Scalarizations of quantum Oppenheimer-Snyder black holes

Myung, Yun Soo
Center for Quantum Spacetime (CQUeST), Sogang University
29 December 2025

(Shanghai-APCTP-Sogang Workshop, CQUeST)

based on arXiv:2505.19450 & 2509.24113

Contents

1. Introduction

No-hair (Kerr) hypothesis implies that the astrophysical black holes are described by the Kerr family with mass(M) and angular momentum (J).

→ So far, all observations are in agreement with the Kerr hypothesis.

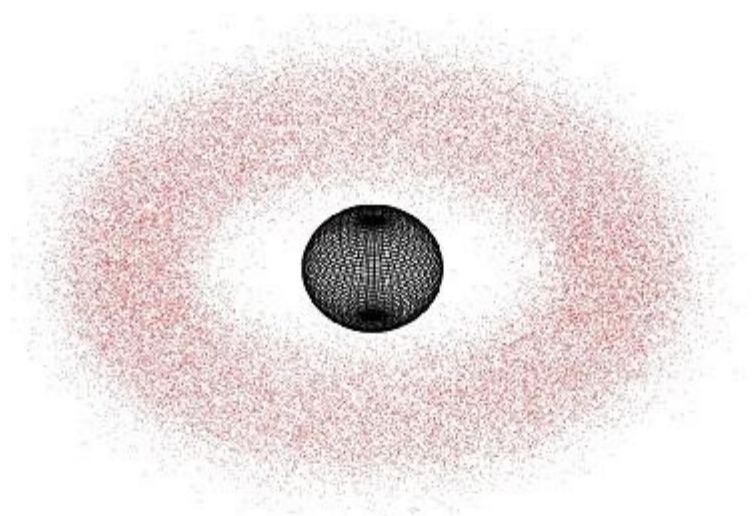
1) Motion of stars around the supermassive black hole in the center of our galaxy (2020 Nobel Prize: Penrose-Genzel-Ghez)

2) Observation of GWs from black hole mergers (2017 Nobel Prize: Weiss-Barish-Thorne)

3) The Shadow of the Supermassive Black Hole (M87*) in the center of M87 (2019, EHT collaboration)

4) The Shadow of the Supermassive Black Hole (SgrA*) in the center of the Milky Way (2023, EHT collaboration)

Black hole with scalar hair
(Photo courtesy of Pedro V. Cunha)



Preview of no-hair theorem

Scalar-tensor theory	No-hair theorem by	Scalar hairy BHs
Scalar-vacuum→ $\frac{1}{4}R-\frac{1}{2}(\partial\phi)^2$	Chase Commun. Math. Phys. 19 (1970) 276	No
Massive scalar-vacuum→ $\frac{1}{4}R-\frac{1}{2}(\partial\phi)^2-\frac{1}{2}m^2\phi^2$	Bekenstein Phys. Rev. Lett. 28 (1972) 452	No
Massive complex scalar-vacuum→ $\frac{1}{4}R-\partial_\mu\phi^*\partial^\mu\phi-m^2\phi^*\phi$	Pena-Sudarsky Class. Quant. Grav. 14 (1997) 3131	Yes, primary & regular (2015)
Conformal scalar-vacuum→ $\frac{1}{4}R-\frac{1}{2}(\partial\phi)^2-\frac{1}{12}R\phi^2$	Xanthopoulos-Zannias J. Math. Phys. 32 (1991) 1875	No, but a secondary BBMB solution(1970,1974)
***V*-scalar-vacuum→** $\frac{1}{4}R-\frac{1}{2}(\partial\phi)^2-V(\phi)$	Heusler-Straumann, Bekenstein Class. Quant. Grav. 9 (1992) 2177 Phys. Rev. D 51 (1995) 6608	**No, but many solutions for negative potentials →Evasion of noble no-hair theorem**
Brans-Dicke theory→ $\varphi R-\frac{\omega(\varphi)}{\varphi}(\partial\varphi)^2-U(\varphi)$	Hawking, Sotiriou-Faraoni Commun. Math. Phys. 25 (1972) 167 Phys. Rev. Lett. 108 (2012) 081103	No
Horndeski/Galileon theories→ **General scalar-tensor theories**	Hui-Nicolis Phys. Rev. Lett. 110 (2013) 24, 241104	Yes, secondary (2014) and primary (2014)

Herdeiro and Radu, Int. J. Mod. Phys. D 24, 1542014 (2015).

- **Minimally coupled (massless complex, massive) scalar**
  → No-hair theorem
  → GR black holes without scalar hair

- **Conformally coupled scalar** : $\boxed{\int d^4x\sqrt{-g}\,\left[R-\alpha\left(6(\partial\phi)^2+\phi^2 R\right)\right]}$

  → Evasion of no-hair theorem
  → Analytic extremal BBMB black hole with scalar hair
  (**but scalar hair is secondary and diverges at *r=m***)

$$\boxed{ds^2_{\rm BBMB}=-\left(1-\frac{m}{r}\right)^2 dt^2+\frac{dr^2}{\left(1-\frac{m}{r}\right)^2}+r^2 d\Omega_2^2 \quad \text{with} \quad \text{scalar hair } \phi(r)=\sqrt{\frac{1}{\alpha}}\frac{m}{r-m}}$$

  → Its series (numerical) solution was also found in eprint:1907.09676.

R. Ruffini and J.A. Wheeler, Phys. Today 24, 30 (1971).
J.D. Bekenstein, Phys. Rev. Lett., 28, 452 (1972).
N.M. Bocharova, K.A. Bronnikov and V.N. Melnikov, Vestn. Mosk. Univ. Ser. III Fiz. Astron. , 706, 6 (1970) .
J.D. Bekenstein, Phys. Rev., D51, R6608 (1995).

Three important works for Spontaneous Scalarization: GB+[1,2] and M+[3]

1. New Gauss-Bonnet Black Holes with Curvature-Induced Scalarization in Extended Scalar-Tensor Theories

Daniela D. Doneva and Stoytcho S. Yazadjiev
PRL120 (2018) 131103-Published 30 March 2018

2. Spontaneous Scalarization of Black Holes and Compact Stars from a Gauss-Bonnet Coupling

Hector O. Silva, Jeremy Sakstein, Leonardo Gualtieri, Thomas P. Sotiriou, and Emanuele Berti
PRL120 (2018) 131104-Published 30 March 2018

3. Spontaneous Scalarization of Charged Black Holes

C. A. R. Herdeiro, E. Radu, N. Sanchis-Gual and J. A. Font,
PRL121 (2018) 101102 -Published 5 September 2018

$\boxed{\text{GB}_{+}\text{ spontaneous scalarization for SBH=OS-BH}}$

$$S_{\rm EGBS}=\frac{1}{16\pi}\int d^4x\sqrt{-g}\left[R-2(\partial\phi)^2+\lambda^2 f(\phi)R^2_{\rm GB}\right],\; R^2_{\rm GB}=R^2-4R_{\mu\nu}R^{\mu\nu}+R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$$

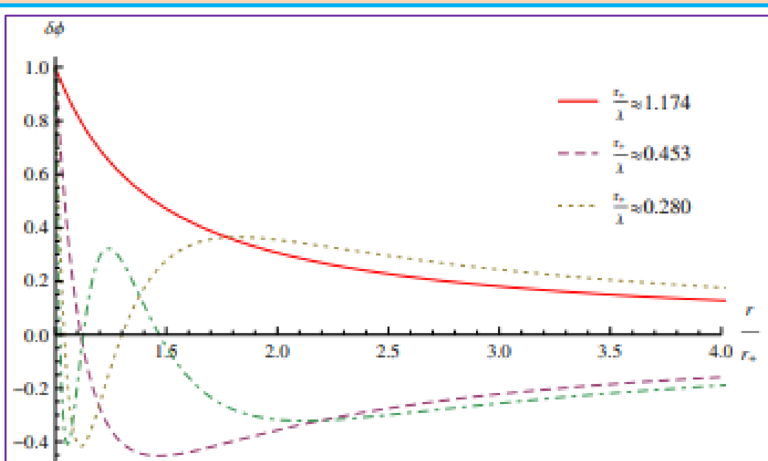
$$\text{Coupling function }\rightarrow f(\phi)=\frac{1-e^{-6\phi^2}}{12},\;\frac{\phi^2}{2},\quad \boxed{\bar{R}^2_{\rm GB}=\frac{48M^2}{r^6}>0}\text{ for SBH}$$

⊙ Onset scalarization based on the linearized scalar theory

- Linearized scalar eq: $\left[\bar{\Box}-m^2_{\rm eff}(r)\right]\delta\varphi=0$ with $m^2_{\rm eff}(r)=-\frac{\lambda^2\bar{R}^2_{\rm GB}}{4}<0$ (tachyon-like mass)
- Tachyonic instability ($\frac{1}{\lambda}\leq\frac{1}{\lambda_{\rm th}}=1.174r_+$) for SBH (destabilization of SBH)
- Infinite scalar clouds predict infinite branches of n=0 ($\frac{1}{\lambda}\leq\frac{1}{\lambda_{\rm th}}$), 1 ($\frac{1}{\lambda}\leq 0.453r_+$), 2 ($\frac{1}{\lambda}\leq 0.280r_+$), $\cdots$

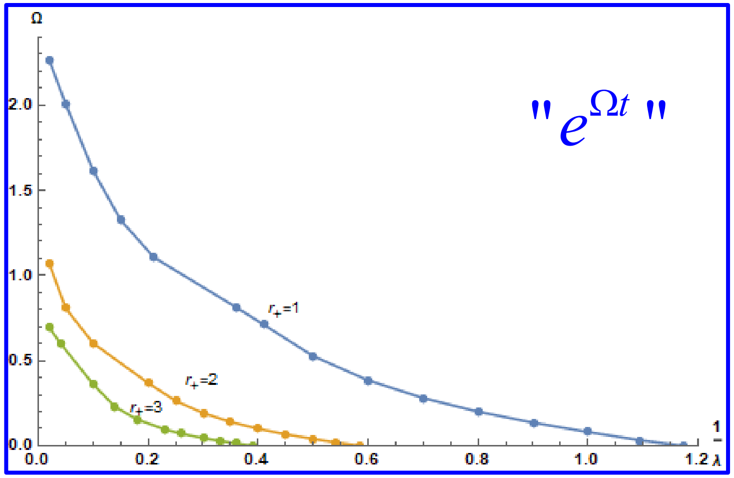
⊙ Solving full eqs with their backreation → scalarized SBHs and performing their stability analysis

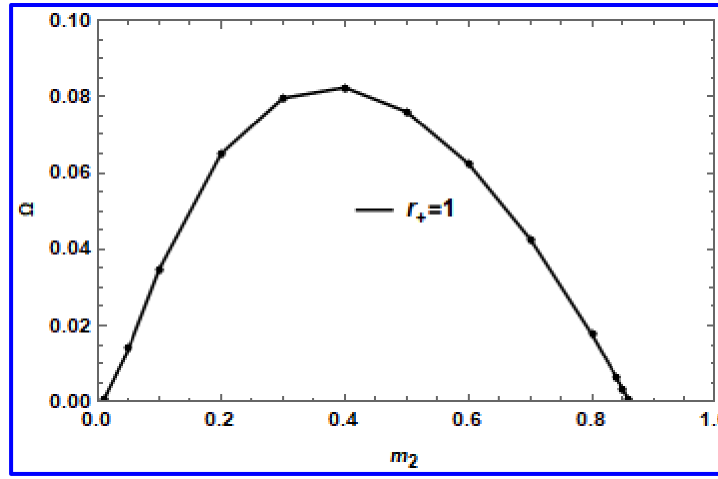
$\rightarrow n=0$: stable, others of $n=1,2,3,\cdots$: unstable

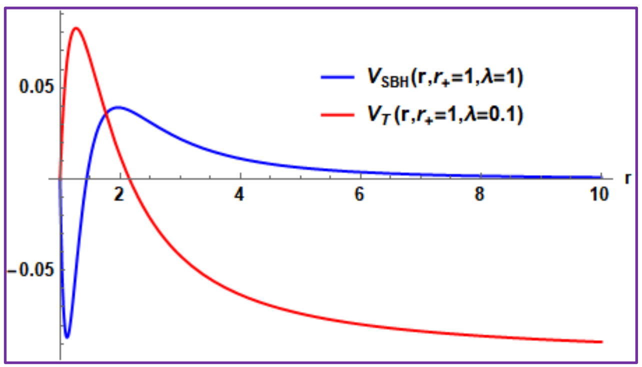


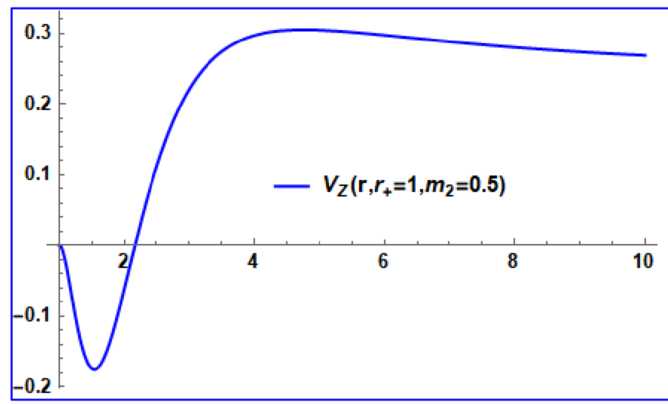
Four scalar clouds→
Seeds to generate four (n=0,1,2,3) branches

Comment #1: Tachyonic Instability for scalar field→ tachyon? → Gregory-Laflamme instability for Ricci tensor









$$\boxed{V_T(r)=\left(1-\frac{r_+}{r}\right)\left(\frac{r_+}{r^3}-\lambda\right)}$$

$$\boxed{V_{\rm SBH}(r)=\left(1-\frac{r_+}{r}\right)\left(\frac{r_+}{r^3}-\frac{3\lambda^2 r_+^2}{r^6}\right)}$$

$$\boxed{\frac{1}{\lambda}\Leftrightarrow\frac{m_2}{2}}$$

$$V_z(r)=\left(1-\frac{r_+}{r}\right)\left[\frac{r_+}{r^3}+m_2^2-\frac{3r^3(r_+-0.5r)+3r_+(r-0.5r_+)/m_2^2}{\frac{m_2^2}{4}(r^3+r_+/m_2^2)^2}\right]$$

→→

Y. S. Myung and D. C. Zou, arXiv:1805.05023

Einstein-Weyl gravity: $S_{\rm EW} = \gamma \int d^4x\sqrt{-g}\left(R - \frac{1}{2m_2^2}C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma}\right),\ m_2^2 = \frac{\gamma}{2\alpha}$

- Linerarized massive tensor eq around SBH with $R_{\mu\nu}$=0:

$$\boxed{\left(\Delta_{\rm L} + m_2^2\right)\delta R_{\mu\nu} = 0}$$

with Licherowicz operator of $\Delta_{\rm L}\delta R_{\mu\nu} = -\bar{\Box}\delta R_{\mu\nu} - 2\bar{R}_{\mu\rho\nu\sigma}\delta R^{\rho\sigma}$.

