposing gauge conditions**.
- Complete gauge fixing gives **a unique representative**, **not automatically an observable.**

Examples of complete scalar gauge choices

- Longitudinal gauge
- Spatially-flat gauge
- Comoving gauge
- Uniform-density gauge
- Uniform-expansion gauge
- …….

What gauge fixing does

- Removes residual coordinate freedom
- Makes expressions unambiguous in that gauge
- Can be tied to a physical slicing
- Does not by itself define a measurement
- A gauge-fixed quantity is **not automatically observable**.

Example

In longitudinal gauge,

$$\alpha_{\mathrm{L}} = \Phi$$

so α coincides with a gauge-invariant quantity **in that gauge**.

However:

- this does **not** mean that every gauge-fixed variable is observable;
- it only means that some gauge-fixed variables may represent a gauge-invariant quantity.

Key statement

Gauge-fixed means "uniquely defined in a chosen gauge," not "automatically measurable."

Step 3: Observables

An observable is tied to an operational measurement procedure.

An observable specifies:

1. a physical observer,
2. a reference frame,
3. a signal propagation process,
4. and a detector response.

Examples

① observed redshift z
② CMB temperature anisotropy $\Delta T/T$
③ luminosity distance D_L
④ angular diameter distance D_A
⑤ observed galaxy number counts

Key statement

> A true observable must ultimately be free of gauge ambiguity,
> but its theoretical expression may involve gauge-dependent intermediate variables.

EXAMPLE

CMB Example: The Observable Is a Complete Combination

➢IThe observed CMB temperature anisotropy is not given by a single perturbation variable.
➢ndividual pieces can be gauge dependent; the full prediction is not.

$$\mathcal{O} = A + B + C + \cdots$$

- A: intrinsic temperature perturbation
- B: gravitational redshift contribution
- C: Doppler contribution
- plus integrated line-of-sight effects

Each contribution may be gauge dependent:

$$\widetilde{A} = A + \delta_{\text{gauge}} A \neq A, \qquad \widetilde{B} = B + \delta_{\text{gauge}} B \neq B, \qquad \widetilde{C} = C + \delta_{\text{gauge}} C \neq C.$$

the gauge-transformed quantity is

$$\begin{aligned} \widetilde{\mathcal{O}} &= \widetilde{A} + \widetilde{B} + \widetilde{C} + \cdots \\ &= (A + \delta_{\text{gauge}} A) + (B + \delta_{\text{gauge}} B) + (C + \delta_{\text{gauge}} C) + \cdots \\ &= \mathcal{O} + \delta_{\text{gauge}} A + \delta_{\text{gauge}} B + \delta_{\text{gauge}} C + \cdots . \end{aligned}$$

A physical observable must satisfy

$$\widetilde{\mathcal{O}} = \mathcal{O}.$$

Therefore,

$$\boxed{\delta_{\text{gauge}} A + \delta_{\text{gauge}} B + \delta_{\text{gauge}} C + \cdots = 0}$$

Message

> Individual pieces may be gauge dependent, while the full observable is gauge invariant.

Lessons from the CMB and Clock Examples

The same logic appears in many cosmological observables.

Lesson 1

A physical observable may be built
from several individually gauge-dependent pieces.

Lesson 2

Gauge invariance of the final observable can emerge
through cancellations among intermediate terms.

Lesson 3

A gauge-fixed value may represent an observable
only when the gauge is tied to a physical clock, slicing,
or measurement prescription.

**The right question is not only "Is A gauge dependent?",
but "Can A be embedded in a complete observable?"**

Observer-Centered Formulation

A general schematic form of an observable

$$\boxed{\mathcal{O}_{\text{obs}} = \mathcal{M}\left[A,\ z^{\mu}(\tau),\ u^{\mu},\ {e_a}^{\mu},\ k^{\mu},\ X^{A},\ W,\ R_{\text{det}}\right]}$$

- A: theoretical field or tensor
- $z^{\mu}(\tau)$: observer worldline
- u^{μ}: observer 4-velocity
- ${e_a}^{\mu}$: observer tetrad
- k^{μ}: signal propagation
- X^{A}: physical clocks and rods
- W: observation/averaging window
- R_{det}: detector response

Not all of these elements are required for every observable; only the relevant subset is selected depe nding on the measurement.

Practical Examples

The same quantity can sit at different levels depending on how it is used.

Bardeen potential Φ	**Gauge invariant**	Enters SW, ISW, and lensing, but is not usually measured alone.
Longitudinal-gauge lapse α_{L}	**Gauge-fixed represen tation**	In longitudinal gauge, $\alpha_{\mathrm{L}} = \Phi$ Useful, but not automatically a direct observable.
CMB anisotropy $\left(\frac{\Delta T}{T}\right)_{\mathrm{obs}}$	**Observable**	Constructed from source + propagation + observer expression.
Observed redshift $1 + z = \frac{(k_\mu u^\mu)_s}{(k_\mu u^\mu)_o}$	**Observable**	Involves source, observer, and photon propagation

Conceptual Position of the 2EMT

A gauge-fixed quadratic EMT is well-defined, but not automatically an observable.

Computing $\tau_{\mu\nu}^{(2)}$ in a complete gauge gives a definite expression.
If different gauges give different results, gauge fixing alone cannot settle observability.
A physical interpretation requires one of the following additional structures:

Gauge-invariant completion

Construct a quantity whose gauge transformation cancels.

Physical slicing or clock

Interpret the gauge as fixed by a real clock, density surface, or expansion surface.

Observer prescription

Specify the observer, local frame, signal, and measurement protocol.

Therefore: study the 2EMT as a gauge-dependent quadratic source first, and then ask how it can be embedded in an observable framework.

