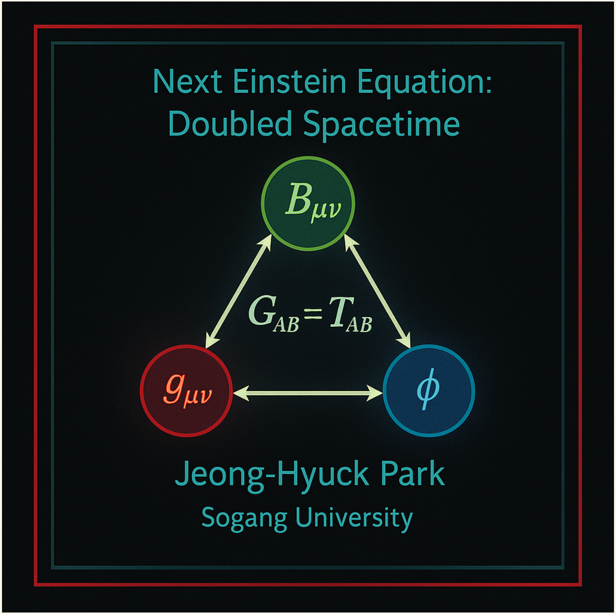

Shanghai-APCTP-Sogang-GIST Workshop, 29 December 2025

∗ Title and Figure made by ChatGPT ∗

Quiz : In electrodynamics, the electric field is denoted by E for obvious reason.

But, the magnetic field is denoted by B or H instead of M. **Why?**

Physics: the History of Unification

– Originally (1861), Maxwell wrote his equations with neighboring nine alphabets,

$$B,\ C,\ D,\ E,\ F,\ G,\ H,\ I,\ J$$

lacking vector notation.

– It was Heaviside (1864), or **SO**(3), who reformulated them into modern four equations,

$$\nabla \cdot \mathbf{E} = \rho\,, \qquad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}\,, \qquad \nabla \cdot \mathbf{B} = 0\,, \qquad \nabla \times \mathbf{B} = \mathbf{J} + \frac{\partial \mathbf{E}}{\partial t}$$

– Minkowski (1908), or **SO**(1, 3), then made further simplification,

$$\partial_\lambda F^{\lambda\mu} = J^\mu\,, \qquad \epsilon^{\kappa\lambda\mu\nu}\partial_\lambda F_{\mu\nu} = 0$$

• Nonetheless, these simplifications are all rewriting of the same 8 equations in component.

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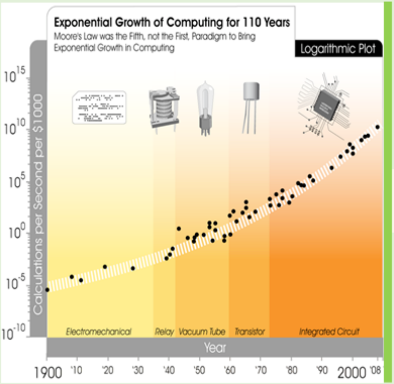
Moore's Law vs. Physics Law



B, *C*, *D*, *E*, *F*, *G*, *H*, *I*, *J*

SO(3)

$$\nabla \cdot \mathbf{E} = \rho\,, \qquad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}\,, \qquad \nabla \cdot \mathbf{B} = 0\,, \qquad \nabla \times \mathbf{B} = \mathbf{J} + \frac{\partial \mathbf{E}}{\partial t}$$

SO(1,3)

$$\partial_\lambda F^{\lambda\mu} = J^\mu\,, \qquad \partial_{[\lambda} F_{\mu\nu]} = 0$$

Not only are transistors becoming smaller,

but the laws of physics are also becoming simpler.

Physics: the History of Unification

– Similar simplification has been made for the gravitational sector in string theory.

The vanishings of the three β-functions on string worldsheet,

$$R_{\mu\nu} + 2\bigtriangledown_{\mu}(\partial_{\nu}\phi) - \tfrac{1}{4}H_{\mu\rho\sigma}H_{\nu}{}^{\rho\sigma} = 0$$

$$\tfrac{1}{2}e^{2\phi}\bigtriangledown^{\rho}\left(e^{-2\phi}H_{\rho\mu\nu}\right) = 0$$

$$R + 4\Box\phi - 4\partial_{\mu}\phi\partial^{\mu}\phi - \tfrac{1}{12}H_{\lambda\mu\nu}H^{\lambda\mu\nu} = 0$$

have been unified, thanks to $\mathbf{O}(D, D)$, into a single formula, w/ S. Rey, W. Rim, Y. Sakatani 2015

$$G_{AB} = 0\,.$$

which is the vacuum case of more general, **Einstein Double Field Equation (EDFE),**

$$G_{AB} = T_{AB}$$

where A, B are $\mathbf{O}(D, D)$ vector indices. w/ S. Angus and K. Cho 2018

In contrast to electrodynamics, this simplification turns out to be more than just rewriting.

Question : **What is the gravitational theory that string theory predicts?**

i) Conventional Answer

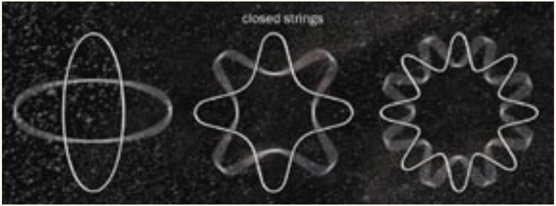
ii) Better Answer

iii) Doubled Answer

What is the gravitational theory that string theory predicts?