- Zerilli-potential for $l = 0$-mode of linearized Ricci tensor

$$\to\to V_{\rm z}(r) = \left(1 - \frac{r_+}{r}\right)\left[\frac{r_+}{r^3} + m_2^2 - \frac{3r^3(r_+ - 0.5r) + 3r_+(r - 0.5r_+)/m_2^2}{\frac{m_2^2}{4}(r^3 + r_+/m_2^2)^2}\right]$$

- GL instability bound on m_2: $0 < m_2 \leq m_2^{\rm th} = \frac{0.896}{r_+}$
- Non-Schwazschild black hole with Ricci tensor hair ($R_{\mu\nu} \neq 0$)

$\to$ Lu-Perkins-Pope-Stelle solution, PRL114 (2015), 171601

Comment #2: What is spontaneous scalarization?

Curvature-induced (GB_+) spontaneous scalarization (SS) in EGBS theory

→ **Scalar coupling $f(\phi)$ to GB term develops tachyonic (GL) instability**

→ **Infinite scalar clouds (static scalar perturbations) are seeds to generate infinite branches of scalarized SBHs.**
(SS: onset scalarization → determines whole scheme for scalarization)

→ **Their backreation originates scalarized SBHs when solving full eqs.**
(Infinite branches: n=0 branch is stable, but all other branches of n=1, 2, 3,⋯ are unstable) .

2. quantum Oppenheimer-Snyder (qOS) model

Ashitekar-Pawlowski-Singh (APS) model for the interior region

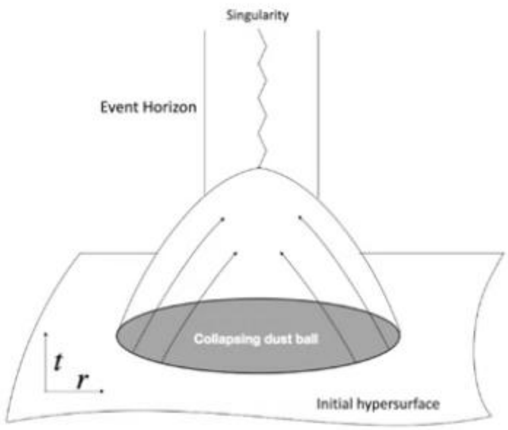
→ A Dust Ball
→ Loop Quantum Cosmology (LQC) goes beyond the Big-Bang singularity
→ Big-Bounce from LQC can avoid Big-Bang singularity in General Relativity

- qOS model describes a collapsing Dust Ball inside a deformed SBH.
- qSC model represents a deformed SBH surrounded by APS model.

Loop Quantum Cosmology, I. Agullo and P. Singh, eprint:1612.01236

Jerzy Lewandowski, Yongge Ma, Jinsong Yang, and Cong Zhang,PRL130 (2023)101501

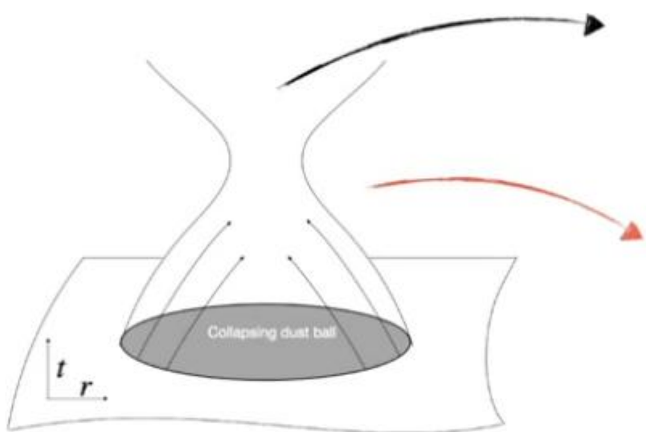
Oppenheimer-Snyder model → OS-BH=SBH



Some facts:

- The dust ball takes the metric $ds^2 = -d\tau^2 + a(\tau)^2 ds_E^2$;
- $a(\tau)$ is governed by: $\mathbb{H}^2 = \frac{8\pi G}{3}\rho$ and $\partial_\tau(\rho a^3) = 0$;
- The Schwarzschild outside is the unique spherically symmetric and stationary metric that can be glued to the dust ball metric by the junction condition. This is the result without necessary to consider the EOM.

Quantum Oppenheimer-Snyder model → qOS-BH



$$ds^2 = -d\tau^2 + a(\tau)^2 ds_E^2$$

$$\mathbb{H}^2 = \frac{8\pi G}{3}\rho\left(1 - \frac{\rho}{\rho_c}\right) \text{ and } \partial_\tau(\rho a^3) = 0$$

by not EOM but Junction Condition

$$ds^2 = -f(r)dt^2 + g(r)^{-1}dr^2 + r^2 d\Omega^2$$

$$f(r) = g(r) = 1 - \frac{2GM}{r} + \frac{\alpha G^2 M^2}{r^4}$$

$$\alpha = 16\sqrt{3}\pi\gamma^3\ell_p^2$$

[Lewandowski, Ma, Yang, CZ 23']

APS model for the interior spacetime

- Semicalssical solution of APS metric: $ds^2_{\mathrm{APS}} = -d\tau^2 + a(\tau)^2\left[d\tilde{r}^2 + \tilde{r}^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right)\right]$

  $\rightarrow$ Nowhere and never singular spacetime

- deformed Friedmann equation by LQC(−)/Braneworld Scenario (+)

  $\boxed{H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho\left(1-\frac{\rho}{\rho_c}\right)/\left(1+\frac{\rho}{\rho_{\mathrm{B}}}\right)}$ $\rightarrow$ classical regime: $\rho \ll \rho_c$ & quantum regime: $\rho \sim \rho_c$

  defomation parameter: $\rho_c = \frac{3}{2\pi G\alpha}$ $\leftarrow$ maximum attainable energy density by LQC

- Conservation-law

  $\nabla_\mu T^{\mathrm{APS},\mu\nu} = 0 \rightarrow T_\mu^{\mathrm{APS},\nu} = \mathrm{diag}\left[\rho,0,0,0\right] \rightarrow \frac{d\rho}{d\tau} + 3\frac{\dot{a}}{a}\rho = 0 \rightarrow \rho(\tau) = \frac{M}{4\pi a^3(\tau)\tilde{r}_0^3/3}$

  $M \rightarrow$ total mass of Dust Ball with radius $a(\tau)\tilde{r}_0$

- Dust Ball describes geodesics with $\tilde{r},\theta,\varphi = \mathrm{constant}$, but it is confined to $0 \le \tilde{r} \le \tilde{r}_0$

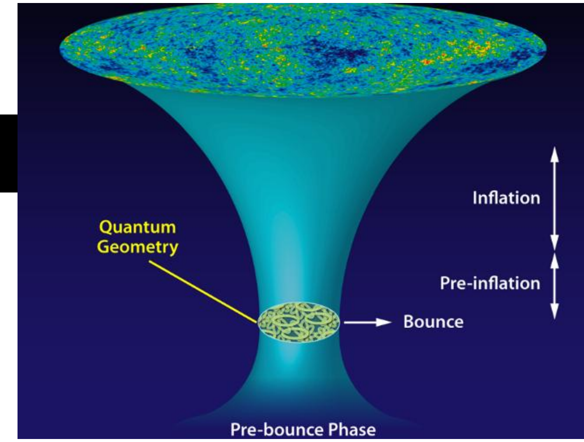
Two (GR/LQC) different cosmological solutions

$$H^2 = F(\rho)$$

with

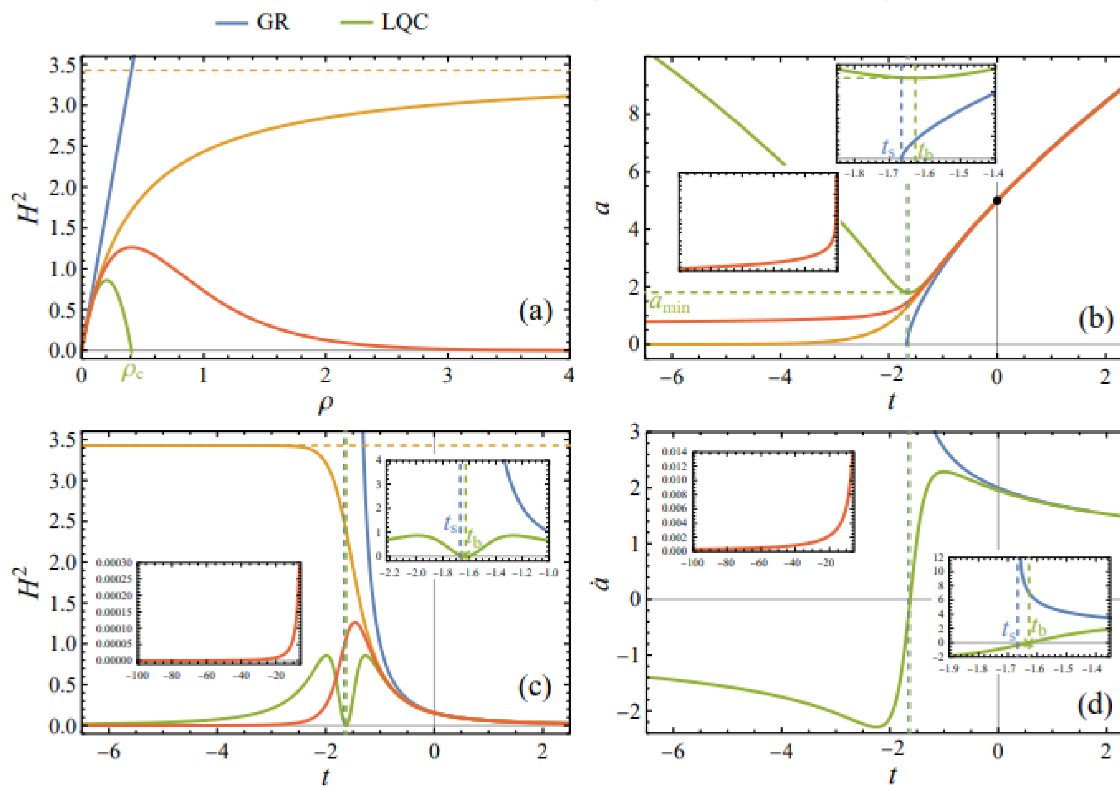
$$\begin{cases} F_{\mathrm{GR}}(\rho) = \dfrac{8\pi G}{3}\rho\,, \\ F_{\mathrm{LQC}}(\rho) = \dfrac{8\pi G}{3}\rho\left(1-\dfrac{\rho}{\rho_c}\right) \end{cases}$$

LQC→
No where and never singular spacetime



$\uparrow (t)$

Sandglass $\uparrow$ (Bouncing universe)



Non-singular cosmologies matching regular black holes, Shulan Li, Jian-Pin Wu, Xian-Hui Ge, e-Print: 2512.00926

Exterior Spacetime→ qOS-BH spacetime

- qOS-BH solution for exterior spacetime by imposing $\boxed{\text{junction conditions}}$

$$ds^2_{\mathrm{qOS}} = \bar{g}_{\mu\nu} dx^\mu dx^\nu = -g(r)dt^2 + \frac{dr^2}{g(r)} + r^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right)$$

$$\text{with} \quad g(r) = 1 - \frac{2GM}{r} + \frac{\alpha G^2 M^2}{r^4} \equiv 1 - \frac{2m_{\mathrm{qOS}}(r)}{r} \quad \left(m_{\mathrm{qOS}}(r): \text{mass function}\right)$$

$\rightarrow$ A defomed SBH, but its Penrose diagram is similar to RNBH

- $(\theta, \varphi) \rightarrow$ joint region for Dust Ball and qOS-BH
- $M \rightarrow$ ADM mass of qOS-BH and $\alpha = 16\sqrt{3}\pi\gamma^3\ell_p^2 \rightarrow$ quantum parameter

  with γ (Barbero-Immirzi parameter) and $\ell_p = \sqrt{G\hbar} = 1$(Planck length)

Jerzy Lewandowski, Yongge Ma, Jinsong Yang, and Cong Zhang, PRL130 (2023)101501

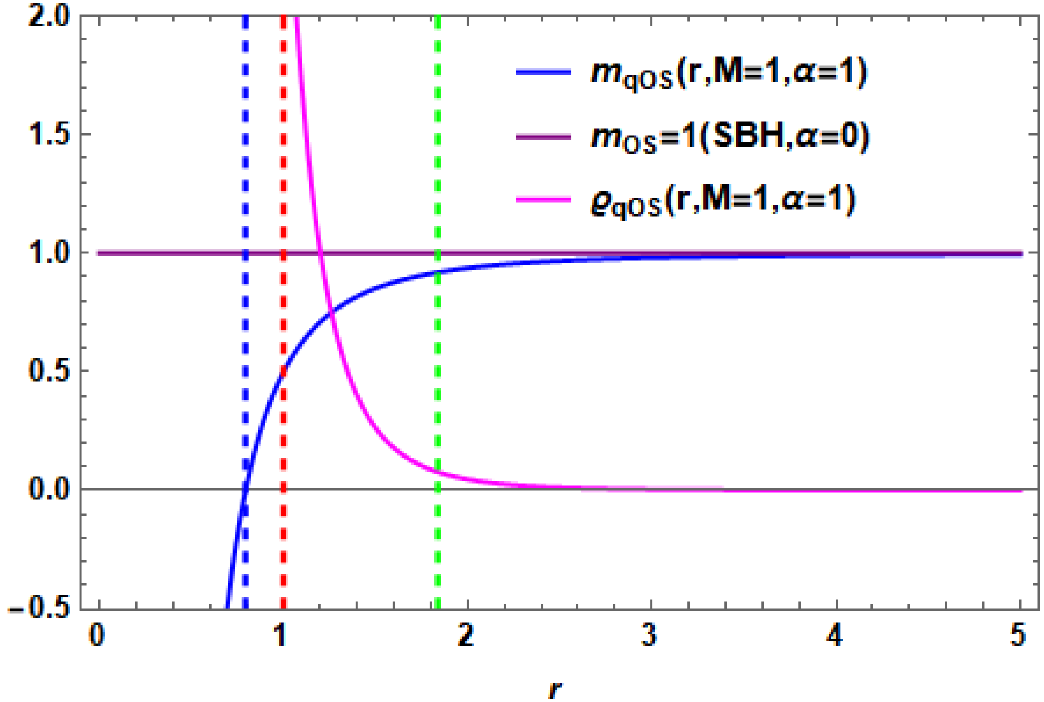
- $\boxed{\dfrac{\alpha M^2}{r^4}}$ $\rightarrow$ a BH mass gap $\leftarrow$ anisotropic EM tensor $\left(T_{\mu\nu}^{\mathrm{qOS}} = \dfrac{\partial L_{\mathrm{qOS}}}{\partial g_{\mu\nu}} \right)$

where $\boxed{L_{\mathrm{qOS}}}$ $\rightarrow$ anisotropic fluid model

because of $T_{\mu}^{\mathrm{qOS},\nu} = \dfrac{3\alpha M^2}{r^6} \mathrm{diag}[-1,-1,2,2] \equiv \mathrm{diag}[-\rho_{\mathrm{qOS}}, p_{\mathrm{qOS}}, p_\theta, p_\varphi]$

- Comparing it with RN-EM tensor $T_{\mu}^{\mathrm{RN},\nu} = \dfrac{2Q^2}{r^4} \mathrm{diag}[-1,-1,1,1]$
- An exterior observer in past A-region may not see any quantum effects.
- But, a Killing observer perceives $\boxed{\rho_{\mathrm{qOS}} = \dfrac{3\alpha M^2}{r^6}}$ due to collapsing of Dust Ball.
- If the quantum deformation of the APS vanishes, one finds $T_{\mu\nu}^{\mathrm{qOS}}=0$ (SBH=OS-BH).