Several Results for 5 Gauge Choices

Gauge	Temporal slicing condition
Longitudinal	$\beta = 0$
Spatially flat	$\psi = 0$
Comoving	$\delta\phi = 0$
Uniform density	$\delta\rho = 0$
Uniform expansion	$\delta\theta = 0$

$(E = 0)$

After gauge fixing, all choices give $\tau_{\mu\nu}(\Psi)$

Fourier-mode expansion

$$\boxed{\Psi\left(\eta,\mathbf{x}\right)=\int \mathrm{d}^3\mathbf{k}\Psi_{\mathbf{k}}\left(\eta\right)\mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{x}}}$$

Slow-roll parameters

$$\boxed{\begin{aligned}\epsilon &\equiv -\frac{\dot H}{H^2}=4\pi G\frac{\dot\phi_0^2}{H^2},\\ \delta &\equiv -\frac{\ddot\phi_0}{H\dot\phi_0}=\epsilon-\frac{\dot\epsilon}{2H\epsilon},\end{aligned}}$$

Solutions

$$\boxed{\Psi_{\mathbf{k}}\left(\eta\right)\approx\begin{cases}A_1\epsilon & \text{(for long wavelength)}\\ 4\pi G\dot\phi_0\left[c_1\sin\left(k\eta\right)+c_2\cos\left(k\eta\right)\right] & \text{(for short wavelength)}\end{cases}}$$

QEMT(k)

$$\boxed{\hat\tau_{\mu\nu}\equiv 8\pi G\tau_{\mu\nu}(\mathbf{k})=8\pi G\left\langle T_{\mu\nu}^{(2,\mathbf{eff})}(\mathbf{k})\right\rangle}$$

For long-wavelength, $\sigma_1 \equiv k/\mathcal{H} \ll 1$.
For short-wavelength, $\sigma_2 \equiv \mathcal{H}/k \ll 1$.

1. Longitudinal Gauge

A. Long-Wavelength

$$\hat{\tau}_{00} = k^2 |A_1|^2 \left[\left(2\epsilon + 4\epsilon\sigma_1^2\right) + \left(-2\epsilon(4\delta - 11\epsilon) + \frac{3\epsilon(\delta - 3\epsilon)}{\sigma_1^2}\right) + \left(8\epsilon(\delta-\epsilon)^2 - \frac{\epsilon(12\delta^2 - 13\epsilon^2)}{\sigma_1^2}\right) + \left(-\frac{12\delta\epsilon(\delta-\epsilon)^2}{\sigma_1^2}\right)\right].$$

$$\hat{\tau}_{ij} = k^2 |A_1|^2 \left[\left(-\frac{2\epsilon}{3} + 4\epsilon\sigma_1^2\right) + \left(\frac{2\epsilon(4\delta + \epsilon)}{3} - \frac{3\epsilon(\delta - 3\epsilon)}{\sigma_1^2}\right) + \left(-\frac{8\epsilon(\delta-\epsilon)^2}{3} + \frac{\epsilon(36\delta^2 - 48\delta\epsilon - 9\epsilon^2)}{3\sigma_1^2}\right) + \left(-\frac{4\epsilon(\delta-\epsilon)^2(3\delta+2\epsilon)}{\sigma_1^2}\right)\right].$$

B. Short-Wavelength

$$\hat{\tau}_{00} = \frac{2\pi G k^4}{a^2}\left(|c_1|^2 + |c_2|^2\right)\left[\left(6 + 2\sigma_2^2\right) + \left((14\epsilon - \delta)\sigma_2^2 + 3(\delta - 3\epsilon)\sigma_2^4\right) + \left(2\delta^2\sigma_2^2 - (6\delta^2 + 6\delta\epsilon - \epsilon^2)\sigma_2^4\right) + \left(3\delta^3\sigma_2^4\right)\right].$$

$$\hat{\tau}_{ij} = \frac{2\pi G k^4}{a^2}\left(|c_1|^2 + |c_2|^2\right)\left[\left(\frac{10}{3} - \frac{2}{3}\sigma_2^2\right) + \left(-\frac{5\delta - 4\epsilon}{3}\sigma_2^2 - 3(\delta - 3\epsilon)\sigma_2^4\right) + \left(-\frac{2\delta^2}{3}\sigma_2^2 + \left(6\delta^2 - 2\delta\epsilon - 7\epsilon^2\right)\sigma_2^4\right) + \left(-\delta^2(3\delta + 2\epsilon)\sigma_2^4\right)\right].$$

2. Spatially-flat Gauge

A. Long-Wavelength

$$\hat{\tau}_{00} = k^2 |A_1|^2 \left[\left(\epsilon + \epsilon\sigma_1^2\right) + \left(-\epsilon(4\delta - 9\epsilon) + \frac{3\epsilon(\delta - 3\epsilon)}{\sigma_1^2}\right) \right.$$
$$\left. + \left(4\epsilon(\delta-\epsilon)(\delta-2\epsilon) - \frac{\epsilon(12\delta^2 - 54\delta\epsilon + 53\epsilon^2)}{\sigma_1^2}\right) + \left(\frac{\epsilon(2\delta-3\epsilon)(6\delta^2 - 27\delta\epsilon + 25\epsilon^2)}{\sigma_1^2}\right)\right].$$

$$\hat{\tau}_{ij} = k^2 |A_1|^2 \left[\left(-\frac{\epsilon}{3} + \epsilon\sigma_1^2\right) + \left(\frac{\epsilon(4\delta - 25\epsilon)}{3} - \frac{3\epsilon(\delta - 3\epsilon)}{\sigma_1^2} - \frac{2\epsilon^2}{3}\sigma_1^2\right) \right.$$
$$+ \left(-\frac{4\epsilon(\delta^2 - 13\delta\epsilon + 16\epsilon^2)}{3} + \frac{\epsilon(12\delta^2 - 54\delta\epsilon + 59\epsilon^2)}{\sigma_1^2}\right) + \left(\frac{2\epsilon^2(2\delta-3\epsilon)(2\delta-5\epsilon)}{3} - \frac{\epsilon(2\delta-3\epsilon)(6\delta^2 - 27\delta\epsilon + 41\epsilon^2)}{\sigma_1^2}\right)$$