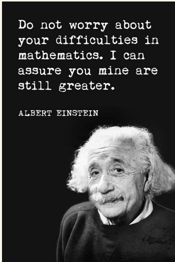
– The conventional answer is General Relativity (GR):

Riemannian metric $g_{\mu\nu}$ appears as a massless mode in the quantization of a closed string.



Different modes of a string correspond to different particles (fields).

Needless to say, ever since the formulation of GR by Einstein, Riemannian geometry has been the mathematical paradigm for theoretical physics where $g_{\mu\nu}$ is privileged to be the only fundamental variable that defines the concept of ‘spacetime’.



What is the gravitational theory that string theory predicts?

However, $g_{\mu\nu}$ is only one segment of the closed string massless sector that should further includes two additional fields, a skew-symmetric B-field and a scalar dilaton ϕ:

$$\{g_{\mu\nu}\,,\,B_{\mu\nu}\,,\phi\}\;\equiv\;\textbf{Closed String Massless Sector}$$

where $g_{\mu\nu}=g_{\nu\mu},\ B_{\mu\nu}=-B_{\nu\mu}$.

This is the universal common sector in all string theories.

What is the gravitational theory that string theory predicts?

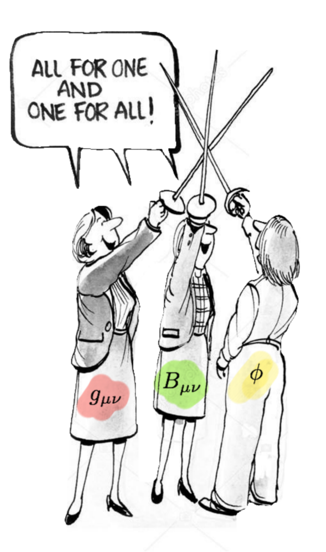
– The better answer is Supergravity:

$$S_{\text{SUGRA}} = \int \mathrm{d}^D x \sqrt{-g} e^{-2\phi} \left(R + 4\partial_\mu \phi \partial^\mu \phi - \tfrac{1}{12} H_{\lambda\mu\nu} H^{\lambda\mu\nu} \right) + \text{ other sectors}$$

where $H_{\lambda\mu\nu} = \partial_\lambda B_{\mu\nu} + \partial_\mu B_{\nu\lambda} + \partial_\nu B_{\lambda\mu}$ is the field strength of B-field, or H-flux.

This action secretly keeps $\mathbf{O}(D, D)$ symmetry which transforms the trio $\{g, B, \phi\}$ to one another, and may suggest to regard the whole sector as gravitational and also geometric.

This suggests a shift beyond the Riemannian paradigm.



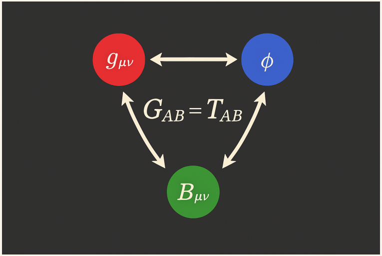
Trinity of the Closed String Massless Sector

What is the gravitational theory that string theory predicts?

This idea has come true through the developments, under the name, **Double Field Theory (DFT)**

Siegel 1993; Hull-Zwiebach 2009

(*c.f.* Generalised Geometry *à la* Hitchin-Gualtieri)



Stringy Trinity

DFT reformulated SUGRA actions in an **O**(D, D) manifest way and further evolved to have its own Einstein equation, *i.e.* Einstein Double Field Equation.

– The doubled answer is Double Field Theory.

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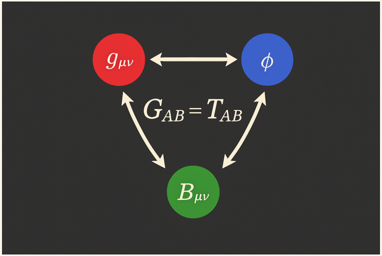


Stringy Trinity

– The original form of the DFT action: Summer School Lecture by Zwiebach, München, 2010

$$S_{\text{DFT 2010}} = \int e^{-2d} \left[\begin{array}{l} \mathcal{H}^{AB}\left(\frac{1}{8}\partial_A\mathcal{H}_{CD}\partial_B\mathcal{H}^{CD} + \frac{1}{2}\partial_C\mathcal{H}_A{}^D\partial_D\mathcal{H}_B{}^C - 4\partial_A d\partial_B d + 4\partial_A\partial_B d\right) \\ -\partial_A\partial_B\mathcal{H}^{AB} + 4\partial_A\mathcal{H}^{AB}\partial_B d \end{array} \right]$$

Holm-Hull-Zwiebach 2010

With the parametrization,

$$\mathcal{H}_{AB} = \begin{pmatrix} g^{-1} & -g^{-1}B \\ Bg^{-1} & g - Bg^{-1}B \end{pmatrix}, \qquad e^{-2d} = \sqrt{|g|}e^{-2\phi}$$

Giveon, Rabinovici, Veneziano '89, Duff '90

and letting the half of the doubled coordinates, $x^A = (\tilde{x}_\mu, x^\nu)$, trivial:

$$\partial_A = \left(\frac{\partial}{\partial\tilde{x}_\mu}, \frac{\partial}{\partial x^\nu}\right) \equiv (0, \partial_\nu)$$

it reproduces the universal part in SUGRAs:

$$S_{\text{DFT 2010}} \quad \Longrightarrow \quad \int \mathrm{d}^D x \sqrt{-g}e^{-2\phi}\left(R + 4\partial_\mu\phi\partial^\mu\phi - \tfrac{1}{12}H_{\lambda\mu\nu}H^{\lambda\mu\nu}\right)$$

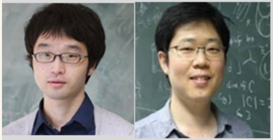
– Geometric Formulation?