Mass function → bouncing point



--------------- → bouncing point from junction condition:

$$m_{qOS}(r) = 0 \rightarrow r_b(M,\alpha) = \left(\frac{\alpha M}{2}\right)^{\frac{1}{3}}$$

--------------- → inner horizon

--------------- → outer horizon

- NEC/WEC/SEC are satisfied, but DEC is not satisfied.

Two (outer/inner) horizons for exterior spacetime

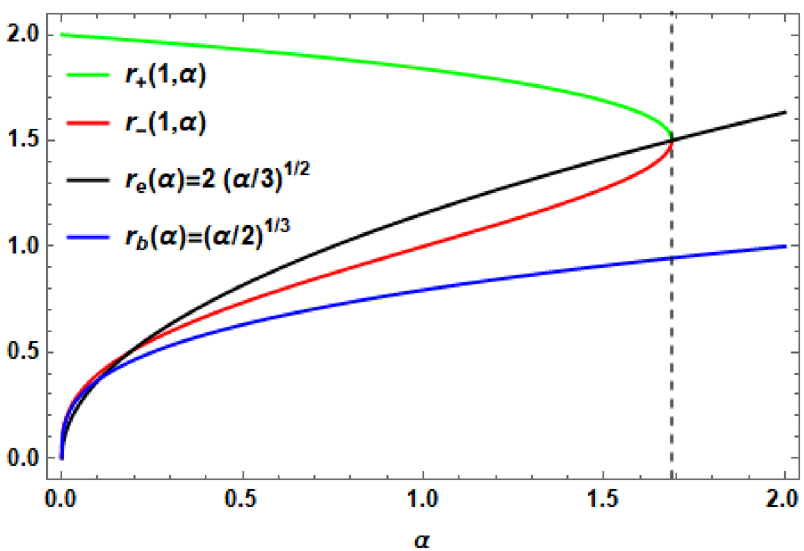
- From $g(r)=0$, four roots $r_k(M,\alpha)$, k=1, 2, 3, 4

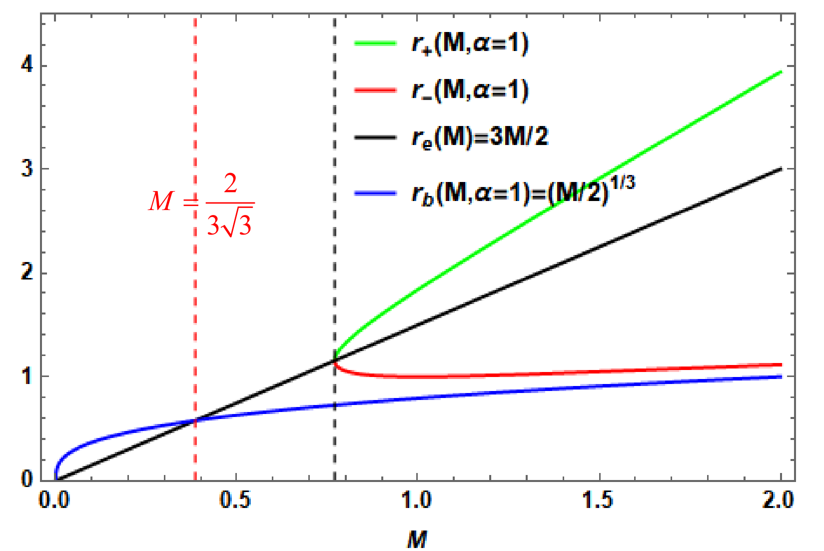
$\to r_3(M,\alpha)\to r_-(M,\alpha)$, $r_4(M,\alpha)\to r_+(M,\alpha)$, $r_{1/2}(M,\alpha)\to$ complex

- $$r_\pm(M,\alpha)=\frac{M}{2}+\frac{1}{2}\left(M^2+\frac{2^{5/3}M^2\alpha}{(3\eta)^{1/3}}+\frac{(2\eta)^{1/3}}{3^{2/3}}\right)^{1/2}\pm\frac{1}{2}\left(2M^2-\frac{2^{5/3}M^2\alpha}{(3\eta)^{1/3}}-\frac{(2\eta)^{1/3}}{3^{2/3}}+\frac{2M^3}{\left(M^2+\frac{2^{5/3}M^2\alpha}{(3\eta)^{1/3}}+\frac{(2\eta)^{1/3}}{3^{2/3}}\right)^{1/2}}\right)^{1/2}$$

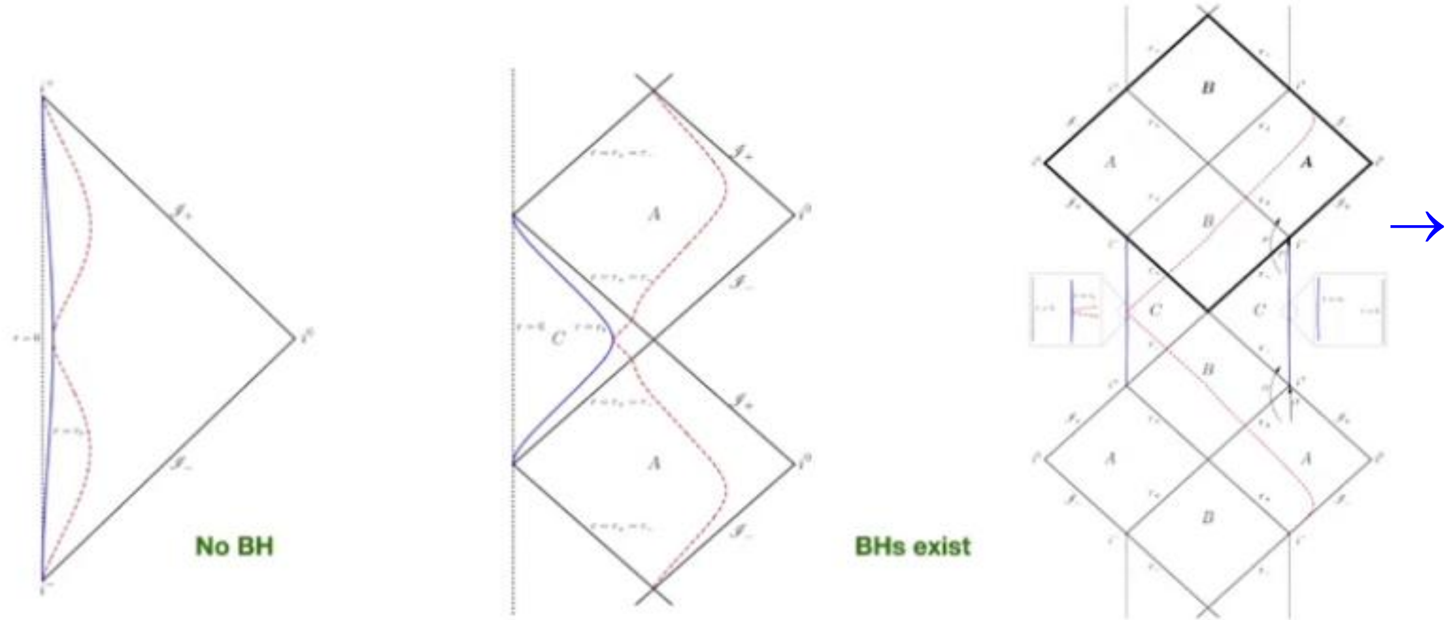
with $\eta(M,\alpha)=\alpha M^3\left(9M+\sqrt{3}\sqrt{27M^2-16\alpha}\right)\xrightarrow[\text{condition of } r_\pm]{} 0<\alpha<\alpha_e\left(=\frac{27M^2}{16}\right)$ or $M>M_{\min}\left(=\frac{4\sqrt{\alpha}}{3\sqrt{3}}\right)$

- Extremal horizons $\to\left(r_e(\alpha)=\frac{2\sqrt{\alpha}}{\sqrt{3}},\ r_e(M)=\frac{3M}{2}\right)$



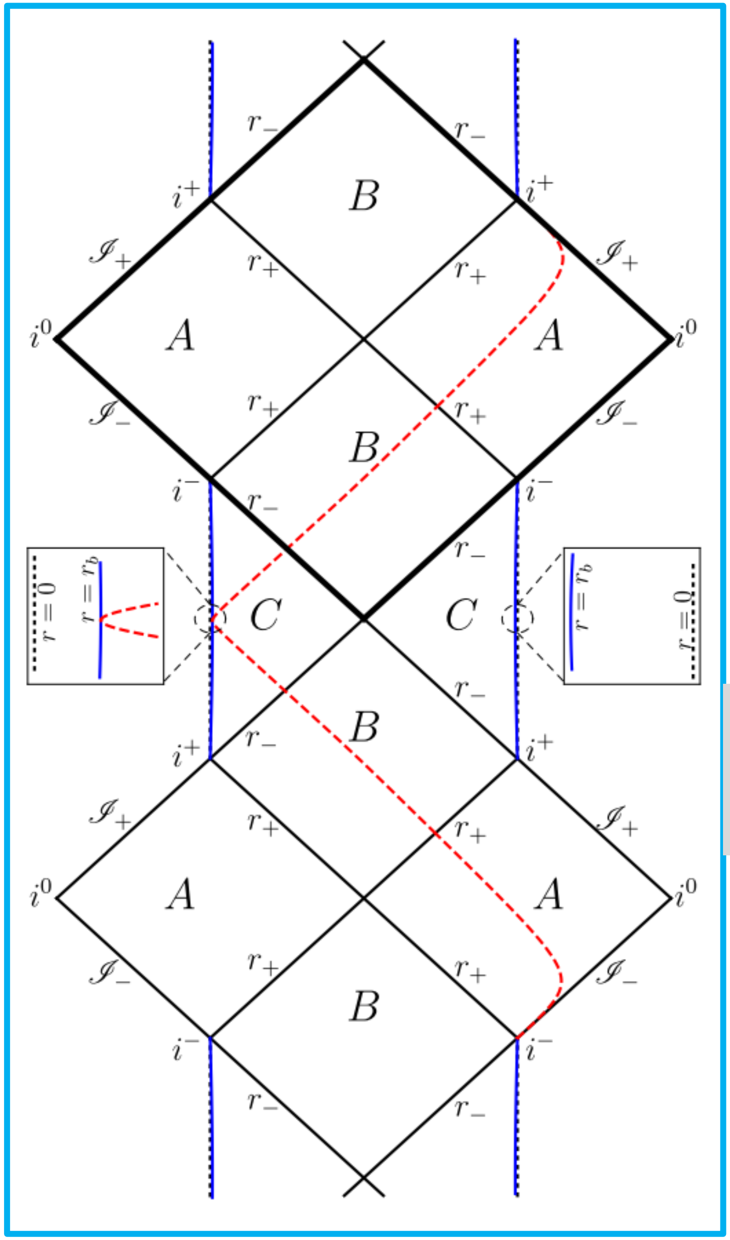


Quantum Oppenheimer-Snyder model with $\alpha \neq 0$



$M < M_{\min}$ $\quad$ $M = M_{\min}$ $\quad$ $M > M_{\min}$

Penrose diagram for qOS model → RNBH



Three regions $\rightarrow \begin{cases} A: \ r > r_+ \\ B: \ r_- < r < r_+ \\ C: 0 < r < r_- \end{cases}$

r_b : ⟶ (bouncing point)

$r = 0$: ········ (singularity)

Timelike geodesics of Dust Ball:

$i^-(A) \rightarrow r_+ \rightarrow r_-(B) \rightarrow r_b(C) \rightarrow r_-(B) \rightarrow r_+ \rightarrow i^+(A)$

Thermodynamics of qOS-BH for *A*-region: prediction?
positive Heat capacity →Allowed region for GB- scalarization

Mass: $m(M,\alpha)=\dfrac{r_+^3-r_+^2\sqrt{r_+^2-\alpha}}{\alpha}\to_{r_+\to\sqrt{S/\pi}} m(S,\alpha),$

Temperature: $T(M,\alpha)=\dfrac{2\alpha-3r_+^2+3r_+\sqrt{r_+^2-\alpha}}{2\pi\alpha\sqrt{r_+^2-\alpha}}\longrightarrow_{r_+\to\sqrt{S/\pi}} T(S,\alpha),$

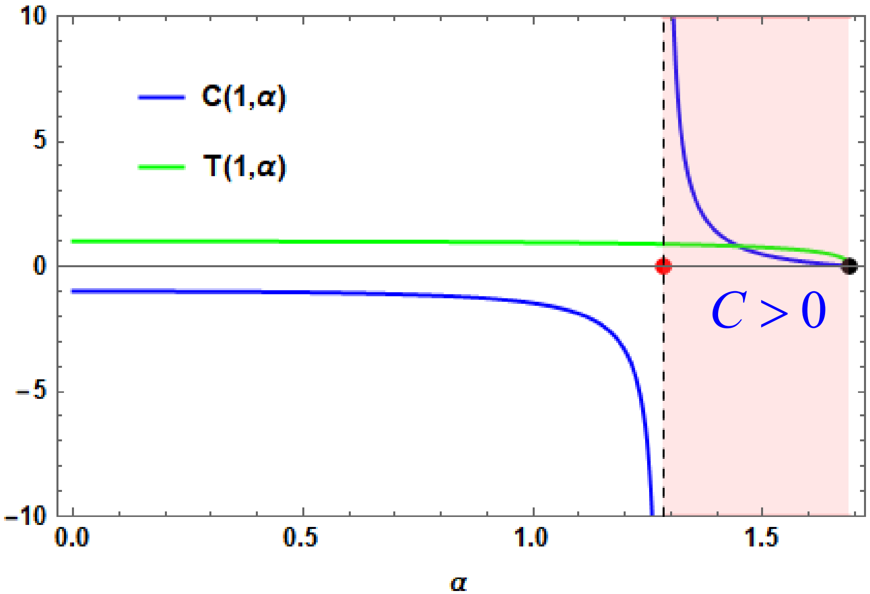
$\Rightarrow\quad T(M,\alpha)=0\to$ Extremal Point