B. Short-Wavelength

$$\hat{\tau}_{00} = \frac{2\pi G k^4}{a^2}\left(|c_1|^2 + |c_2|^2\right) \left[\left(2 + \sigma_2^2\right) + \left((\delta - 4\epsilon)\sigma_2^2 + 3(\delta - 3\epsilon)\sigma_2^4\right) \right.$$
$$\left. + \left((\delta-\epsilon)^2\sigma_2^2 - \left(6\delta^2 - 24\delta\epsilon + 17\epsilon^2\right)\sigma_2^4\right) + \left((\delta-\epsilon)\left(3\delta^2 - 12\delta\epsilon + 7\epsilon^2\right)\sigma_2^4\right)\right].$$

$$\hat{\tau}_{ij} = \frac{2\pi G k^4}{a^2}\left(|c_1|^2 + |c_2|^2\right) \left[\left(\frac{2}{3} - \frac{1}{3}\sigma_2^2\right) + \left(-\frac{7\delta - 6\epsilon}{3}\sigma_2^2 - 3(\delta - 3\epsilon)\sigma_2^4\right) \right.$$
$$+ \left(\frac{-\delta^2 + 22\delta\epsilon + 11\epsilon^2}{3}\sigma_2^2 + \left(6\delta^2 - 24\delta\epsilon + 23\epsilon^2\right)\sigma_2^4\right)$$
$$\left. + \left(\frac{2\epsilon(\delta-\epsilon)(\delta-3\epsilon)}{3}\sigma_2^2 - (\delta-\epsilon)\left(3\delta^2 - 12\delta\epsilon + 23\epsilon^2\right)\sigma_2^4\right)\right].$$

3. Comoving Gauge

A. Long-Wavelength

$$\hat{\tau}_{00} = k^2 \left|A_1\right|^2 \left[\left(-6 + \frac{9}{\sigma_1^2} - \sigma_1^2\right) + \left(12\delta - 25\epsilon - \frac{36\delta - 49\epsilon}{\sigma_1^2} - 4(\delta - 2\epsilon)\sigma_1^2\right) \right.$$
$$\left. + \left(\epsilon(4\delta + \epsilon) + \frac{4(\delta - \epsilon)(9\delta - 16\epsilon)}{\sigma_1^2}\right) + \left(-4\epsilon(\delta - \epsilon)^2 + \frac{4\epsilon(\delta - \epsilon)^2}{\sigma_1^2}\right) \right].$$

$$\hat{\tau}_{ij} = k^2 \left|A_1\right|^2 \left[\left(20 - \frac{3}{\sigma_1^2} - \frac{5}{3}\sigma_1^2 + \frac{2}{3}\sigma_1^4\right) + \left(-9(4\delta - 5\epsilon) + \frac{12\delta + 5\epsilon}{\sigma_1^2} - \frac{4\delta}{3}\sigma_1^2\right) \right.$$
$$\left. + \left(-\frac{24\delta^2 - 28\delta\epsilon + 15\epsilon^2}{3} - \frac{4(3\delta^2 + 8\delta\epsilon - 13\epsilon^2)}{\sigma_1^2} - \frac{8(\delta - \epsilon)^2}{3}\sigma_1^2\right) + \left(\frac{4\epsilon(\delta - \epsilon)^2}{3} + \frac{4\epsilon(\delta - \epsilon)(11\delta - 17\epsilon)}{\sigma_1^2}\right) \right].$$

B. Short-Wavelength

$$\hat{\tau}_{00} = \frac{2\pi G k^4}{a^2} \left(\left|c_1\right|^2 + \left|c_2\right|^2\right) \left[\left(-\frac{1}{\epsilon} + \frac{3}{\epsilon}\sigma_2^2 + \frac{9}{\epsilon}\sigma_2^4\right) + \left(-\frac{2\delta - 3\epsilon}{\epsilon} + \frac{6(\delta - 2\epsilon)}{\epsilon}\sigma_2^2 - \frac{18\delta - 13\epsilon}{\epsilon}\sigma_2^4\right) \right.$$
$$\left. + \left((2\delta + 5\epsilon)\sigma_2^2 + \frac{\delta(9\delta - 14\epsilon)}{\epsilon}\sigma_2^4\right) + \left(-\delta^2\sigma_2^2 + \delta^2\sigma_2^4\right) \right].$$

$$\hat{\tau}_{ij} = \frac{2\pi G k^4}{a^2} \left(\left|c_1\right|^2 + \left|c_2\right|^2\right) \left[\left(-\frac{11}{3\epsilon} + \frac{17}{\epsilon}\sigma_2^2 - \frac{3}{\epsilon}\sigma_2^4\right) + \left(-\frac{2\delta + 3\epsilon}{3\epsilon} - \frac{2(9\delta - 10\epsilon)}{\epsilon}\sigma_2^2 + \frac{6\delta + 17\epsilon}{\epsilon}\sigma_2^4\right) \right.$$
$$\left. + \left(-\frac{2\delta^2}{3\epsilon} - \frac{6\delta^2 + 10\delta\epsilon + 11\epsilon^2}{3\epsilon}\sigma_2^2 - \frac{3\delta^2 + 28\delta\epsilon - 8\epsilon^2}{\epsilon}\sigma_2^4\right) + \left(\frac{\delta^2}{3}\sigma_2^2 + \delta(11\delta - 12\epsilon)\sigma_2^4\right) \right].$$