Collaborators on DFT since 2010

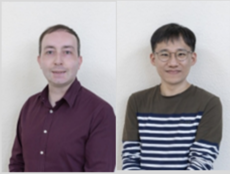
Lab7616

IHS

Differential Geometry & Suersymmetry:
Imtak Jeon, Kanghoon Lee



Einstein Equation:
Stephen Angus, Kyungho Cho



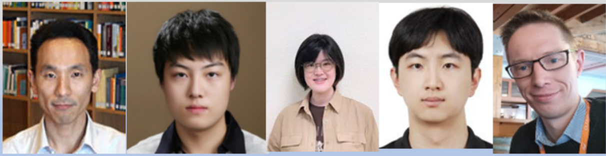
Non-Riemannian Geometry:
Kevin Morand, Miok Park, Shigeki Sugimoto

Phenomenology:
- Standard Model
- Solar System Test

Kangsin Choi

Wormhole/Fracton/Box Operator:
Hun Jang, Minkyoo Kim, Kawon Lee

Cosmology alternative to de Sitter:
Shinji Mukohyama, Hocheol Lee, Lu Yin, Minjae Cho, Nils Nilsson



33 SCI papers = { JHEP 13, PLB 5, PRD 3, NPB, JCAP, EPJC 5, PRR, PRL 4 }

Lecture Note: arXiv:2505.10163 (EPJC invited review article)

Contents Hereafter:

I. Geometric Formulation of DFT and EDFE, $G_{AB} = T_{AB}$

II. Riemannian *vs.* Non-Riemannian Geometries in DFT

III. Phenomenological Implication: Test of DFT

- Solar System Test (PPN)
- Cosmological Test (alternative to de Sitter)

Yet, due to limited time, I will skip technical details. See arXiv: 2505.10163 for review.

I. Geometric Formulation of DFT:

$$G_{AB} = T_{AB}$$

– Its Autonomous Structure –

DFT = $\mathbf{O}(D,D)$ completion of GR

– GR is characterised by

$$\mathcal{L}_{\xi}\,,\quad g_{\mu\nu}\,,\quad \nabla_{\lambda}g_{\mu\nu}=0\;\;\Rightarrow\;\;\gamma^{\lambda}_{\mu\nu}=\tfrac{1}{2}g^{\lambda\rho}(\partial_{\mu}g_{\rho\nu}+\partial_{\nu}g_{\mu\rho}-\partial_{\rho}g_{\mu\nu})\,,\quad G_{\mu\nu}=\kappa T_{\mu\nu}$$

– Dictated by $\mathbf{O}(D,D)$ Symmetry Principle, DFT has its own version of each item above.

O(D, D) Symmetry Principle

– The **O**(D, D) symmetry is characterized by an invariant metric:

$$\mathcal{J}_{AB} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

which, with its inverse, raises and lowers the **O**(D, D) indices, $A, B, \cdots, M, N, \cdots$:

$$\partial^A = \mathcal{J}^{AB}\partial_B\,, \qquad\qquad \mathcal{J}_{AB}\mathcal{J}^{BC} = \delta_A{}^C$$

– The **O**(D, D) metric $\mathcal{J}_{AB}$ splits the doubled coordinates of DFT into two parts:

$$x^A = (\tilde{x}_\mu, x^\nu)\,, \qquad \partial_A = (\tilde{\partial}^\mu, \partial_\nu)\,, \qquad \partial^A = \mathcal{J}^{AB}\partial_B = (\partial_\mu, \tilde{\partial}^\nu)\,.$$

Section Condition & Generalised Lie Derivative

– In order to halve the doubled dimensionality, it is necessary to impose **section condition**:

$$\partial_A\partial^A = \partial_\mu\tilde{\partial}^\mu + \tilde{\partial}^\mu\partial_\mu = 0\,.$$

Namely, all the functions in DFT $\{\Phi, \Psi, \Upsilon, \cdot\cdot\}$ must satisfy

$$\partial_A\partial^A\Phi = 0 \quad \& \quad \partial_A\partial^A(\Phi\Psi) = 0 \quad \Longrightarrow \quad \partial_A\Phi\partial^A\Psi = 0\,,$$

which can be solved by setting $\tilde{\partial}^\mu = 0$ up to $\mathbf{O}(D,D)$ rotations $\Rightarrow$ choice of section.

– DFT-diffeomorphisms are then given by generalised Lie derivative: Siegel 1993

$$\hat{\mathcal{L}}_\xi T_{M_1\cdots M_n} = \underbrace{\xi^N\partial_N T_{M_1\cdots M_n}}_{\text{transport}} + \underbrace{\omega_T\,\partial_N\xi^N T_{M_1\cdots M_n}}_{\text{weight}} + \sum_{i=1}^{n}\underbrace{(\partial_{M_i}\xi_N - \partial_N\xi_{M_i})}_{\mathbf{so}(D,D)\text{ rotation}} T_{M_1\cdots M_{i-1}}{}^N{}_{M_{i+1}\cdots M_n}\,,$$

whose commutators are only closed under the section condition.