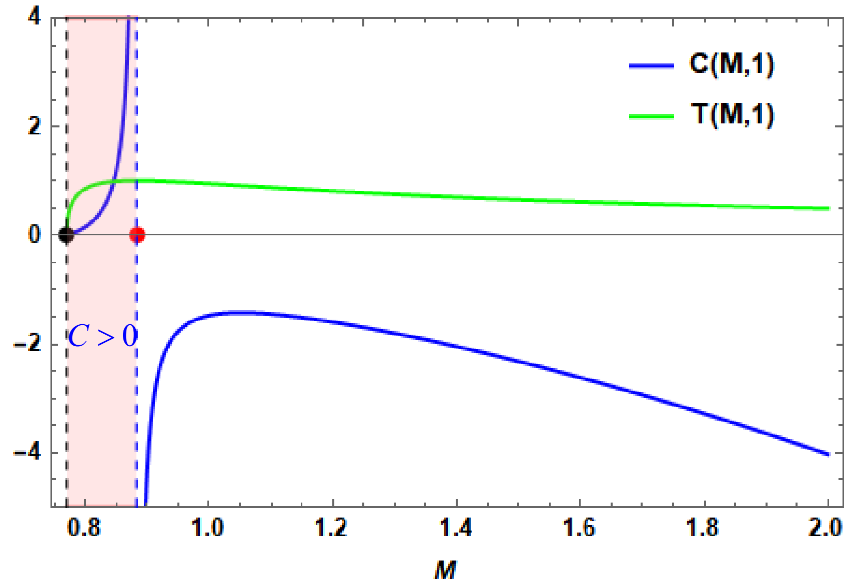
Heat capacity: $C(M,\alpha)\equiv\dfrac{nc(M,\alpha)}{dc(M,\alpha)}=-\dfrac{2\pi r_+(r_+^2-\alpha)[2\alpha-3r_+^2+3r_+\sqrt{r_+^2-\alpha}\,]}{3(\alpha-r_+^2)\sqrt{r_+^2-\alpha}+3r_+^3-4\alpha r_+},$

$$\Rightarrow\begin{cases} nc(M,\alpha)=0\to\text{Extremal Point}\\ dc(M,\alpha)=0\to\text{Davies Point}\\ \qquad\text{(double horizon)}\end{cases}$$

Chemical potential: $W_\alpha(M,\alpha)=-\dfrac{r_+^2[\alpha-2r_+^2+2r_+\sqrt{r_+^2-\alpha}\,]}{2\alpha^2\sqrt{r_+^2-\alpha}}\to_{r_+\to\sqrt{S/\pi}} W_\alpha(S,\alpha),$

First-law and Smarr formula: $dm=TdS+W_\alpha d\alpha,\quad m=2TS+2W_\alpha\alpha$





3. GB-onset scalarization of qOS-BHs

- $\mathcal{L}_{\rm EGBSq} = \frac{1}{16\pi}\left[R - 2(\partial\phi)^2 + \lambda f(\phi) R_{\rm GB}^2 + L_{\rm qOS}\right]$ with coupling function $f(\phi) = 2\left(\phi^2 - \frac{a\phi^4}{4}\right)$
- Einstein eq:

$$G_{\mu\nu} = 2\partial_\mu\phi\partial_\nu\phi - (\partial\phi)^2 g_{\mu\nu} + \Gamma_{\mu\nu} + T_{\mu\nu}^{\rm qOS}$$

with $\Gamma_{\mu\nu} = 2R\nabla_{(\mu}\Psi_{\nu)} + 4\nabla^\alpha\Psi_\alpha G_{\mu\nu} - 8R_{(\mu|\alpha|}\nabla^\alpha\Psi_{\nu)} + 4R^{\alpha\beta}\nabla_\alpha\Psi_\beta g_{\mu\nu} - 4R^\beta_{\mu\alpha\nu}\nabla^\alpha\Psi_\beta$,

$\Psi_\mu = \lambda f'(\phi)\partial_\mu\phi$ and $T_\mu^{{\rm qOS},\nu} = \frac{3\alpha M^2}{r^6}{\rm diag}[-1,-1,2,2]$.

- Scalar eq: $\Box\phi + \frac{\lambda}{4} f'(\phi) R_{\rm GB}^2 = 0$

$\rightarrow$ qOS-BH solution with ϕ=0 for the background spacetime

$$ds_{\rm qOS}^2 = -g(r)dt^2 + \frac{dr^2}{g(r)} + r^2 d\Omega_2^2 \text{ with } g(r) = 1 - \frac{2M}{r} + \frac{\alpha M^2}{r^4}$$

Spontaneous scalarization of charged black holes at the approach to extremality, Y. Brihaye and Betti Hartmann, Phys.Lett.B792 (2019) 244

Aspects of Gauss-Bonnet scalarization of charged black holes, Carlos A. R. Herdeiro, Alexandre M. Pombo, Eugen Radu, Universe 7(2021), 483

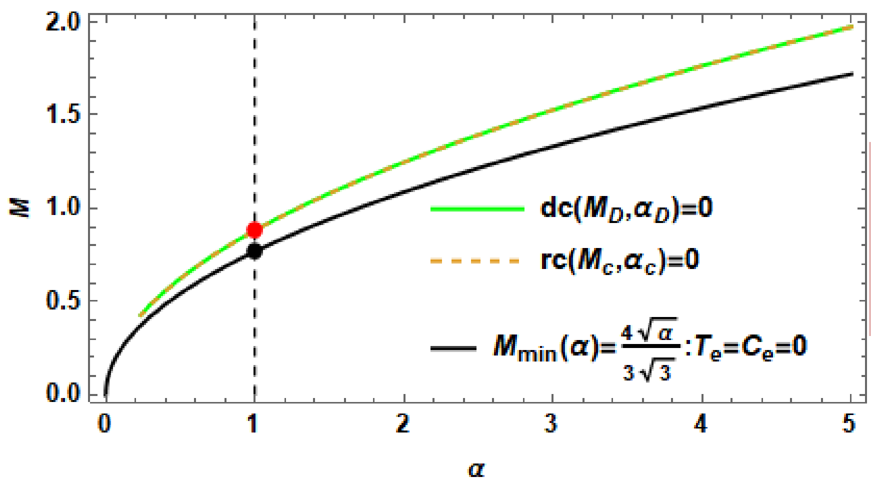
GB- onset scalarization→ Thermodynamics of qOS-BH

- Linearized scalar eq → $\left(\bar{\Box}-\tilde{m}_{\rm eff}^2\right)\delta\phi=0$ with $\tilde{m}_{\rm eff}^2=-\lambda R_{GB}^2$

$$\delta\phi(t,r_*,\theta,\varphi)=\sum_m\sum_{l=|m|}^{\infty}\frac{\psi_{lm}(t,r_*)}{r}Y_{lm}(\theta,\varphi)\xrightarrow{\text{Sch-eq}}\boxed{\frac{\partial^2\psi_{lm}(t,r_*)}{\partial r_*^2}-\frac{\partial^2\psi_{lm}(t,r_*)}{\partial t^2}=V(r)\psi_{lm}(t,r_*)}$$

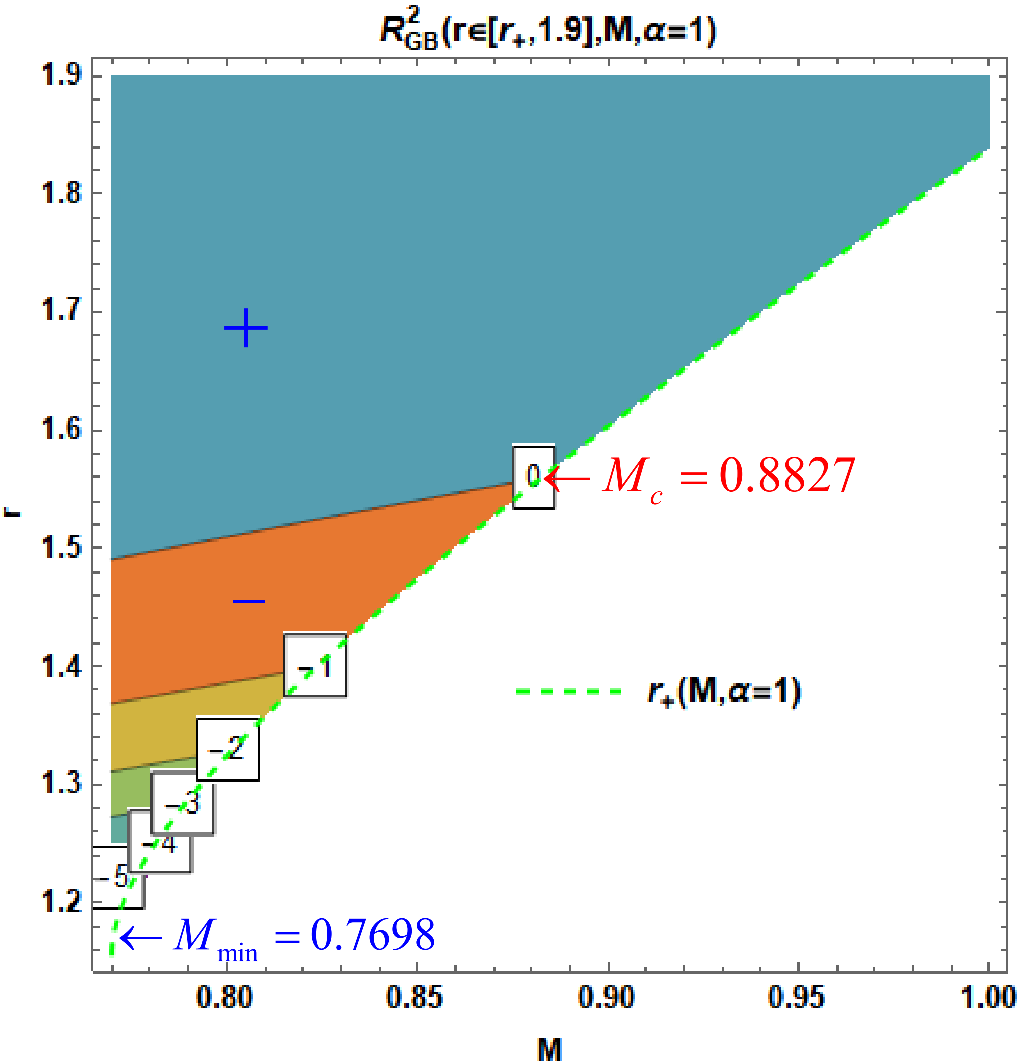
with $V(r)=g(r)\left(\frac{2M}{r^3}-\frac{4\alpha M^2}{r^6}+\frac{l(l+1)}{r^2}+\tilde{m}_{\rm eff}^2\right)$ and $\tilde{m}_{\rm eff}^2=-\frac{48\lambda M^2}{r^6}\left[\frac{3\alpha^2M^2}{r^6}-\frac{5\alpha M}{r^3}+1\right]$

- Critical onset condition → $V_{l=0}(r_+)\psi_{00}(t,r_*)=0\xrightarrow[\lambda\to-\infty]{}rc(M,\alpha)\psi_{00}(t,r_*)=0$
- Critical onset curve → $rc(M,\alpha)\equiv3\tilde{\alpha}_c^2-5\tilde{\alpha}_c+1=0$ with $\tilde{\alpha}_c=\frac{\alpha M}{r_+^3}$
- Critical onset parameters → $\tilde{\alpha}_c=0.2324\to\alpha_c=1.2835=\alpha_D$ for $M_c=1$ (M_c=0.8827=M_D for α_c=1)

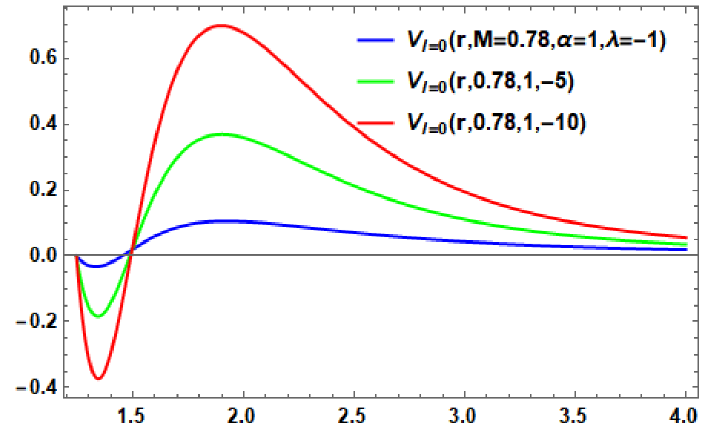


Comment #3:
Davies curve=Critical onset curve
→Feature of qOS-BH

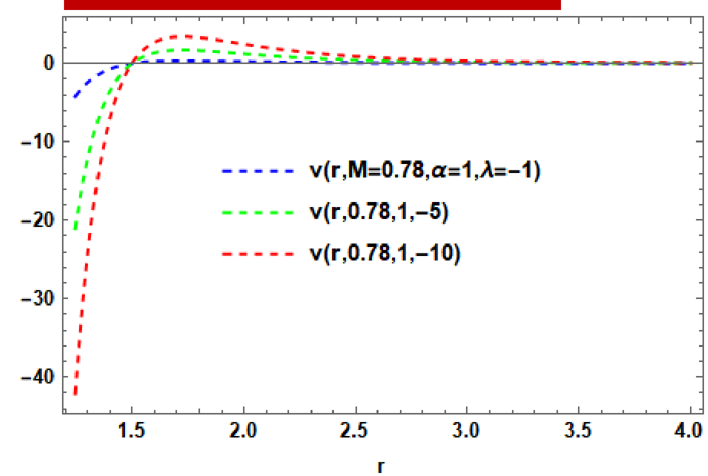
Sign flip-flop of GB-term in the near-horizon: Tachyonic instability → $-: \lambda < 0 \rightarrow \tilde{m}^2_{\text{eff}} = -\lambda R^2_{\text{GB}} < 0$ (tachyonic term)



Potentials



Reduced Potentials



GB- onset scalarization→ positive heat capacity

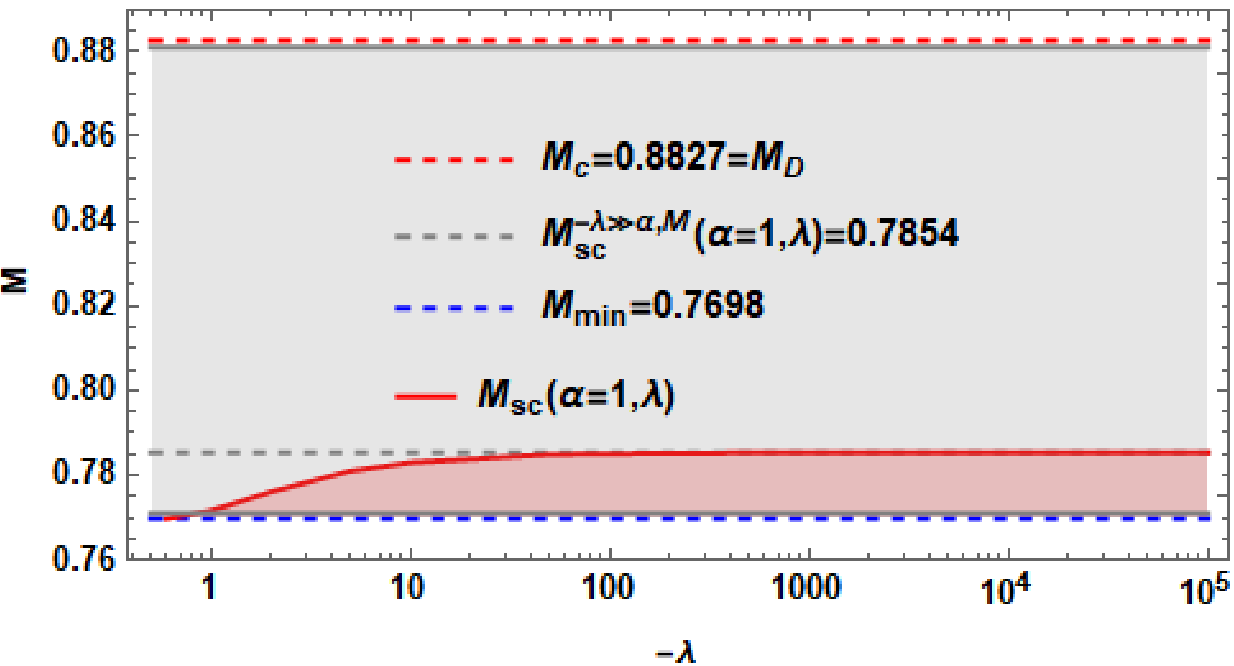
- Sufficient condition of instability $\rightarrow \int_{r_+(M,\alpha)}^{\infty} v(r)\, dr \equiv I_v < 0$ with reduced potential $v(r) \equiv \dfrac{V_{l=0}(r)}{g(r)}$