4. Uniform-Density Gauge

A. Long-Wavelength

$$\hat{\tau}_{00} = k^2 |A_1|^2 \Bigg[\left(-4 + \frac{9}{\sigma_1^2} - \frac{22}{3}\sigma_1^2 + \frac{10}{9}\sigma_1^4 \right) + \left(\frac{12\delta - 77\epsilon}{3} - \frac{36\delta - 49\epsilon}{\sigma_1^2} - \frac{120\delta - 217\epsilon}{9}\sigma_1^2 + \frac{2(2\delta - 3\epsilon)}{9}\sigma_1^4 \right) + \left(\frac{24\delta^2 - 36\delta\epsilon + 29\epsilon^2}{3} + \frac{4(\delta - \epsilon)(9\delta - 16\epsilon)}{\sigma_1^2} - \frac{36\delta^2 - 151\delta\epsilon + 135\epsilon^2}{9}\sigma_1^2 \right) + \left(-\frac{4}{3}\epsilon\delta(\delta - \epsilon) + \frac{4\epsilon(\delta - \epsilon)^2}{\sigma_1^2} + \frac{\epsilon(2\delta - 3\epsilon)^2}{9}\sigma_1^2 \right) \Bigg].$$

$$\hat{\tau}_{ij} = k^2 |A_1|^2 \Bigg[\left(34 - \frac{3}{\sigma_1^2} + \frac{64}{9}\sigma_1^2 + \frac{58}{27}\sigma_1^4 - \frac{2}{27}\sigma_1^6 \right) + \left(-\frac{168\delta - 205\epsilon}{3} + \frac{12\delta + 5\epsilon}{\sigma_1^2} - \frac{100\delta + 47\epsilon}{9}\sigma_1^2 + \frac{16(6\delta - 7\epsilon)}{27}\sigma_1^4 \right) + \left(-\frac{72\delta^2 - 120\delta\epsilon + 85\epsilon^2}{3} - \frac{4(3\delta^2 + 8\delta\epsilon - 13\epsilon^2)}{\sigma_1^2} - \frac{5(28\delta^2 - 57\delta\epsilon + 25\epsilon^2)}{9}\sigma_1^2 + \frac{2(2\delta - 5\epsilon)(2\delta - 3\epsilon)}{27}\sigma_1^4 \right) + \left(\frac{4\epsilon(\delta - \epsilon)(5\delta - 6\epsilon)}{3} + \frac{4\epsilon(\delta - \epsilon)(11\delta - 17\epsilon)}{\sigma_1^2} + \epsilon(2\delta - 3\epsilon)^2\sigma_1^2 \right) \Bigg].$$

B. Short-Wavelength

$$\hat{\tau}_{00} = \frac{2\pi G k^4}{a^2}\left(|c_1|^2 + |c_2|^2\right)\Bigg[\left(-\frac{16}{3\epsilon} + \frac{1}{9\epsilon\,\sigma_2^2} + \frac{5}{\epsilon}\sigma_2^2 + \frac{9}{\epsilon}\sigma_2^4\right)$$
$$+\left(-\frac{2(30\delta - 47\epsilon)}{9\epsilon} + \frac{2\delta - \epsilon}{9\epsilon\,\sigma_2^2} + \frac{2(3\delta - 31\epsilon)}{3\epsilon}\sigma_2^2 - \frac{18\delta - 13\epsilon}{\epsilon}\sigma_2^4\right)$$
$$+\left(\frac{-9\delta^2 + 41\delta\epsilon - 23\epsilon^2}{9\epsilon} + \frac{6\delta^2 + 6\delta\epsilon + 17\epsilon^2}{3\epsilon}\sigma_2^2 + \frac{\delta(9\delta - 14\epsilon)}{\epsilon}\sigma_2^4\right)$$
$$+\left(\frac{(\delta - \epsilon)^2}{9} - \frac{\delta(\delta + 2\epsilon)}{3}\sigma_2^2 + \delta^2\sigma_2^4\right)\Bigg].$$

$$\hat{\tau}_{ij} = \frac{2\pi G k^4}{a^2}\left(|c_1|^2 + |c_2|^2\right)\Bigg[\left(-\frac{8}{9\epsilon} - \frac{11}{27\epsilon\,\sigma_2^2} + \frac{31}{\epsilon}\sigma_2^2 - \frac{3}{\epsilon}\sigma_2^4\right)$$
$$+\left(-\frac{8(10\delta + 9\epsilon)}{9\epsilon} + \frac{54\delta - \epsilon}{27\epsilon\,\sigma_2^2} - \frac{2(45\delta - 41\epsilon)}{3\epsilon}\sigma_2^2 + \frac{6\delta + 17\epsilon}{\epsilon}\sigma_2^4\right)$$
$$+\left(-\frac{47\delta^2 - 37\delta\epsilon + \epsilon^2}{9\epsilon} + \frac{2(\delta - \epsilon)(\delta - 3\epsilon)}{27\epsilon\,\sigma_2^2} - \frac{12\delta^2 + 36\delta\epsilon + 13\epsilon^2}{3\epsilon}\sigma_2^2 - \frac{3\delta^2 + 28\delta\epsilon - 8\epsilon^2}{\epsilon}\sigma_2^4\right)$$
$$+\left((\delta - \epsilon)^2 + \frac{\delta(5\delta - 2\epsilon)}{3}\sigma_2^2 + \delta(11\delta - 12\epsilon)\sigma_2^4\right)\Bigg].$$