With $\xi^M = (\lambda_\mu, \zeta^\nu)$, it unifies B-field gauge symmetry $\delta B = \mathrm{d}\lambda$ and ordinary Lie derivative $\mathcal{L}_\zeta$.

Doubled-yet-Gauged Coordinates: Geometric Meaning of Section Condition

– The section condition is mathematically equivalent to a certain translational invariance:

$$\Phi(x) = \Phi(x + \Delta)\,, \qquad \Delta^M = \Psi\partial^M\Upsilon\,,$$

where Δ^M is said to be 'derivative-index-valued'.

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– Physics should be invariant under such a shift of the doubled coordinates, suggesting

The doubled coordinates are gauged by derivative-index-valued shifts, satisfying $\Delta^M \partial_M = 0$,

$$x^M \sim x^M + \Delta^M(x) \quad : \quad \text{Coordinate Gauge Symmetry}$$

Each equivalence class, or gauge orbit in $\mathbb{R}^{D+D}$, corresponds to a single physical point.

JHP 2013

Proper length can be defined by gauged differential one-forms: $\mathrm{D}x^M = \mathrm{d}x^M - \mathfrak{a}^M$ where $\mathfrak{a}^M \partial_M = 0$.

Fundamental Fields $\{\, \mathcal{H}_{MN}\,,\, d\,\}$ and Projectors

– DFT has its own dynamical metric $\mathcal{H}_{MN}$ ("generalised metric") satisfying two defining properties,

$$\mathcal{H}_{MN} = \mathcal{H}_{NM}\,, \qquad\qquad \mathcal{H}_M{}^K \mathcal{H}_N{}^L \mathcal{J}_{KL} = \mathcal{J}_{MN}$$

Combined with $\mathcal{J}_{MN} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, it generates a pair of projectors (orthogonal and complete),

$$P_{MN} = \tfrac{1}{2}(\mathcal{J}_{MN} + \mathcal{H}_{MN})\,, \qquad \bar{P}_{MN} = \tfrac{1}{2}(\mathcal{J}_{MN} - \mathcal{H}_{MN})\,; \qquad \begin{array}{ll} P_L{}^M P_M{}^N = P_L{}^N\,, & \bar{P}_L{}^M \bar{P}_M{}^N = \bar{P}_L{}^N \\ P_L{}^M \bar{P}_M{}^N = 0\,, & P_M{}^N + \bar{P}_M{}^N = \delta_M{}^N \end{array}$$

– The **O**(D, D) singlet dilaton d sets the DFT-integral measure e^{-2d} (unit diffeomorphic weight).

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Vielbeins for Twofold Spin Group: Spin$(1, D-1) \times$ Spin$(D-1, 1)$

– Taking the ‘square root’ of each projector,

$$P_{MN} = V_M{}^p V_N{}^q \eta_{pq} \,, \qquad \bar{P}_{MN} = \bar{V}_M{}^{\bar{p}} \bar{V}_N{}^{\bar{q}} \bar{\eta}_{\bar{p}\bar{q}} \,,$$

we obtain a pair of DFT-vielbeins for the twofold spin groups:

$$V_{Mp} V^M{}_q = \eta_{pq} \,, \qquad \bar{V}_{M\bar{p}} \bar{V}^M{}_{\bar{q}} = \bar{\eta}_{\bar{p}\bar{q}} \,, \qquad V_{Mp} \bar{V}^M{}_{\bar{q}} = 0 \,.$$

Namely, $\mathcal{J}_{MN}$ and $\mathcal{H}_{MN}$ are simultaneously diagonalisable as **diag**$(\eta, \bar{\eta})$ and **diag**$(\eta, -\bar{\eta})$.

Index	Representation	Metric (raising/lowering indices)
$p, q, \cdots$	**Spin**$(1, D-1)$ vector	$\eta_{pq} = \mathrm{diag}(- + + \cdots +)$
$\alpha, \beta, \cdots$	**Spin**$(1, D-1)$ spinor	$C_{\alpha\beta}, \qquad (\gamma^p)^T = C\gamma^p C^{-1}$
$\bar{p}, \bar{q}, \cdots$	**Spin**$(D-1, 1)$ vector	$\bar{\eta}_{\bar{p}\bar{q}} = \mathrm{diag}(+ - - \cdots -)$
$\bar{\alpha}, \bar{\beta}, \cdots$	**Spin**$(D-1, 1)$ spinor	$\bar{C}_{\bar{\alpha}\bar{\beta}}, \qquad (\bar{\gamma}^{\bar{p}})^T = \bar{C}\bar{\gamma}^{\bar{p}}\bar{C}^{-1}$

* **Twofold Local Lorentz Symmetries, Spin$(1, D{-}1) \times$ Spin$(D{-}1, 1)$, implies**

  - $\exists$ Two separate local inertial frames for the left- and right-moving closed-string modes.

    Duff 1986

  - Diagonal gauging fixing reduces to a single spin group in conventional gravity (SUGRA).

  - It predicts that there are two distinct types of spinors, *i.e.* fermions.

  - It implies the unification of type IIA and IIB superstrings.

Christoffel & Spin Connections

w/ Imtak Jeon & Kanghoon Lee 2010, 2011

- In GR, the Christoffel symbol is the unique metric-compatible connection, $\nabla_\lambda g_{\mu\nu} = 0$, which satisfies either a torsionless condition, or an alternative condition that the metric is the only ingredient to form the connection.