  $I_v = 0 \longrightarrow M_{\mathrm{sc}}(\alpha = 1, \lambda)$: Numerical data

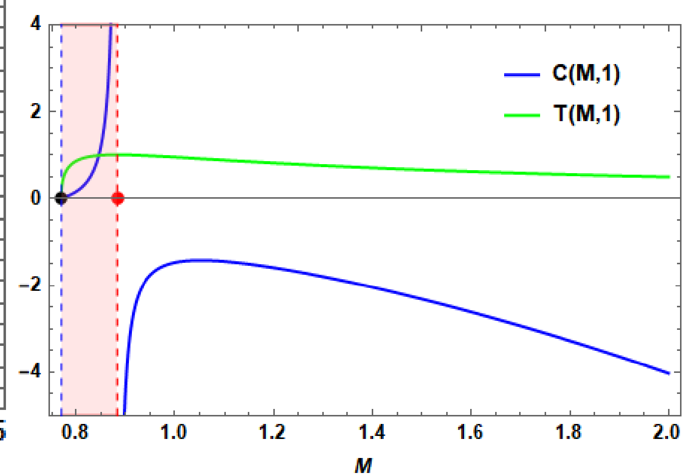
- For $-\lambda \gg \alpha, M \rightarrow \int_{r_+(M,\alpha)}^{\infty} \tilde{m}_{\mathrm{eff}}^2 dr \equiv I(M,\alpha) < 0 \rightarrow I(M,\alpha) = \dfrac{6M^2[120M^2\alpha^2 - 275M\alpha r_+^3 + 88r_+^6]}{55 r_+^{11}} = 0$

  $\longrightarrow M_{sc}^{-\lambda \gg \alpha, M} = 0.7854$

- $M_{\mathrm{th}}(\alpha = 1, \lambda) \in [M_{\min}, M_{\mathrm{c}}]$ could be obtained by computing time-domain profile $\psi_{00}(t, r_*)$ and taking Gaussian wave packet $\psi_{00}(0, r_*) = e^{-(r_* - r_*^c)^2/\lambda}$ as an initial wave.



Allowed region for GB- =positive Heat capacity



Boundary potentials→ ill-defined (unable to find scalar clouds), being seeds to generate finite branches for scalarized qOS-BHs

- Boundary potentials to obtain starting branch points (λ_n) by standard WKB method

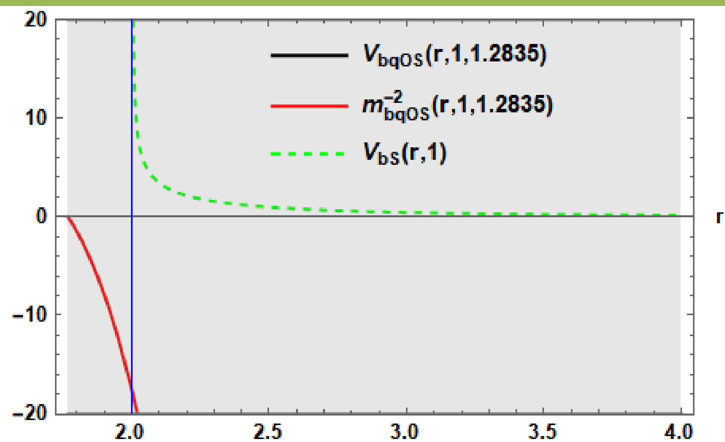
- $$\boxed{\frac{\partial^2 \psi_{lm}(r_*)}{\partial r_*^2} - V(r_*)\psi_{lm}(r_*) = 0} \xrightarrow[\text{WKB-QC}]{} \int_{r_{*-}}^{r_{*+}} dr_* \sqrt{-V(r_*)} = \left(n + \frac{3}{4}\right)\pi, \ n = 0,\ 1,\ 2, \cdots$$

- $$\lambda < 0 \to \sqrt{-\lambda}\int_{r_+}^{\infty} V^-_{\rm bqOS}(r,M,\alpha)\, dr, \ V^-_{\rm bqOS}(r,M,\alpha) = \frac{\sqrt{48}M}{r^6}\frac{\sqrt{m^{-2}_{\rm bqOS}(r,M,\alpha)}}{\sqrt{1-\frac{2M}{r}+\frac{\alpha M^2}{r^4}}}$$

$$\text{with} \quad m^{-2}_{\rm bqOS}(r,M,\alpha) = -(3\alpha^2 M^2 - 5\alpha M r^3 + r^6) \le 0 \ \text{for} \ r \ge r_+$$

- $$\text{GB}_+ \to V_{bS}(r,M) = \frac{\sqrt{48}M}{r^3}\frac{1}{\sqrt{1-\frac{2M}{r}}} \to \sqrt{\lambda_n}\int_{r_+=2M}^{\infty} V_{bS}(r,M)dr \equiv \sqrt{\lambda_n}\, I_n(M) = \pi(n+3/4)$$

$$\Rightarrow \text{Infinite branches for SBH=OS-BH} \to \lambda_n(M) = \frac{\pi^2(n+3/4)^2}{I_n^2(M)}, \quad n = 0,\ 1,\ 2, \cdots$$



→ $V_{\rm bqOS}(r,M,\alpha)$ is not properly defined outside the horizon

→ Unabling to obtain scalar clouds for scalarized qOS-BHs

In this case, we directly construct scalarized qOS-BHs by solving full EOMs.
However, we could not obtain scalarized qOS-BHs because $\boxed{L_{\rm qOS} = ?}$ is still unknown.

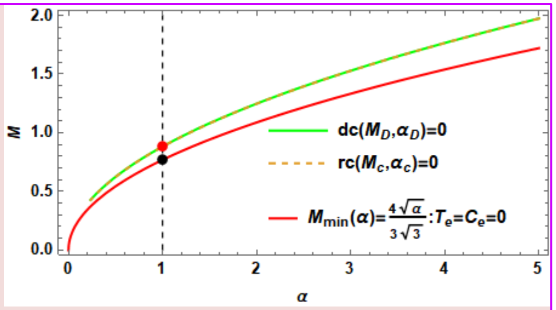
→ Comment #4: A handicap of qOS-BHs obtained by junction condition.

4. GBe onset scalarization of e(extremal) qOS-BHs

eqOS-BHs →degenerate state

→zero temperature/zero heat capacity but non-vanishing entropy

- $g(r) = 1 - \frac{2M}{r} + \frac{\alpha M^2}{r^4} \xrightarrow[\alpha \to \alpha_e = 27M^2/16]{} \left(1 - \frac{3M}{2r}\right)^2 \left(1 + \frac{M}{r} + \frac{3M^2}{4r^2}\right)$



- eqOS- BH spacetime

$$ds^2_{\text{eqOS}} = -g_e(r)dt^2 + \frac{dr^2}{g_e(r)} + r^2 d\Omega_2^2 \,,\; g_e(r) \equiv \left(1 - \frac{3M}{2r}\right)^2$$

- $s(l=0)$-mode scalar potential

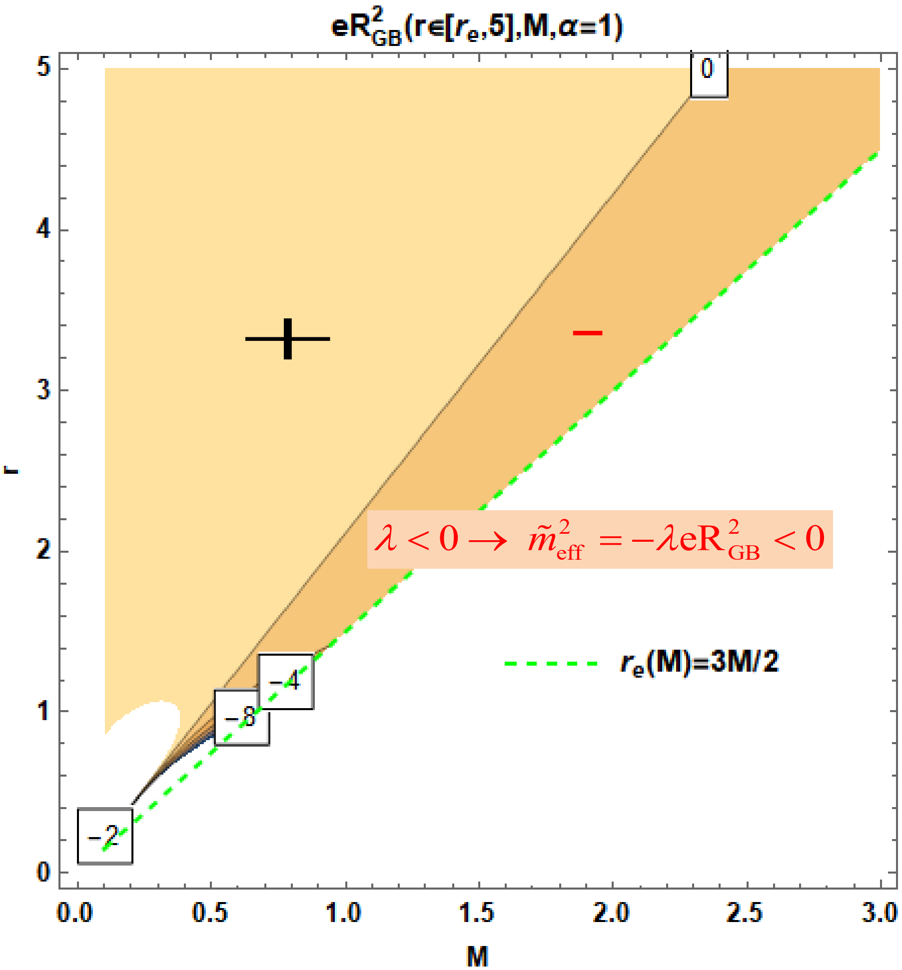
$$V_e(r) = g_e(r)\left[\frac{3M}{r^3} - \frac{9M^2}{2r^4} + \tilde{m}^2_{\text{eff}}\right] \text{with } \tilde{m}^2_{\text{eff}} = -\lambda \text{eR}^2_{\text{GB}} = -\frac{27\lambda M^2}{2r^6}\left[\frac{15M^2}{r^2} - \frac{24M}{r} + 8\right]$$

- Sufficient condition for instability

$$\int_{r_e=3M/2}^{\infty}\left[\frac{V_e(r)}{g_e(r)}\right]dr \equiv I_e(M,\lambda) = \frac{70M^2 + 64\lambda}{315M^3} < 0 \to \boxed{0 < M \le M_{\text{eqOS}}(\lambda), \quad M_{\text{eqOS}}(\lambda) \simeq 0.96\sqrt{-\lambda}}$$

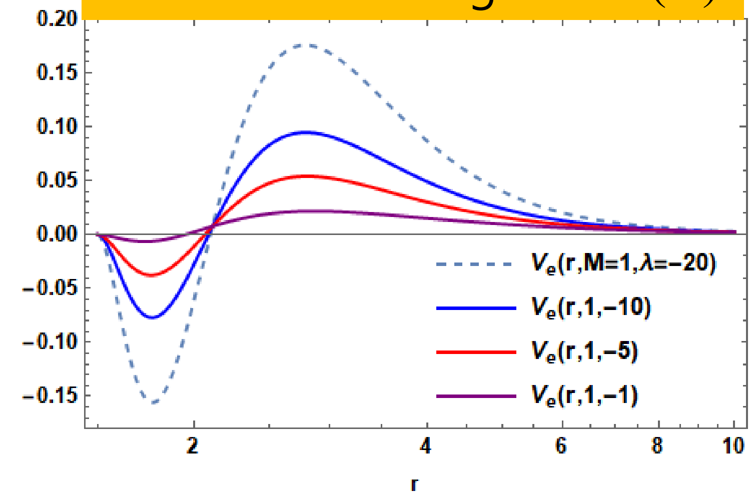
Whole negative of eGB-term in the near-horizon of eqOS-BH:

Tachyonic instability is available for any $\lambda < 0$

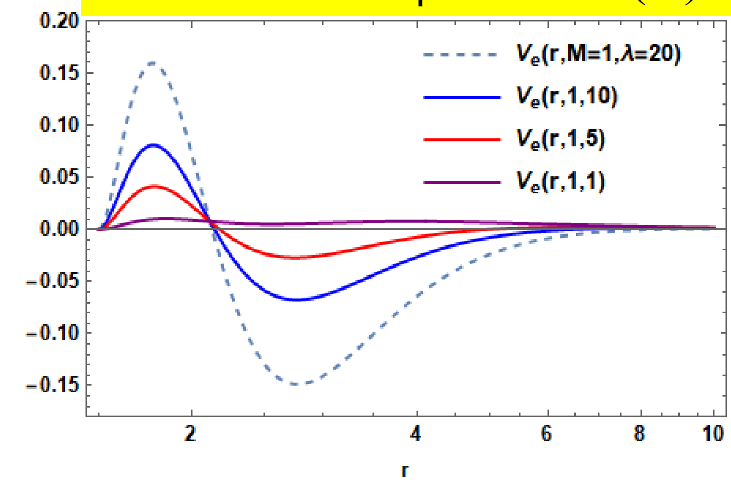


eqOS-BH

Potential with negative λ (O)



Potential with positive λ (X)