5. Uniform-Expansion Gauge

A. Long-Wavelength

$$\hat{\tau}_{00} = k^2 \left|A_1\right|^2 \Bigg[\left(\epsilon + \epsilon\sigma_1^2\right) + \left(-\epsilon(4\delta - 17\epsilon) + \frac{108\epsilon^2}{\sigma_1^6} + \frac{126\epsilon^2}{\sigma_1^4} + \frac{3\epsilon(\delta - 30\epsilon)}{\sigma_1^2}\right) + \left(4\epsilon(\delta-\epsilon)^2 - \frac{648\epsilon^3}{\sigma_1^8} - \frac{36\epsilon^2(12\delta + 11\epsilon)}{\sigma_1^6} - \frac{6\epsilon^2(63\delta - 124\epsilon)}{\sigma_1^4} - \frac{4\epsilon(3\delta^2 - 3\delta\epsilon + 2\epsilon^2)}{\sigma_1^2}\right) \Bigg].$$

$$\hat{\tau}_{00} = k^2 \left|A_1\right|^2 \Bigg[\left(-\frac{\epsilon}{3} + \epsilon\,\sigma_1^2\right) + \left(\frac{\epsilon(4\delta + \epsilon)}{3} - \frac{252\epsilon^2}{\sigma_1^6} - \frac{138\epsilon^2}{\sigma_1^4} - \frac{3\epsilon(\delta - 50\epsilon)}{\sigma_1^2}\right) + \left(-\frac{4\epsilon(\delta-\epsilon)^2}{3} + \frac{1728\epsilon^3}{\sigma_1^8} + \frac{36\epsilon^2(28\delta + 29\epsilon)}{\sigma_1^6} + \frac{18\epsilon^2(41\delta - 81\epsilon)}{\sigma_1^4} + \frac{4\epsilon\left(3\delta^2 - 49\delta\epsilon + 36\epsilon^2\right)}{\sigma_1^2}\right) \Bigg].$$

B. Short-Wavelength

$$\hat{\tau}_{00} = \frac{2\pi G k^4}{a^2}\left(|c_1|^2 + |c_2|^2\right)\Bigg[\left(2+\sigma_2^2\right)$$
$$+\left((\delta+7\epsilon)\sigma_2^2 + 3(\delta-12\epsilon)\sigma_2^4 + 234\epsilon\sigma_2^6 + 108\epsilon\sigma_2^8\right)$$
$$+\left(\delta^2\sigma_2^2 - 2(3\delta^2+9\delta\epsilon-5\epsilon^2)\sigma_2^4 - 6\epsilon(33\delta-16\epsilon)\sigma_2^6\right.$$
$$\left.-36\epsilon(6\delta+41\epsilon)\sigma_2^8 - 648\epsilon^2\sigma_2^{10}\right)$$
$$+\left(3\delta^2(\delta-2\epsilon)\sigma_2^4 + 3\epsilon(2\delta-\epsilon)(15\delta+26\epsilon)\sigma_2^6\right.$$
$$\left.+9\epsilon(12\delta^2+133\delta\epsilon-32\epsilon^2)\sigma_2^8 + 81\epsilon^2(16\delta+87\epsilon)\sigma_2^{10} + 2673\epsilon^3\sigma_2^{12}\right)\Bigg].$$

$$\hat{\tau}_{ij} = \frac{2\pi G k^4}{a^2}\left(|c_1|^2 + |c_2|^2\right)\Bigg[\left(\frac{2}{3} - \frac{1}{3}\sigma_2^2\right)$$
$$+\left(-\frac{7\delta+\epsilon}{3}\sigma_2^2 - 3(\delta-12\epsilon)\sigma_2^4 - 390\epsilon\sigma_2^6 - 252\epsilon\sigma_2^8\right)$$
$$+\left(-\frac{\delta^2}{3}\sigma_2^2 + 2(3\delta^2-37\delta\epsilon-8\epsilon^2)\sigma_2^4 + 54\epsilon(9\delta+\epsilon)\sigma_2^6\right.$$
$$\left.+252\epsilon(2\delta+15\epsilon)\sigma_2^8 + 1728\epsilon^2\sigma_2^{10}\right)$$
$$+\left(-\delta^2(3\delta+2\epsilon)\sigma_2^4 - 3\epsilon(122\delta^2-119\delta\epsilon-14\epsilon^2)\sigma_2^6\right.$$
$$\left.-9\epsilon(28\delta^2+505\delta\epsilon+82\epsilon^2)\sigma_2^8 - 27\epsilon^2(116\delta+819\epsilon)\sigma_2^{10} - 7047\epsilon^3\sigma_2^{12}\right)\Bigg].$$

Analyses

$$\mathcal{F}_{\rm LW} \equiv \mathcal{H}^2 |A_1|^2,$$

$$\mathcal{F}_{\rm SW} \equiv \frac{2\pi G k^4}{a^2}\left(|c_1|^2 + |c_2|^2\right).$$

$$w_{\rm raw} \equiv \frac{\hat{\tau}_{ij}}{\hat{\tau}_{00}}$$

Overall Pattern

Long-wavelength limit

Gauge	$\hat{\tau}_{00}$	$\hat{\tau}_{ij}$	$w_{\rm raw}$
Longitudinal	$3\epsilon(\delta - 3\epsilon)\mathcal{F}_{\rm LW}$	$-3\epsilon(\delta - 3\epsilon)\mathcal{F}_{\rm LW}$	-1
Spatially flat	$3\epsilon(\delta - 3\epsilon)\mathcal{F}_{\rm LW}$	$-3\epsilon(\delta - 3\epsilon)\mathcal{F}_{\rm LW}$	-1
Comoving	$9\mathcal{F}_{\rm LW}$	$-3\mathcal{F}_{\rm LW}$	$-\frac{1}{3}$
Uniform density	$9\mathcal{F}_{\rm LW}$	$-3\mathcal{F}_{\rm LW}$	$-\frac{1}{3}$

The uniform-expansion result contains increasingly strong inverse powers of σ_1, so its strict infrared limit cannot be reliably inferred from the truncated slow-roll series.