- Similarly, the DFT-Christoffel connection can be uniquely fixed,

$$\Gamma_{LMN} = 2\left(P\partial_L P\bar{P}\right)_{[MN]} + 2\left(\bar{P}_{[M}{}^J\bar{P}_{N]}{}^K - P_{[M}{}^J P_{N]}{}^K\right)\partial_J P_{KL} - \tfrac{4}{D-1}\left(\bar{P}_{L[M}\bar{P}_{N]}{}^K + P_{L[M}P_{N]}{}^K\right)\left(\partial_K d + (P\partial^J P\bar{P})_{[JK]}\right)$$

satisfying, among others, the compatibility:

$$\nabla_L \mathcal{J}_{MN} = 0\,, \qquad \nabla_L \mathcal{H}_{MN} = 0\,, \qquad \nabla_L d = -\tfrac{1}{2}e^{2d}\nabla_L\left(e^{-2d}\right) = 0$$

where $\nabla_L = \partial_L + \Gamma_L$ is defined by

$$\nabla_L T_{M_1\cdots M_n} := \partial_L T_{M_1\cdots M_n} - \omega_T \Gamma^K{}_{KL} T_{M_1\cdots M_n} + \sum_{i=1}^{n} \Gamma_{LM_i}{}^N T_{M_1\cdots M_{i-1}NM_{i+1}\cdots M_n}\,.$$

- One can further obtain the twofold spin connections,

$$\Phi_{Mpq} = V^N{}_p \nabla_M V_{Nq}\,, \qquad \bar{\Phi}_{M\bar{p}\bar{q}} = \bar{V}^N{}_{\bar{p}}\nabla_M \bar{V}_{N\bar{q}}$$

from the requirement that the 'master' covariant derivative

$$\mathcal{D}_M = \partial_M + \Gamma_M + \Phi_M + \bar{\Phi}_M = \nabla_M + \Phi_M + \bar{\Phi}_M$$

should be compatible with the DFT-vielbeins,

$$\mathcal{D}_M V_{Np} = \nabla_M V_{Np} + \Phi_{Mp}{}^q V_{Nq} = 0\,, \qquad \mathcal{D}_M \bar{V}_{N\bar{p}} = \nabla_M \bar{V}_{N\bar{p}} + \bar{\Phi}_{M\bar{p}}{}^{\bar{q}}\bar{V}_{N\bar{q}} = 0\,.$$

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$$\mathcal{D}_M = \partial_M + \Gamma_M + \Phi_M + \bar{\Phi}_M = \nabla_M + \Phi_M + \bar{\Phi}_M$$

should be compatible with the DFT-vielbeins,

$$\mathcal{D}_M V_{Np} = \nabla_M V_{Np} + \Phi_{Mp}{}^{q} V_{Nq} = 0\,, \qquad \mathcal{D}_M \bar{V}_{N\bar{p}} = \nabla_M \bar{V}_{N\bar{p}} + \bar{\Phi}_{M\bar{p}}{}^{\bar{q}} \bar{V}_{N\bar{q}} = 0\,.$$

- In GR, the Christoffel symbol is the unique metric-compatible connection, $\nabla_\lambda g_{\mu\nu}=0$, which satisfies either a torsionless condition, or an alternative condition that the metric is the only ingredient to form the connection.

- Similarly, the DFT-Christoffel connection can be uniquely fixed,

$$\Gamma_{LMN}=2\left(P\partial_L P\bar{P}\right)_{[MN]}+2\Big(\bar{P}_{[M}{}^J\bar{P}_{N]}{}^K-P_{[M}{}^J P_{N]}{}^K\Big)\partial_J P_{KL}-\tfrac{4}{D-1}\Big(\bar{P}_{L[M}\bar{P}_{N]}{}^K+P_{L[M}P_{N]}{}^K\Big)\Big(\partial_K d+(P\partial^J P\bar{P})_{[JK]}\Big)$$

satisfying, among others, the compatibility:

$$\nabla_L\mathcal{J}_{MN}=0\,,\qquad \nabla_L\mathcal{H}_{MN}=0\,,\qquad \nabla_L d=-\tfrac{1}{2}e^{2d}\nabla_L\left(e^{-2d}\right)=0$$

where $\nabla_L=\partial_L+\Gamma_L$ is defined by

$$\nabla_L T_{M_1\cdots M_n}:=\partial_L T_{M_1\cdots M_n}\;-\;\omega_T\,\Gamma^K{}_{KL}T_{M_1\cdots M_n}\;+\;\sum_{i=1}^{n}\Gamma_{LM_i}{}^N T_{M_1\cdots M_{i-1}NM_{i+1}\cdots M_n}\,.$$

- One can further obtain the twofold spin connections,

$$\Phi_{Mpq}=V^N{}_p\nabla_M V_{Nq}\,,\qquad \bar{\Phi}_{M\bar{p}\bar{q}}=\bar{V}^N{}_{\bar{p}}\nabla_M\bar{V}_{N\bar{q}}$$

from the requirement that the 'master' covariant derivative

$$\mathcal{D}_M=\partial_M+\Gamma_M+\Phi_M+\bar{\Phi}_M=\nabla_M+\Phi_M+\bar{\Phi}_M$$

should be compatible with the DFT-vielbeins,

$$\mathcal{D}_M V_{Np}=\nabla_M V_{Np}+\Phi_{Mp}{}^q V_{Nq}=0\,,\qquad \mathcal{D}_M\bar{V}_{N\bar{p}}=\nabla_M\bar{V}_{N\bar{p}}+\bar{\Phi}_{M\bar{p}}{}^{\bar{q}}\bar{V}_{N\bar{q}}=0\,.$$