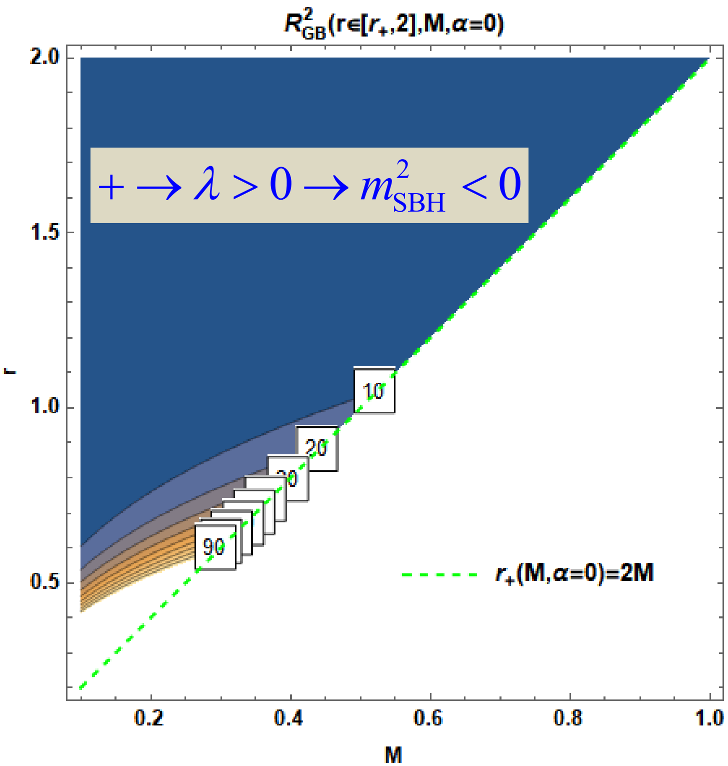
Whole positive of GB-term in the near-horizon of SBH=QS-BH:

Contrastively, tachyonic instability is allowed for $\lambda > 0$

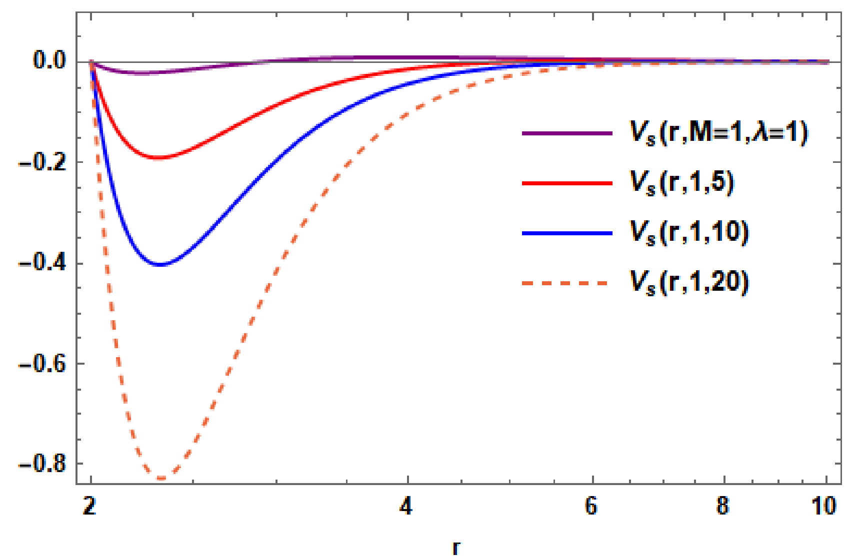
GB$_+$ scalarization for SBH with $\lambda > 0$

$$V_S(r) = g_S(r)\left[\frac{2M}{r^3} + m_{SBH}^2\right], \quad g_S(r) = 1 - \frac{2M}{r}, \quad m_{SBH}^2 = -\lambda R_{GB}^2 = -\frac{48\lambda M^2}{r^6}$$

$$\int_{r_+=2M}^{\infty} (V_S / g_S)\, dr \equiv I_S(M,\lambda) = \frac{5M^2 - 6\lambda}{20M^3} < 0 \rightarrow \boxed{0 < M \le M_S(\lambda), \quad M_S(\lambda) \simeq 1.1\sqrt{\lambda}}$$



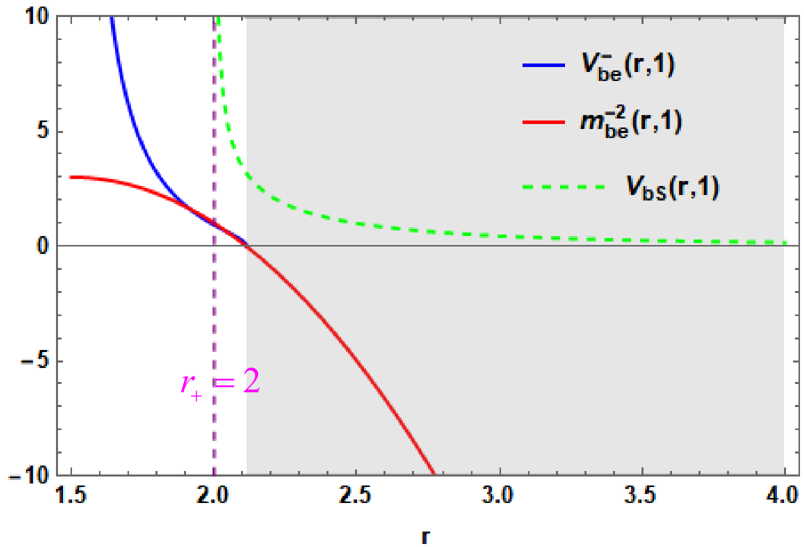
Potentials for SBH



Boundary Potentials→ handicap (unable to find scalar clouds), being seeds to generate infinite branches for scalarized eqOS-BHs

- Boundary potentials to obtain starting branch points (λ_n / λ_n) by standard WKB method
- $\lambda < 0 \to V^-_{\rm be}(r,M) = \dfrac{\sqrt{27}M}{\sqrt{2}r^3}\dfrac{\sqrt{m^{-2}_{\rm be}(r,M)}}{1-\dfrac{3M}{2r}}$, $\quad m^{-2}_{\rm be}(r,M) = -\dfrac{15M^2}{r^2} + \dfrac{24M}{r} - 8 \le 0$ for $r > 2.11$
- GB₊ $\to V_{\rm bS}(r,M) = \dfrac{\sqrt{48}M}{r^3}\dfrac{1}{\sqrt{1-\dfrac{2M}{r}}} \to \sqrt{\lambda_n}\int_{r_+=2M}^{\infty} V_{\rm bS}(r,M)dr \equiv \sqrt{\lambda_n} I_n(M) = \pi(n+3/4)$

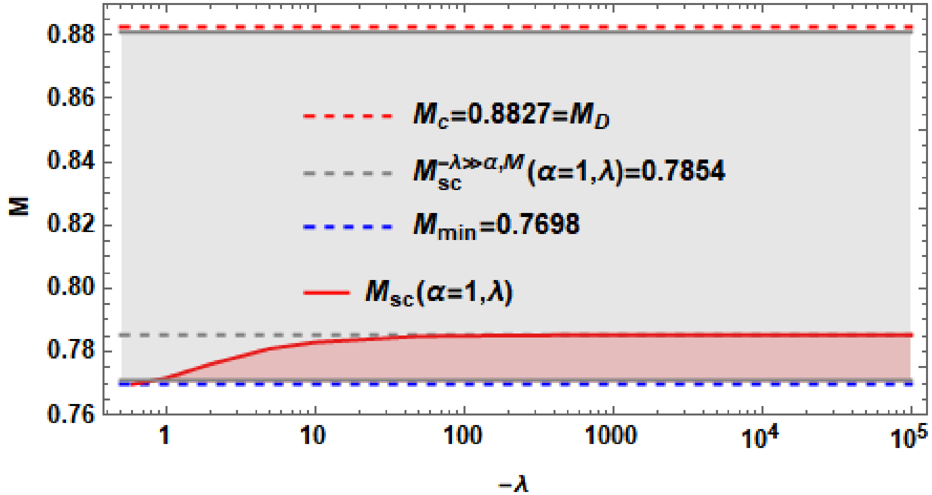
$\Rightarrow$ Infinite branches for SBH=OS-BH $\to \lambda_n(M) = \dfrac{\pi^2(n+3/4)^2}{I_n^2(M)}$, $\quad n = 0,\ 1,\ 2,\cdots$



$\to V^-_{\rm be}(r,1)$ is not properly defined for $r > 2.11$

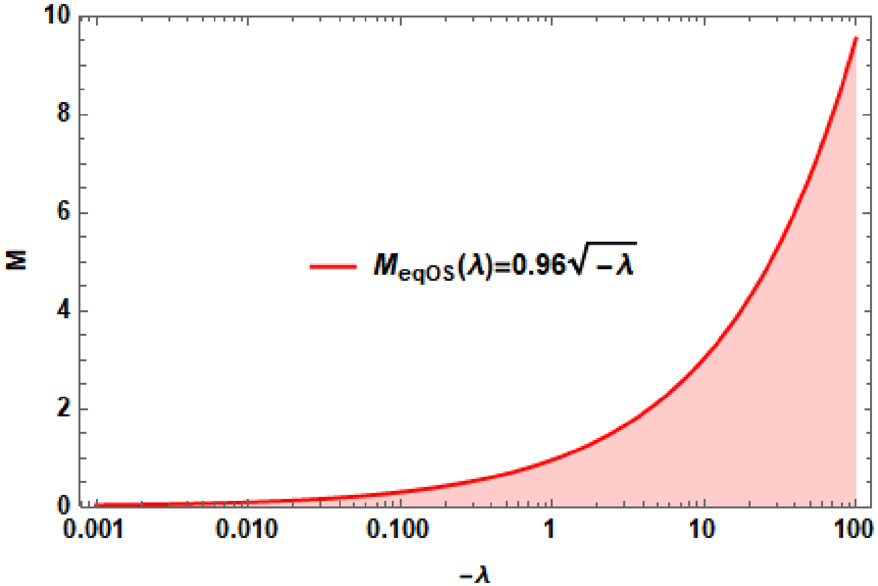
$\to$ Unabling to obtain scalar clouds for scalarized eqOS-BH

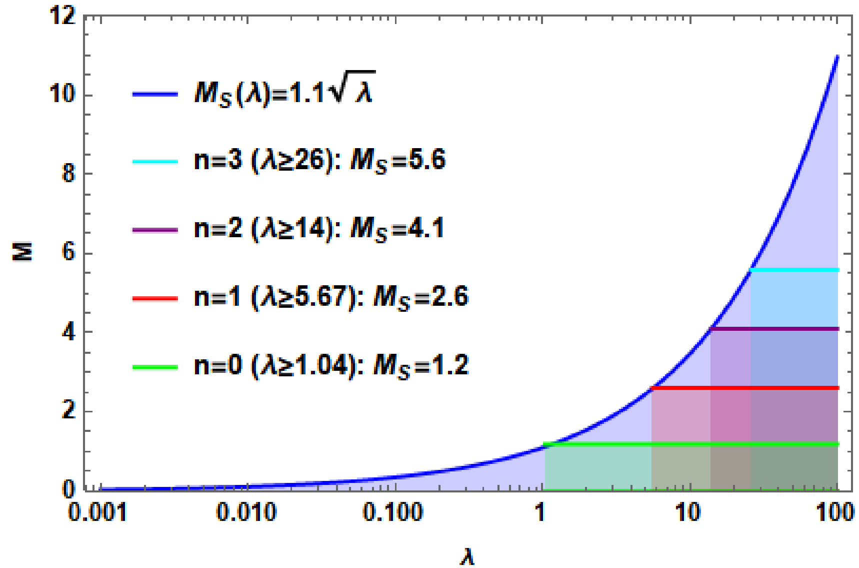
Comparison between GB-/GBe /GB+ onset scalarizations



Allowed region for GB-scalarization with $\boxed{\lambda<0}$
= positive heat capacity (single branch)

$$M_{\rm min}(=0.7698)\ \le M_{\rm th}(\alpha=1,\lambda<0)\le\ M_c(=0.8827)$$





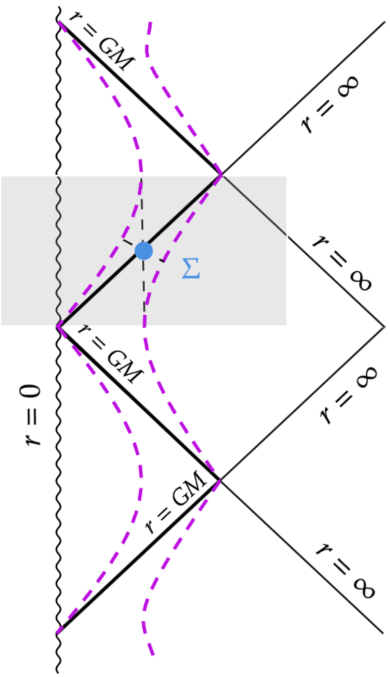
GBe scalarization with $\boxed{\lambda<0}$
→ Single branch of scalarized qOS-BHs

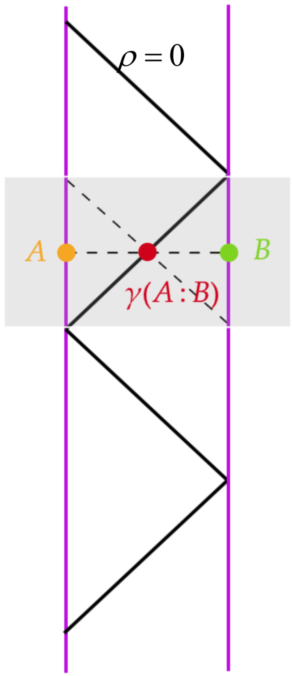
GB+scalarization with $\boxed{\lambda>0}$
→Infinite branches of scalarized SBHs

Comment #5: No scalar clouds are found for eqOS-BHs.
→This requires our investigation to confine to the near-extremal limit or the near-horizon approximation.