Short-wavelength limit

Gauge	$\hat{\tau}_{00}$	$\hat{\tau}_{ij}$	$w_{\rm raw}$
Longitudinal	$6\mathcal{F}_{\rm SW}$	$\frac{10}{3}\mathcal{F}_{\rm SW}$	$\frac{5}{9}$
Spatially flat	$2\mathcal{F}_{\rm SW}$	$\frac{2}{3}\mathcal{F}_{\rm SW}$	$\frac{1}{3}$
Comoving	$-\frac{1}{\epsilon}\mathcal{F}_{\rm SW}$	$-\frac{11}{3\epsilon}\mathcal{F}_{\rm SW}$	$\frac{11}{3}$
Uniform density	$\frac{1}{9\epsilon\sigma_2^2}\mathcal{F}_{\rm SW}$	$-\frac{11}{27\epsilon\sigma_2^2}\mathcal{F}_{\rm SW}$	$-\frac{11}{3}$
Uniform expansion	$2\mathcal{F}_{\rm SW}$	$\frac{2}{3}\mathcal{F}_{\rm SW}$	$\frac{1}{3}$

1. Longitudinal Gauge

$$\beta = 0, \qquad E = 0.$$

Physical character

- Zero-shear gauge.
- Metric perturbations are directly represented by the Bardeen potentials.
- Provides the clearest gravitational-potential description.

Long-wavelength result

$$\hat{\tau}^{\mathrm{L}}_{ij} \simeq -\hat{\tau}^{\mathrm{L}}_{00},$$

with

$$\hat{\tau}^{\mathrm{L}}_{00} = O(\epsilon\delta, \epsilon^2).$$

Therefore, the infrared 2EMT is vacuum-like but slow-roll suppressed.

Short-wavelength result

$$\hat{\tau}^{\mathrm{L}}_{00} \simeq 6\mathcal{F}_{\mathrm{SW}}, \qquad \hat{\tau}^{\mathrm{L}}_{ij} \simeq \frac{10}{3}\mathcal{F}_{\mathrm{SW}}.$$

Thus,

$$w^{\mathrm{L}}_{\mathrm{raw}} \simeq \frac{5}{9}.$$

Interpretation

The longitudinal-gauge result is regular in both limits and provides a useful gravitational benchmark. However, it is tied to a zero-shear slicing rather than to a direct physical clock.

2. Spatially-Flat Gauge

$$\psi = 0, \qquad E = 0.$$

Physical character

- Vanishing intrinsic spatial-curvature perturbation.
- The scalar degree of freedom is carried mainly by the inflaton fluctuation.
- Particularly natural for quantizing scalar perturbations.

Long-wavelength result

At the lowest nonvanishing slow-roll order,

$$\hat{\tau}^{\mathrm{SF}}_{\mu\nu} \simeq \hat{\tau}^{\mathrm{L}}_{\mu\nu}.$$

Hence,

$$w^{\mathrm{SF}}_{\mathrm{raw}} \simeq -1,$$

with a slow-roll-suppressed effective density.

The equality with the longitudinal result does not persist at all higher slow-roll orders.

Short-wavelength result

$$\hat{\tau}^{\mathrm{SF}}_{00} \simeq 2\mathcal{F}_{\mathrm{SW}},$$

$$\hat{\tau}^{\mathrm{SF}}_{ij} \simeq \frac{2}{3}\mathcal{F}_{\mathrm{SW}}.$$

Therefore,

$$\boxed{w^{\mathrm{SF}}_{\mathrm{raw}} \simeq \frac{1}{3}}$$

Interpretation

This is the cleanest short-wavelength result. It is positive and radiation-like, consistent with kinetic and gradient energy dominating sub-Hubble inflaton fluctuations.

3. Comoving Gauge

$$\delta\phi = 0, \qquad E = 0.$$

Physical character

- Inflaton rest-frame slicing.
- The inflaton acts as a physical clock.
- Quantities are evaluated at fixed inflaton value.

Long-wavelength result

$$\hat{\tau}_{00}^{\mathrm{C}} \simeq 9\mathcal{F}_{\mathrm{LW}},$$

$$\hat{\tau}_{ij}^{\mathrm{C}} \simeq -3\mathcal{F}_{\mathrm{LW}}.$$

Thus,

$$w_{\mathrm{raw}}^{\mathrm{C}} \simeq -\frac{1}{3}.$$

Unlike the longitudinal and spatially-flat results, this contribution is not slow-roll suppressed.

Short-wavelength result

$$\hat{\tau}_{00}^{\mathrm{C}} \simeq -\frac{1}{\epsilon}\mathcal{F}_{\mathrm{SW}},$$

$$\hat{\tau}_{ij}^{\mathrm{C}} \simeq -\frac{11}{3\epsilon}\mathcal{F}_{\mathrm{SW}}.$$

Relation to uniform-density gauge

All dominant $1/\sigma_1^2$ coefficients agree with the uniform-density result. Therefore,

$$\boxed{\hat{\tau}_{\mu\nu}^{\mathrm{C}} \longrightarrow \hat{\tau}_{\mu\nu}^{\mathrm{UD}} \qquad (k/\mathcal{H} \to 0)}$$

This is consistent with the equivalence of comoving and uniform-density slicings for single-field adiabatic perturbations on super-Hubble scales.

Interpretation

The $1/\epsilon$ enhancement reflects the fact that

$$T_{\mathrm{C}} \sim \frac{\delta\phi}{\phi_0'}, \qquad \phi_0'^2 \propto \epsilon.$$

As $\epsilon \to 0$, the inflaton becomes a poor clock. The large negative short-wavelength result should therefore be interpreted as a slicing effect rather than as direct evidence for negative physical energy.

4. Uniform-Density Gauge

$$\delta\rho = 0, \qquad E = 0.$$

Physical character

- Fixed-density slicing.
- The local energy density acts as a physical clock.
- Natural for super-Hubble curvature perturbations and transition hypersurfaces.