Curvature & (Semi-)covariance

w/ Imtak Jeon & Kanghoon Lee 2010, 2011

– Semi-covariant Riemann curvature :

$$S_{KLMN}=S_{[KL][MN]}=S_{MNKL}:=\tfrac{1}{2}\left(\mathfrak{R}_{KLMN}+\mathfrak{R}_{MNKL}-\Gamma^{J}{}_{KL}\Gamma_{JMN}\right)\,,\qquad S_{[KLM]N}=0\,,$$

where $\mathfrak{R}_{ABCD}$ denotes the ordinary "field strength",

$$\mathfrak{R}_{CDAB}=\partial_A\Gamma_{BCD}-\partial_B\Gamma_{ACD}+\Gamma_{AC}{}^{E}\Gamma_{BED}-\Gamma_{BC}{}^{E}\Gamma_{AED}\,.$$

By construction, like in GR, it varies as <u>total derivative</u>:

$$\delta S_{ABCD}=\nabla_{[A}\delta\Gamma_{B]CD}+\nabla_{[C}\delta\Gamma_{D]AB}\qquad\Longrightarrow$$

hence good for <u>variational principle</u>

– The 'semi-covariance', means

$$\delta_\xi\left(\nabla_L T_{M_1\cdots M_n}\right)=\hat{\mathcal{L}}_\xi\left(\nabla_L T_{M_1\cdots M_n}\right)+\sum_{i=1}^{n}\,2(\mathcal{P}+\bar{\mathcal{P}})_{LM_i}{}^{NEFG}\partial_E\partial_F\xi_G\,T_{M_1\cdots M_{i-1}NM_{i+1}\cdots M_n}$$

$$\delta_\xi S_{KLMN}=\hat{\mathcal{L}}_\xi S_{KLMN}+2\nabla_{[K}\left[(\mathcal{P}+\bar{\mathcal{P}})_{L][MN]}{}^{EFG}\partial_E\partial_F\xi_G\right]+2\nabla_{[M}\left[(\mathcal{P}+\bar{\mathcal{P}})_{N][KL]}{}^{EFG}\partial_E\partial_F\xi_G\right]$$

$$\delta_\xi\Gamma_{CAB}=\hat{\mathcal{L}}_\xi\Gamma_{CAB}+2\left[(\mathcal{P}+\bar{\mathcal{P}})_{CAB}{}^{FDE}-\delta_C^F\delta_A^D\delta_B^E\right]\partial_F\partial_{[D}\xi_{E]}$$

where $\mathcal{P}_{LMN}{}^{EFG}=P_L{}^{E}P_{[M}{}^{[F}P_{N]}{}^{G]}+\frac{2}{P_K{}^{K}-1}P_{L[M}P_{N]}{}^{[F}P^{G]E}$ and similarly $\bar{\mathcal{P}}_{LMN}{}^{EFG}$ is set with $\bar{P}_M{}^{N}$.

► The red-colored anomalies can be easily projected out to give fully covariant quantities, e.g.

$$P_K{}^{L}\bar{P}_{N_1}{}^{M_1}\cdots\bar{P}_{N_n}{}^{M_n}\nabla_L T_{M_1\cdots M_n}$$

– Semi-covariant Riemann curvature :

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Fully-Covariant Quantities

w/ Imtak Jeon & Kanghoon Lee 2010, 2011

– Tensors:

$$\mathcal{D}_p T_{\bar{q}_1\bar{q}_2\cdots\bar{q}_n}\,,\qquad \mathcal{D}_{\bar{p}} T_{q_1 q_2\cdots q_n}\,;\qquad \mathcal{D}^p T_{p\bar{q}_1\bar{q}_2\cdots\bar{q}_n}\,,\qquad \mathcal{D}^{\bar{p}} T_{\bar{p}q_1 q_2\cdots q_n}\qquad \text{(divergence)}$$

– Yang–Mills:

$$\mathcal{F}_{p\bar{q}} = \mathcal{F}_{AB} V^A{}_p \bar{V}^B{}_{\bar{q}}\qquad \text{where}\qquad \mathcal{F}_{AB} = \nabla_A W_B - \nabla_B W_A - i\,[W_A, W_B]$$

– Spinors, ρ^α, $\psi^\alpha_{\bar{p}}$:

$$\gamma^p \mathcal{D}_p \rho\,,\qquad \mathcal{D}_{\bar{p}}\rho\,,\qquad \gamma^p \mathcal{D}_p \psi_{\bar{q}}\,,\qquad \mathcal{D}_{\bar{p}}\psi^{\bar{p}}\qquad \text{(Dirac Operators)}$$

– Ramond–Ramond Sector, $\mathcal{C}^\alpha{}_{\bar{\alpha}}$:

$$\mathcal{D}_\pm \mathcal{C} := \gamma^p \mathcal{D}_p \mathcal{C} \pm \gamma^{(D+1)} \mathcal{D}_{\bar{p}} \mathcal{C} \bar{\gamma}^{\bar{p}}\,,\quad (\mathcal{D}_\pm)^2 = 0 \quad\Longrightarrow\quad \mathcal{F} := \mathcal{D}_+ \mathcal{C}\qquad (\text{RR flux})$$

– Curvatures:

$$S_{p\bar{q}} := S_{AB} V^A{}_p \bar{V}^B{}_{\bar{q}}\quad (\text{Ricci})\,,\qquad S_{(0)} := (P^{AC}P^{BD} - \bar{P}^{AC}\bar{P}^{BD}) S_{ABCD}\quad (\text{scalar} \Rightarrow \text{‘pure’ DFT})$$

▶ The original DFT action by Hohm-Hull-Zwiebach 2010 matches $\int_{\Sigma_D} e^{-2d} S_{(0)}$.