Penrose Diagram for eRNBH

$$g^{\mathrm{e}}_{\mathrm{RN}}(r) \equiv \left(1 - \frac{GM}{r}\right)^2$$





↑The strip enclosed by dashed purple curves represents its near-horizon region, $\boxed{AdS_2 \times S^2}$ (eprint: 2505.08012)

5. GB$_{\mathrm{BR}}$ onset scalarization of eqOS-BHs

- Near-horizon geometry of Bertotti-Robinson (BR) spacetime ($AdS_2 \times S^2$)

$$ds^2_{\mathrm{BR}} = \left(\frac{3M}{2}\right)^2 \left(-\rho^2 d\tau^2 + \frac{d\rho^2}{\rho^2}\right) + \left(\frac{3M}{2}\right)^2 (d\theta^2 + \sin^2\theta d\varphi^2) \text{ with } M = \frac{2}{3}$$

- GB term: $-\lambda \mathcal{R}^2_{\mathrm{GB}} = 8\lambda \to \mu^2$ (mass term)
- s-mode ($l = 0$) linearized equation

$$-\frac{1}{\rho^2}\partial^2_\tau \delta\phi + \partial_\rho(\rho^2 \partial_\rho \delta\phi) - \mu^2 \delta\phi = 0 \xrightarrow[\text{Sch-eq}]{\rho \to 1/\rho_*} \boxed{-\frac{\partial^2 \delta\phi(\tau,\rho_*)}{\partial \tau^2} + \frac{\partial^2 \delta\phi(\tau,\rho_*)}{\partial \rho_*^2} = V_{\mathrm{GB}}(\rho_*,\lambda)\delta\phi(\tau,\rho_*)}$$

with GB-potential $V_{\mathrm{GB}}(\rho_*,\lambda) = \dfrac{\mu^2}{\rho_*^2} \to V_{\mathrm{GB}}(\rho,\lambda) = \mu^2 \rho^2$

- Breitenlohner-Freedman (BF) bound around AdS_2

$\mu^2 \geq \mu^2_{\mathrm{BF}} = -\dfrac{1}{4}$ to have a regular (massive) scalar & $\mu^2 < \mu^2_{BF} \to$ a tachyonic scalar

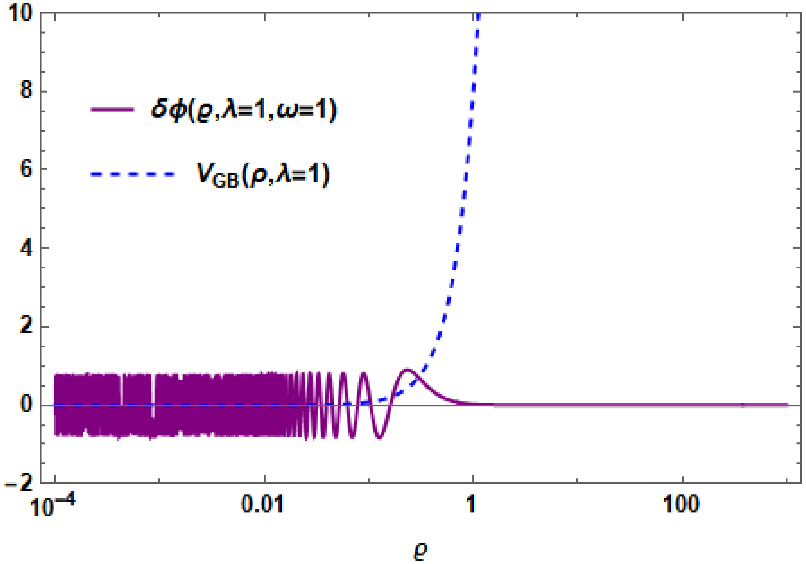
- $$\delta\phi(\tau,\rho_*) = e^{-i\omega\tau}\delta\phi(\rho_*) \xrightarrow[\text{Sch-eq}]{\text{Time-indepedent}} \frac{\partial^2\delta\phi(\rho_*)}{\partial\rho_*^2} + \left[\omega^2 - V_{\mathrm{GB}}(\rho_*,\lambda)\right]\delta\phi(\rho_*) = 0$$

Normalizable solution $\rightarrow \delta\phi(\rho_*) = \sqrt{\rho_*}J_\nu(\omega\rho_*) \rightarrow \delta\phi(\rho) = \frac{1}{\sqrt{\rho}}J_\nu\left(\frac{\omega}{\rho}\right), \quad \nu = \frac{\sqrt{32\lambda+1}}{2}$

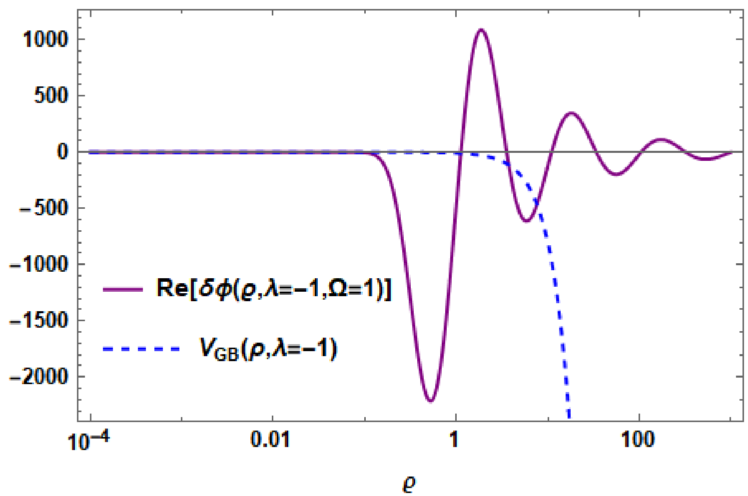
- $$\delta\phi(\tau,\rho_*) = e^{\Omega\tau}\delta\phi(\rho_*) \xrightarrow[\text{Sch-eq}]{\text{Time-indepedent}} \frac{\partial^2\delta\phi(\rho_*)}{\partial\rho_*^2} - \left[\Omega^2 + V_{\mathrm{GB}}(\rho_*,\lambda)\right]\delta\phi(\rho_*) = 0$$

Tachyonic solution $\rightarrow \delta\phi(\rho_*) = \sqrt{\rho_*}Y_\nu(i\Omega\rho_*) \rightarrow \delta\phi(\rho) = \frac{1}{\sqrt{\rho}}Y_\nu\left(\frac{i\Omega}{\rho}\right)$

Normalizable solution

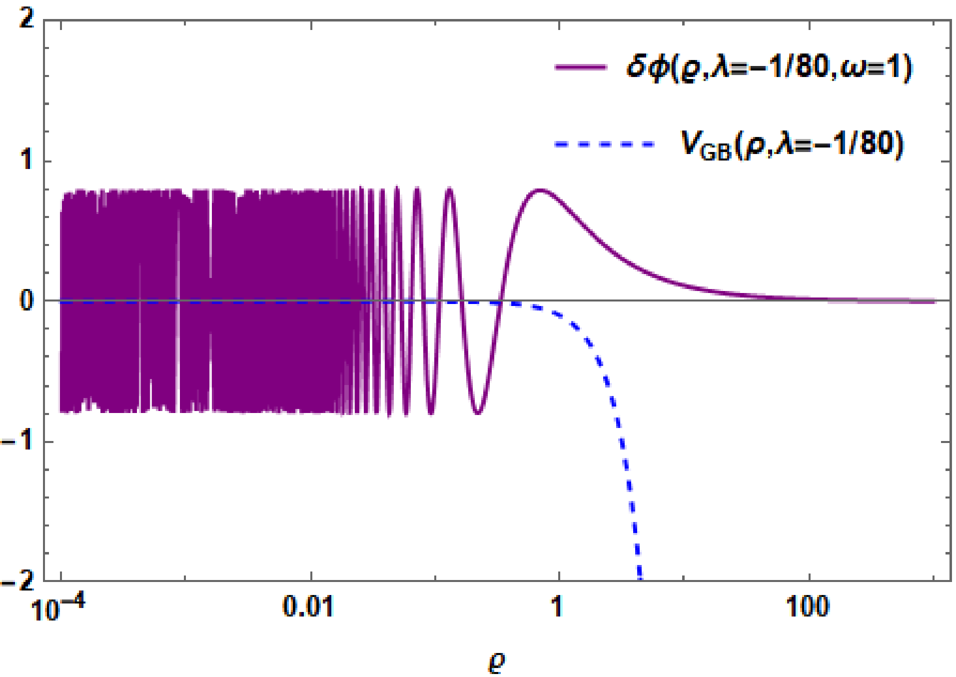


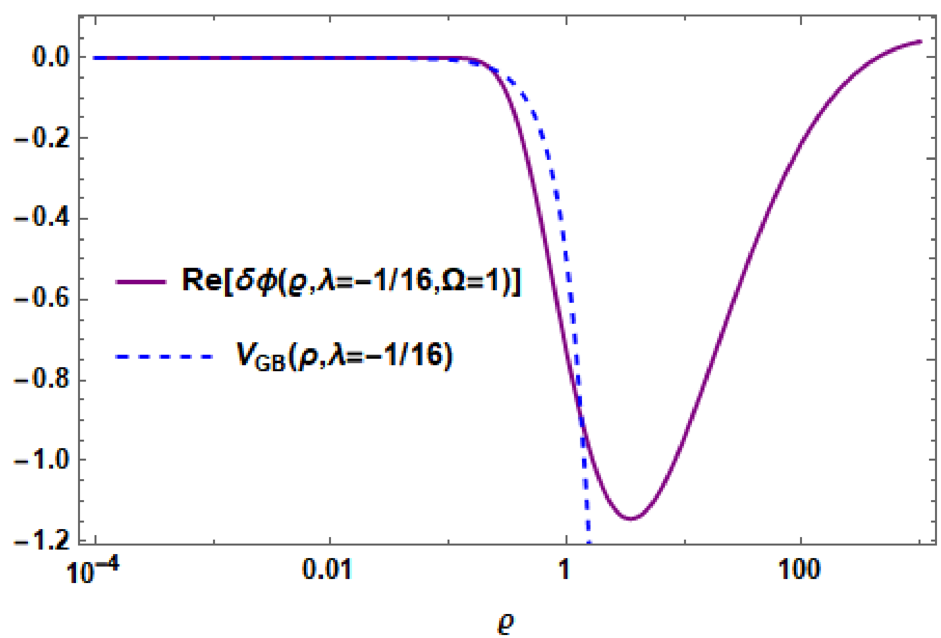
Tachyonic solution



Regular and tachyonic solutions with BF bound

$\mu^2 = -0.1 > \mu^2_{\mathrm{BF}} = -0.25$: $\mu^2 = -0.5 < \mu^2_{\mathrm{BF}}$





- Scalar clouds from static (linearized) equation with $\omega=0$ and $\nu=\dfrac{\sqrt{32\lambda+1}}{2}$

$$\boxed{\frac{\partial^2\delta\phi(\rho_*)}{\partial\rho_*^2}-V_{\rm GB}(\rho_*,\lambda)\delta\phi(\rho_*)=0}$$

$$\to\delta\phi(\rho_*,\lambda)=c_1(\rho_*)^{\nu+\frac{1}{2}}+c_2(\rho_*)^{-\nu+\frac{1}{2}}\to\delta\phi(\rho,\lambda)=c_1(\rho)^{-\nu-\frac{1}{2}}+c_2(\rho)^{\nu-\frac{1}{2}}$$

- Tachyonic cloud with $c_1=c_2=0.5$ and its negative potential

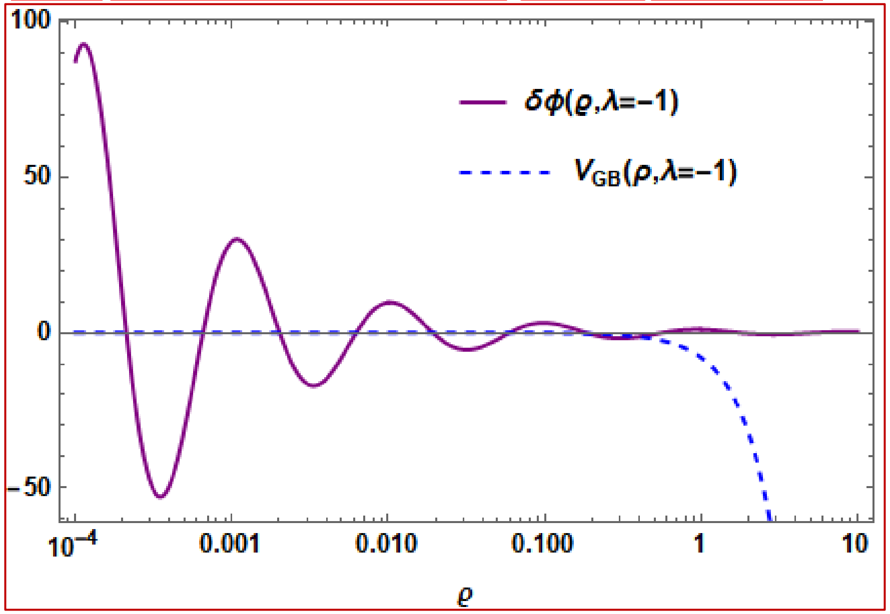
$$\text{Analytic sol}\to\boxed{\delta\phi(\rho,\lambda=-1)=\frac{1}{\sqrt{\rho}}\cos\left(\frac{\sqrt{31}\ln[\rho]}{2}\right)},\quad V_{\rm GB}(\rho,\lambda=-1)=-8\rho^2$$

- Scalar cloud with $c_1=1$, $c_2=0$ and its positive potential

$$\text{Analytic sol}\to\boxed{\delta\phi_{\rm inf}(\rho,\lambda=1)=\left(\frac{1}{\rho}\right)^{\frac{\sqrt{33}+1}{2}}},\quad V_{\rm GB}(\rho,\lambda=1)=8\rho^2$$

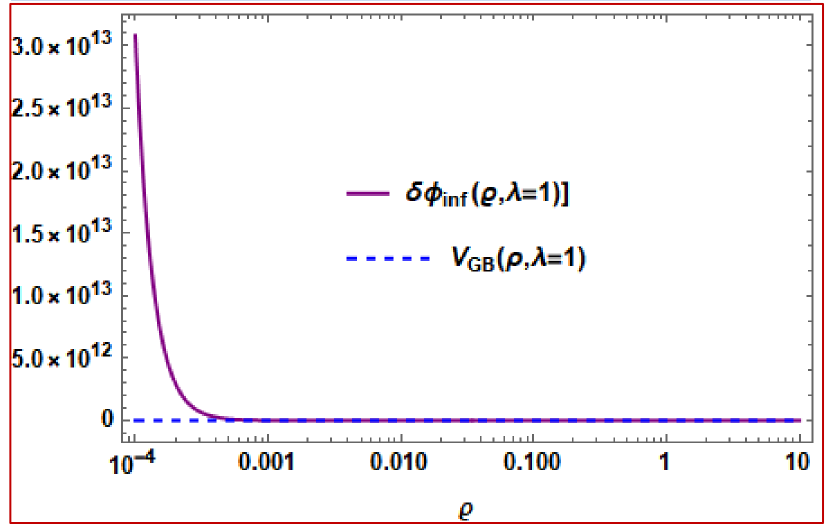
Comment #6: Two branches (analytic scalar clouds with negative/positive λ) and comparing them with GB+ scalarization

Tachyonic cloud with negative potential

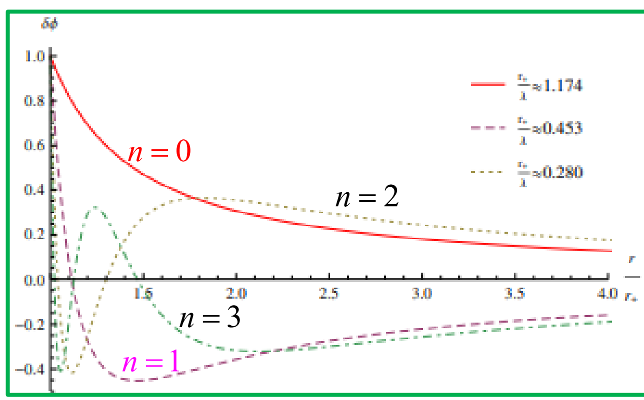


→ Scalar cloud with many nodes for GB+
but it is large at $\rho = 0$

Scalar cloud with positive potential



→ n=0 scalar cloud for GB+
but it blows up at $\rho = 0$
→ Aretakis Instability of eqOS-BH



Four scalar clouds
in GB+ scalarization
for SBH=OS-BH

6. Entropy function approach to GBBR scalarization

We found two branches of analytic scalar clouds with negative/positive λ

1. $L_{\text{qOS}} = ? \rightarrow \mathcal{L}_{\text{NED}} = -2\xi(\mathcal{F})^{\frac{3}{2}}$ with $\mathcal{F}=F_{\mu\nu}F^{\mu\nu} \xrightarrow[\text{mc}]{} \frac{2P^2}{r^4}$ and $\xi = \frac{3\alpha}{2^{3/2}P}$

$\leftarrow$ $\boxed{\text{charged qOS model, EPJC85(2025), 667}}$

Its anisotropic EM tensor $\rightarrow$ $\boxed{T_{\mu}^{\text{NED},\nu} = \frac{3\alpha P^2}{r^6}\text{diag}[-1,-1,2,2] \xrightarrow[P\to M]{} \text{qOS-BH}}$

2. Einstein-Euler-Heisenberg (EEH) action $\rightarrow \mathcal{L}_{\text{EEH}} = R - \mathcal{F} + \mu\mathcal{F}^2$

3. Its extremal BH $\rightarrow r_e = \sqrt{P\sqrt{3\alpha}} \xrightarrow[M(\alpha,P)=\frac{2}{3}\sqrt{P\sqrt{3\alpha}}]{} r_e = \frac{3M}{2}$

$\rightarrow$ the same as eqOS-BH with $g_e(r) = \left(1 - \frac{3M}{2r}\right)^2$

4. A quartic coupling function for GB_{BR} scalarization $\rightarrow f(\phi) = 2\left(\phi^2 - \frac{a}{4}\phi^4\right)$

5. Sen's entropy function approach based on the attractor formalism may be used to obtain
scalarized eqOS-BHs with scalar hair.