Long-wavelength result

$$\hat{\tau}_{00}^{\mathrm{UD}} \simeq 9\mathcal{F}_{\mathrm{LW}},$$

$$\hat{\tau}_{ij}^{\mathrm{UD}} \simeq -3\mathcal{F}_{\mathrm{LW}}.$$

Therefore,

$$w_{\mathrm{raw}}^{\mathrm{UD}} \simeq -\frac{1}{3}.$$

The strict long-wavelength result coincides with the comoving-gauge result.

Short-wavelength result

The dominant behavior is

$$\hat{\tau}_{00}^{\mathrm{UD}} \simeq \frac{1}{9\epsilon\sigma_2^2}\mathcal{F}_{\mathrm{SW}},$$

$$\hat{\tau}_{ij}^{\mathrm{UD}} \simeq -\frac{11}{27\epsilon\sigma_2^2}\mathcal{F}_{\mathrm{SW}}.$$

Thus,

$$w_{\mathrm{raw}}^{\mathrm{UD}} \simeq -\frac{11}{3}.$$

Interpretation

The transformation to uniform-density gauge involves

$$T_\rho \sim \frac{\delta\rho}{\rho_0'}, \qquad \rho_0' \propto -\epsilon.$$

For short wavelengths, derivatives in $\delta\rho$ generate additional $k/\mathcal{H}$ enhancement. The resulting

$$\frac{1}{\epsilon\sigma_2^2}$$

behavior indicates that density is a poorly conditioned clock for high-frequency perturbations.

5. Uniform-Expansion Gauge

$$\delta\theta = 0, \qquad E = 0.$$

Physical character

- Fixed local-expansion slicing.
- Closely related to constant-mean-curvature slicing.
- Particularly relevant to cosmological backreaction.

Short-wavelength result

$$\hat{\tau}_{00}^{\mathrm{UE}} \simeq 2\mathcal{F}_{\mathrm{SW}},$$

$$\hat{\tau}_{ij}^{\mathrm{UE}} \simeq \frac{2}{3}\mathcal{F}_{\mathrm{SW}}.$$

Therefore,

$$\boxed{w_{\mathrm{raw}}^{\mathrm{UE}} \simeq \frac{1}{3}}$$

and

$$\boxed{\hat{\tau}_{\mu\nu}^{\mathrm{UE}} \longrightarrow \hat{\tau}_{\mu\nu}^{\mathrm{SF}} \qquad (\mathcal{H}/k \to 0)}$$

Long-wavelength result

The slow-roll-expanded expressions contain terms such as

$$\frac{\epsilon^2}{\sigma_1^6}, \qquad \frac{\epsilon^3}{\sigma_1^8}, \qquad \frac{\epsilon^2}{\sigma_1^4}.$$

These terms suggest that the slow-roll and long-wavelength expansions are nonuniform.

A structure of the form

$$\frac{1}{k^2 + O(\epsilon\mathcal{H}^2)}$$

can generate, after a premature slow-roll expansion,

$$\frac{1}{\sigma_1^2}, \qquad \frac{\epsilon}{\sigma_1^4}, \qquad \frac{\epsilon^2}{\sigma_1^6}, \cdots$$

Interpretation

The apparent infrared enhancement should not immediately be interpreted as physical backreaction. The exact, unexpanded, or resummed expression must be examined before taking the strict $k \to 0$ limit.

Physical Classification of the Five Gauges

Geometrical gauges

$$\text{Longitudinal}, \qquad \text{Spatially flat}.$$

- Regular in both wavelength limits.
- Do not rely directly on a slowly evolving matter clock.
- Useful as benchmark descriptions.

Matter-clock gauges

$$\text{Comoving}, \qquad \text{Uniform density}.$$

- Physically natural on super-Hubble scales.
- Coincide in the long-wavelength limit.
- Become strongly enhanced or poorly conditioned in the short-wavelength regime.

Geometrical-clock gauge

$$\text{Uniform expansion}.$$

- Directly relevant to expansion and backreaction.
- Agrees with the spatially-flat result in the short-wavelength limit.
- Requires special care in the infrared.

Main Physical Conclusions

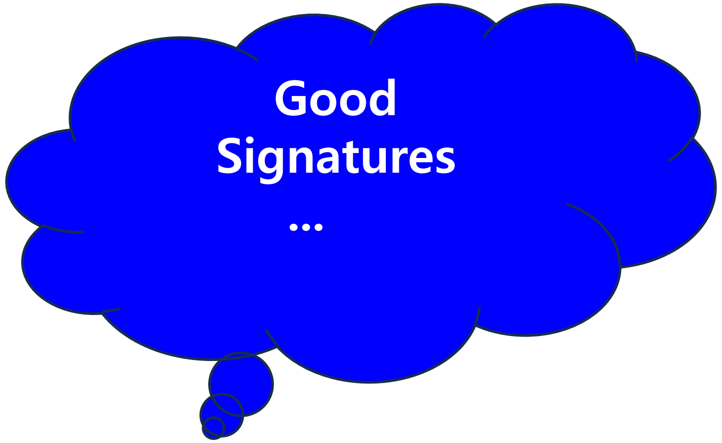
1. The gauge dependence is highly structured

$$\boxed{\text{Comoving} \simeq \text{Uniform density} \quad \text{in the infrared}}$$

and

$$\boxed{\text{Spatially flat} \simeq \text{Uniform expansion} \quad \text{in the ultraviolet}}$$

These agreements reflect the physical properties of the corresponding slicings.



2. Comoving–uniform-density agreement is an important consistency check

For single-field adiabatic perturbations,

$$\text{fixed inflaton} \simeq \text{fixed density}$$

on super-Hubble scales.

The 2EMT reproduces this expected relation.