– Fully-covariant Second-Order Differential Operator:

$$\begin{aligned}\boldsymbol{\Delta} T_{A_1A_2\cdots A_s} :=\; & P^{BC}\nabla_B\nabla_C T_{A_1A_2\cdots A_s}\\ & +\sum_{i=1}^{s} 2P_{A_i}{}^{C}P_B{}^{D}\Big(\mathfrak{R}_{[CD]}-\tfrac{1}{2}\Gamma^{EF}{}_{C}\Gamma_{EFD}-\Gamma^{E}{}_{CD}\nabla_E\Big)T_{A_1\cdots A_{i-1}}{}^{B}{}_{A_{i+1}\cdots A_s}\\ +\sum_{i<j} 2\Big(& P_{A_i}{}^{D}P_B{}^{E}\mathfrak{R}_{A_jCDE}+P_{A_j}{}^{D}P_C{}^{E}\mathfrak{R}_{A_iBDE}-2P_{A_i}{}^{D}P_B{}^{E}P_{A_j}{}^{F}P_C{}^{G}S_{DEFG}\Big)T_{A_1\cdots A_{i-1}}{}^{B}{}_{A_{i+1}\cdots A_{j-1}}{}^{C}{}_{A_{j+1}\cdots A_s}\end{aligned}$$

and similarly, replacing P by $\bar{P}$, $\bar{\boldsymbol{\Delta}} := \bar{P}^{BC}\nabla_B\nabla_C T_{A_1A_2\cdots A_s}+\sum_i\cdots+\sum_{i<j}\cdots$.

– While their sum vanishes identically, their difference defines a fully-covariant Box Operator:

$$\boldsymbol{\Delta}+\bar{\boldsymbol{\Delta}}=\mathcal{J}^{AB}\partial_A\partial_B+\cdots=0\,,\qquad \Box=\boldsymbol{\Delta}-\bar{\boldsymbol{\Delta}}=\mathcal{H}^{AB}\partial_A\partial_B+\cdots+\text{Riemann Curvature}\cdots$$

The Riemmann curvature is **O**(D, D)-completed as a differential operator.

– $\Box$ provides the universal kinetic term for every string mode (mass-shell condition, $p^2+m^2=0$):

- Stringy gravitational wave equations for the massless sector: $\Box(P\delta\mathcal{H}\bar{P})_{AB}=0$ and $\Box\delta d=0$.
- Integrating out massive modes, one obtains Wilsonian α'-corrections with Riemann curvature.

– Fully-covariant Second-Order Differential Operator:

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– While their sum vanishes identically, their difference defines a fully-covariant Box Operator:

$$\mathbf{\Delta}+\bar{\mathbf{\Delta}} = \mathcal{J}^{AB}\partial_A\partial_B + \cdots = 0\,, \qquad \square = \mathbf{\Delta}-\bar{\mathbf{\Delta}} = \mathcal{H}^{AB}\partial_A\partial_B + \cdots + \text{Riemann Curvature}\cdots$$

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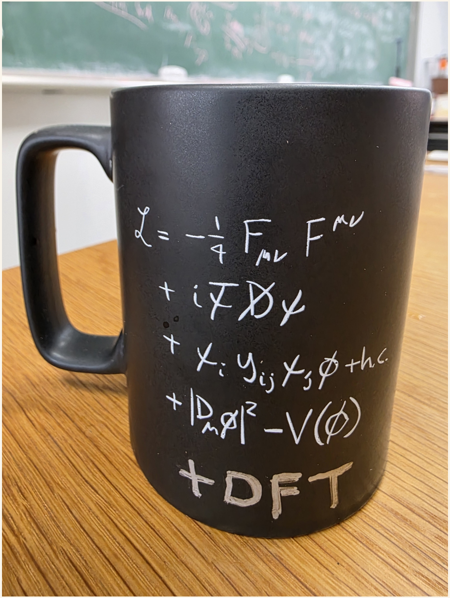
$\mathbf{O}(D, D)$-Symmetric Minimal Coupling

– The pure DFT action is given by

$$S_{\text{DFT}} = \int_{\Sigma_D} e^{-2d} S_{(0)}$$

and can further minimally couple to ‘matter’ governed by $\mathbf{O}(D, D)$ Symmetry Principle through covariant derivatives and $\{\mathcal{H}_{AB}, d\}$.

Consequently, the coupling of $\{g_{\mu\nu}, B_{\mu\nu}, \phi\}$ to each matter is completely fixed.