- BR spacetime: $\boxed{ds^2_{AdS_2\times S^2} = v_1\left(-\rho^2 d\tau^2 + \frac{d\rho^2}{\rho^2}\right) + v_2\left(d\theta^2 + \sin^2\theta d\varphi^2\right) \text{ with } \phi = u}$

- GB term with scalar hair : $\lambda f(\phi) R^2_{\mathrm{GB}} \to -\frac{16\lambda}{v_1 v_2}\left(u^2 - \frac{a}{4}u^4\right)$

- Ricci scalar: $R = -\frac{2}{v_1} + \frac{2}{v_2}$

- NED term : $\mathcal{L}_{\mathrm{NED}} \equiv -2\xi\left(\mathcal{F}\right)^{\frac{3}{2}} \xrightarrow[\mathrm{mc}]{} -\frac{3\alpha}{\sqrt{2}p}\left(\frac{2^{\frac{3}{2}} p^3}{v_2^3}\right) = -\frac{6\alpha p^2}{v_2^3}$

- Entropy function for a magentically charged NED:

$$\mathrm{Ent}(v_1, v_2, p, u) = \pi\left[v_2 - v_1 + \frac{3v_1\alpha p^2}{v_2^2} + 8\lambda\left(u^2 - \frac{a}{4}u^4\right)\right]$$

- $$\frac{d\mathrm{Ent}(v_1,v_2,p,u)}{dv_1}=0\to\pi\left(-1+\frac{3p^2\alpha}{v_2{}^2}\right)=0\to v_2=\sqrt{3\alpha}\,p=r_e^2$$
- $$\frac{d\mathrm{Ent}(v_1,v_2,p,u)}{dv_2}=0\to\pi\left(1-\frac{6p^2v_1\alpha}{v_2{}^3}\right)=0\to v_1=\frac{\sqrt{3\alpha}\,p}{2}=\frac{v_2}{2}$$
- Reduced entropy function by eliminating v_1 :

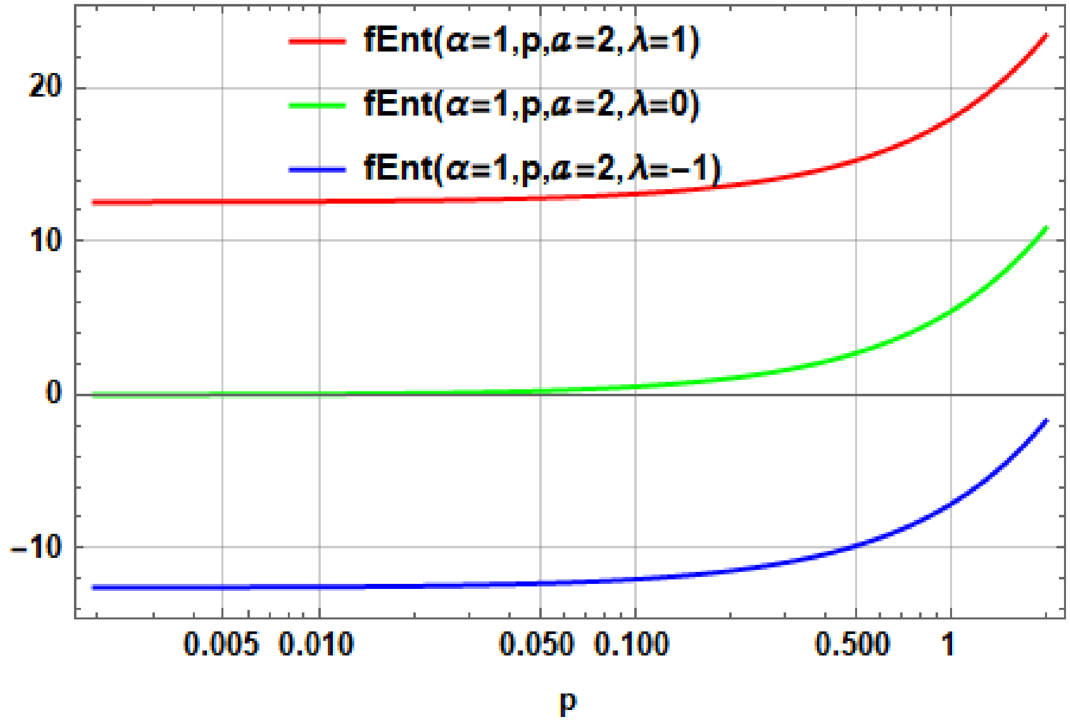
$$\to\mathrm{REnt}(v_2,u)=\pi\left[v_2+8\lambda\left(u^2-\frac{a}{4}u^4\right)\right]$$

$$\Rightarrow\text{Wald entropy: }\boxed{S_{\mathrm{Wald}}=-2\pi\oint\frac{\partial L}{\partial R_{\mu\nu\rho\sigma}}\epsilon_{\mu\nu}\epsilon_{\rho\sigma}r^2d\Omega_2^2\,|_{r=r_e}=\pi\left[r_e^2+8\lambda\left(u^2-\frac{a}{4}u^4\right)\right]}$$

- $$\frac{d\mathrm{Ent}(v_1,v_2,p,u)}{du}=0\quad\to 2u-au^3=0\quad\xrightarrow[\text{scalar hair solution}]{}u=\sqrt{\frac{2}{a}}$$
- Final Entropy function:

$$\to\boxed{\mathrm{fEnt}(\alpha,p,a,\lambda)=\pi\left(v_2+\frac{8\lambda}{a}\right)\to\pi\left(\sqrt{3\alpha}\,p+\frac{8\lambda}{a}\right)}$$

Final Entropy function



Comment #7: Entropy with $\lambda = 1$ (> Entropy with $\lambda = -1$)

→ Stable scalarized eqOS-BH

7. Discussions

- **GB+ spontaneous scalarization on SBH=OS-BH**
  1. **Tachyonic instability triggers onset of scalarization.**
  2. **Infinite branches of scalarized SBHs from infinite scalar clouds (n=0 branch: stable, other branches : unstable).**

- **GB- onset scalarization on qOS-BH**
  1. **Single branch of scalarized qOS-BHs.**
  2. **Allowed region for GB- scalarization= positive heat capacity**
     → $M_{\rm min}(=0.7698) \le M_{\rm th}(\alpha=1,\lambda<0) \le M_c(=0.8827)$
  3. **However, one cannot obtain any scalar clouds.**
  4. **We could not obtain scalarized qOS-BHs because of unknown** $\boxed{L_{qOS}}$

- **GB_e onset scalarization on eqOS-BHs**

1. Single branch of scalarized eqOS-BHs.
2. GB_e scalarization is allowed for negative λ.
3. However, one cannot obtain any scalar clouds.

- **GB_{BR} onset scalarization on eqOS-BHs→ two branches**

GB_{BR}-scalarization is allowed for any negative/positive λ

→ Tachyonic cloud is obtained with negative potential.

→ Scalar cloud is found for positive potential.

· Entropy function approach to GB_{BR} scalarization

Entropy with positive λ (>Entropy with negative λ)
→Stable scalarized eqOS-BH with primary scalar hair $u=\sqrt{2/a}$

■ Thank you for attending

Appendix: Aretakis instability for eqOS-BHs

→Classical instability of eBHs due to degeneracy of the redshift

- Ingoing Eddington-Finkelstein coordinates with $V = \tau - \frac{1}{\rho}$

$$ds^2_{\mathrm{EF}} = -\rho^2 dV^2 + 2dVd\rho + (d\theta^2 + \sin^2\theta d\varphi^2)$$

- Linearized scalar equation around AdS_2

$$\left[2\partial_V\partial_\rho\delta\phi + \partial_\rho\left(\rho^2\partial_\rho\delta\phi\right) - \mu^2\delta\phi = 0\right], \quad \mu^2 = 8\lambda$$

- Acting ∂_ρ^N (transverse derivative) on [---]

$$\rightarrow 2\partial_V\partial_\rho^{N+1}\delta\phi = [8\lambda - N(N+1)]\partial_\rho^N\delta\phi \rightarrow 0$$

- Aretakis constant on the horizon: $\boxed{H_N = \partial_\rho^{N+1}\delta\phi}$ iff $8\lambda = N(N+1) \rightarrow N = \nu - \frac{1}{2}$

Stability and Instability of Extreme Reissner-Nordström Black Hole Spacetimes for Linear Scalar Perturbations I, Stefanos Aretakis, Comm. Math. Phys. 307 (2011), 17-63

Gravitational instability of an extreme Kerr black hole, James Luciettia and Harvey S. Reall, **PRD** 86 (2012), 104030

- Two lowering and raising operators : $L_{-} = V^2\partial_V - 2(\rho V + 1)\partial_\rho$ and $L_{+} = \partial_V$

$\rightarrow$ $\boxed{[L_{+}, L_{-}] = -2L_0,\ [L_{\pm}, L_0] = \mp L_{\pm}}$ with $L_0 = V\partial_V - \rho\partial_\rho \rightarrow \mathrm{sl}(2,\Re)$ conformal algebra

- Lowest-weight condition $L_{-}\delta\phi_{N,h} = 0$ for the lowest-weight $h = N+1$

Lowest-weight state $\rightarrow$ $\delta\phi_{N,N+1}(V,\rho) \propto V^{-N-1}(V\rho + 2)^{-N-1} \xrightarrow[\rho=0]{} V^{-N-1}$ (on the horizon)

- Hightest-weight states by acting $(L_{+})^n = \partial_V^n$ on LWS: $\delta\phi_{N,N+1}(V,\rho)$

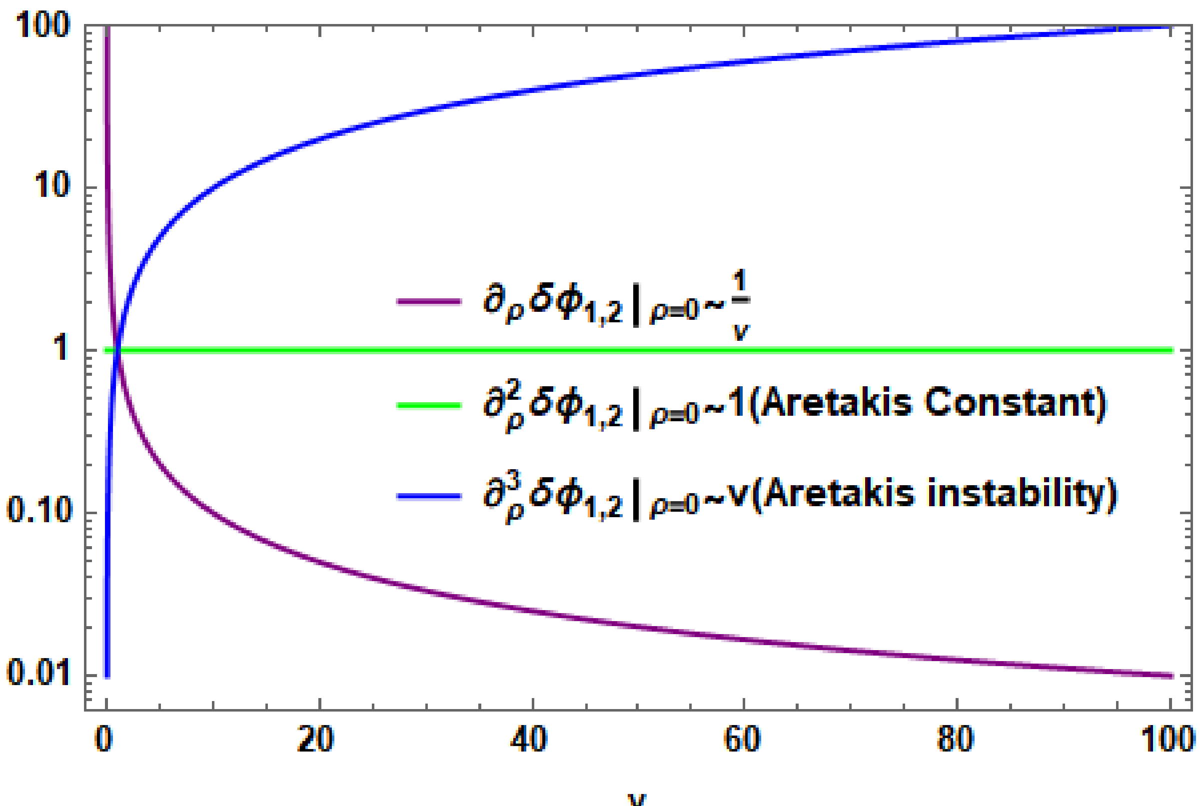
$\rightarrow$ $\delta\phi_{N,n+N+1} = (L_{+})^n\,\delta\phi_{N,N+1}|_{\rho=0} \propto V^{-n-N-1}$ (on the horizon)

- Furthermore, we have transverse derivatives

$$\rightarrow \quad \partial_\rho^k \delta\phi_{N,N+1}|_{\rho\to 0} \propto \; V^{k-N-1} \text{(on the horizon)}$$

- For $\lambda = 1/4 (N=1, \mu^2 = 2)$ with standard mass, one finds

$$\rightarrow \quad \partial_\rho \delta\phi_{1,2}|_{\rho\to 0} \propto \frac{1}{V}, \quad \partial_\rho^2 \delta\phi_{1,2}|_{\rho\to 0} \propto 1 \; (= H_1), \quad \partial_\rho^3 \delta\phi_{1,2}|_{\rho\to 0} \propto \; V$$



Tachyonic cloud with negative potential
→has nothing to do with Aretakis instability

Scalar cloud with positive potential (standard mass)
→ is related to Aretakis instability
→ Degeneracy of the redshift around AdS_2