3. Spatially-flat–uniform-expansion agreement identifies a common UV behavior

Both gauges give

$$\rho_{\mathrm{eff}} > 0, \qquad p_{\mathrm{eff}} \simeq \frac{1}{3}\rho_{\mathrm{eff}}.$$

This radiation-like behavior is the most natural candidate for the common short-wavelength field-energy contribution.

4. Large $1/\epsilon$ factors indicate clock degeneracy

In the de Sitter limit,

$$\phi_0' \to 0, \qquad \rho_0' \to 0, \qquad \theta_0' \to 0.$$

Therefore, comoving, uniform-density, and uniform-expansion slicings become increasingly ill-conditioned as physical clocks.

5. The raw gauge-fixed 2EMTs are not yet observables

The same perturbation mode can give positive, negative, finite, or strongly enhanced effective energy densities depending on the slicing.

Therefore,

$$\boxed{\text{The five gauge-fixed 2EMTs are slicing-dependent quadratic effective sources.}}$$

They should not yet be regarded as five equivalent representations of an established observable.

A true observable would require a common operational prescription involving, for example,

$$\text{a physical clock, an observer, and—when relevant—an averaging prescription.}$$

Next Step 1 — Identify the Gauge-Robust Content

- The gauge dependence of the quadratic EMT is **structured rather than arbitrary**:

$$\text{Comoving} \simeq \text{Uniform density} \quad (k \ll \mathcal{H}),$$

$$\text{Spatially flat} \simeq \text{Uniform expansion} \quad (k \gg \mathcal{H}).$$

- These agreements indicate potentially **gauge-robust limiting behaviors**.
- Large $1/\epsilon$ and inverse-gradient terms instead signal sensitivity to the physical clock, spacetime slicing, and the order of limits.
- The raw gauge-fixed 2EMTs should therefore be interpreted as **quadratic effective sources defined on different spacetime slicings**, rather than as direct observables.

Next Step 2 — Construct the Same Relational Quantity in Every Gauge

For each gauge G, define

$$\mathcal{O}^{G}_{\mathrm{2EMT}}(X_*) \equiv \left[\tau^{(2)}_{\mu\nu,G} u^{\mu}_{G} u^{\nu}_{G} + \Delta_{\mathrm{clock},G} + \Delta_{\mathrm{observer},G}\right]_{X_G = X_*}.$$

Here X_G and u^{μ}_{G} are gauge-dependent coordinate representations of the **same physical clock and observer**. The individual terms,

$$\tau^{(2)}_{\mu\nu,G} u^{\mu}_{G} u^{\nu}_{G}, \qquad \Delta_{\mathrm{clock},G}, \qquad \Delta_{\mathrm{observer},G},$$

generally differ among gauges, and the hypersurface $X_G = X_*$ occurs at different coordinate locations.

Nevertheless, with the same physical clock, observer, hypersurface, and averaging prescription,

$$\mathcal{O}^{\mathrm{L}}_{\mathrm{2EMT}}(X_*) = \mathcal{O}^{\mathrm{SF}}_{\mathrm{2EMT}}(X_*) = \mathcal{O}^{\mathrm{C}}_{\mathrm{2EMT}}(X_*) = \mathcal{O}^{\mathrm{UD}}_{\mathrm{2EMT}}(X_*) = \mathcal{O}^{\mathrm{UE}}_{\mathrm{2EMT}}(X_*),$$

up to the perturbative order considered.

The raw gauge-fixed 2EMTs need not agree; only different gauge representations of the same fully specified relational quantity must agree.

Next Step 3 — Connect the 2EMT to Observable Consequences

Rather than observing the raw 2EMT directly, investigate its gauge-independent effects on

$$H_{\mathrm{eff}}, \qquad N_{\mathrm{eff}}, \qquad \Delta P_{\mathcal{R}}(k), \qquad \Omega_{\mathrm{GW}}(k).$$

Central Message

> **The purpose is not to select one "correct" gauge.**
> The five gauges form a diagnostic framework for determining
> **which contributions are slicing dependent, which represent different relational clocks, and which survive as observable physical effects.**

$$\boxed{\begin{array}{c}\text{Gauge dependence is not the end of the analysis—}\\ \text{it is the starting point for identifying observable physics.}\end{array}}$$

Cosmological Gauge vs. Gauge Symmetry in Quantum Field Theory

Common structure

Both theories contain redundant field descriptions:

$$\text{Gauge orbit} = \{\text{all gauge-related representations of one physical state}\}.$$

- **Gauge fixing:** selects one representative from the orbit.
- **Gauge invariance:** identifies quantities that are unchanged along the orbit.
- **Physical predictions:** must not depend on the chosen representative.

Key difference

Internal gauge theory	Cosmological perturbation theory
$U(1), SU(N), \ldots$	$\mathrm{Diff}(\mathcal{M})$
Acts in internal field space	Acts on spacetime slicing and threading
x^μ remains fixed	$x^\mu \to x^\mu + \xi^\mu$
$A_\mu \to A_\mu + \partial_\mu \alpha$	$\delta g_{\mu\nu} \to \delta g_{\mu\nu} - \mathcal{L}_\xi g^{(0)}_{\mu\nu}$

Gauge-Invariant Quantities and Observables

Internal gauge theory	Cosmological perturbation theory
A_μ is gauge dependent	α, β, ψ, E are gauge dependent
$F_{\mu\nu}$ is gauge invariant	$\Phi, \Psi, \mathcal{R}$ are gauge invariant
$E_\mu = F_{\mu\nu}u^\nu, \quad B_\mu = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}u^\nu F^{\rho\sigma}$	$Q\|_{X=X_*}$ with a physical clock or reference

In spacetime perturbation theory

- less directly observable
- usually requires a relational, observer-based prescription.

Conclusions

- Evaluated QEMT in inflation in 5 gauge conditions
- QEMT is gauge dependent
- Gauge invariant ≠ Gauge fixed ≠ Observable
- Search possible Observables in the future