$\mathbf{O}(D,D)$-Symmetric Actions for Particle and String

Analogous to GR, DFT couples minimally to 'matter', while preserving $\mathbf{O}(D,D)$ symmetry:

i) To point particle with $\mathrm{D}_\tau x^A = \dot{x}^A - \mathfrak{a}^A$

$$\int \mathrm{d}\tau\ \tfrac{1}{2}e^{-1}\mathrm{D}_\tau x^A \mathrm{D}_\tau x^B \mathcal{H}_{AB}(x) - \tfrac{1}{2}m^2 e$$

$$\Longrightarrow \quad \int \mathrm{d}\tau\ \tfrac{1}{2}e^{-1}\dot{x}^\mu \dot{x}^\nu g_{\mu\nu}(x) - \tfrac{1}{2}m^2 e$$

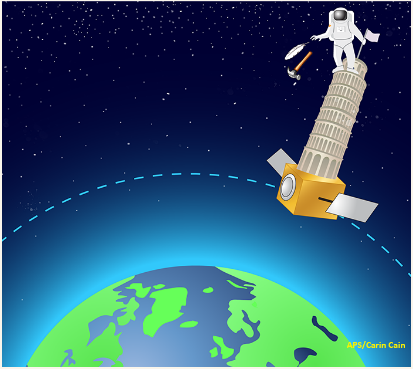
Hence, minimal coupling to string frame metric only.

ii) To string with $\mathrm{D}_\alpha x^A = \partial_\alpha x^A - \mathfrak{a}^A_\alpha$

$$\frac{1}{4\pi\alpha'}\int \mathrm{d}^2\sigma\ -\tfrac{1}{2}\sqrt{-h}h^{\alpha\beta}\mathrm{D}_\alpha x^A \mathrm{D}_\beta x^B \mathcal{H}_{AB}(x) - \epsilon^{\alpha\beta}\mathrm{D}_\alpha x^A \mathfrak{a}_{\beta A}$$

$$\Longrightarrow \quad \frac{1}{2\pi\alpha'}\int \mathrm{d}^2\sigma \left[\begin{array}{c} -\tfrac{1}{2}\sqrt{-h}h^{\alpha\beta}\partial_\alpha x^\mu \partial_\beta x^\nu g_{\mu\nu}(x) \\ +\tfrac{1}{2}\epsilon^{\alpha\beta}\partial_\alpha x^\mu \partial_\beta x^\nu B_{\mu\nu}(x) \end{array}\right]$$

which extends to κ-symmetric superstring. JHP 2016



Equivalence Principle holds for Particle

$$\ddot{x}^\mu + \gamma^\mu_{\rho\sigma}\dot{x}^\rho \dot{x}^\sigma = 0$$

not in Einstein but in String Frame.

Supersymmetric DFT and Coupling to the Standard Model

iii) $D=10$, Type II SDFT (Full Order 32 SUSY, Pseudo-Action) w/ I. Jeon, K. Lee & Y. Suh 2012

$$\mathcal{L}_{\text{type II}} = e^{-2d}\Big[\tfrac{1}{8}S_{(0)} + \tfrac{1}{2}\mathrm{Tr}(\mathcal{F}\bar{\mathcal{F}}) + i\bar{\rho}\mathcal{F}\rho' + i\bar{\psi}_{\bar{p}}\gamma_{q}\mathcal{F}\bar{\gamma}^{\bar{p}}\psi'^{q} + i\tfrac{1}{2}\bar{\rho}\gamma^{p}\mathcal{D}_{p}\rho - i\tfrac{1}{2}\bar{\rho}'\bar{\gamma}^{\bar{p}}\mathcal{D}_{\bar{p}}\rho'$$
$$-i\bar{\psi}^{\bar{p}}\mathcal{D}_{\bar{p}}\rho - i\tfrac{1}{2}\bar{\psi}^{\bar{p}}\gamma^{q}\mathcal{D}_{q}\psi_{\bar{p}} + i\bar{\psi}'^{p}\mathcal{D}_{p}\rho' + i\tfrac{1}{2}\bar{\psi}'^{p}\bar{\gamma}^{\bar{q}}\mathcal{D}_{\bar{q}}\psi'_{p}\Big]$$

which unifies IIA and IIB SUGRAs as different solution sectors.

The full order SUSY, *i.e.* quartic order in fermions, has been recently verified by D. Butter 2022.

iv) $D=4$ DFT minimally coupled to the Standard Model w/ K. Choi 2015 PRL

$$\mathcal{L}_{\text{DFT–SM}} = e^{-2d}\Big[\tfrac{1}{16\pi G_N}S_{(0)} + \textstyle\sum_{A}\mathrm{Tr}(F_{p\bar{q}}F^{p\bar{q}}) - \mathcal{H}^{MN}(\mathcal{D}_M\phi)^{\dagger}\mathcal{D}_N\phi - V(\phi)\Big]$$
$$+\textstyle\sum_{\psi}\bar{\psi}\gamma^{p}\mathcal{D}_{p}\psi + \sum_{\psi'}\bar{\psi}'\bar{\gamma}^{\bar{p}}\mathcal{D}_{\bar{p}}\psi' + y_d\,\bar{q}\cdot\phi\,d + y_u\,\bar{q}\cdot\tilde{\phi}\,u + y_e\,\bar{l}'\cdot\phi\,e'$$

Conjecture: quarks and leptons are distinct kinds of spinors, one for **Spin**$(1,3)$ and the other for **Spin**$(3,1)$.

- Every single term in the above Lagrangians is fully-covariant, w.r.t. global $\mathbf{O}(D,D)$ rotations, DFT-diffeomorphisms, and twofold local Lorentz symmetries.

Einstein Equation from Variational Principle

w/ S. Angus and K. Cho 2018

– Now we consider a general DFT action coupled to generic matter, say Υ's,

$$\text{Action} = \int_\Sigma e^{-2d}\Big[\, \tfrac{1}{2\kappa} S_{(0)} + L_{\text{matter}}\big(\Upsilon, \mathcal{D}_M \Upsilon\big)\Big]\,.$$

The variational principle leads us to define for the matter part,

$$K_{p\bar{q}} := \tfrac{1}{2}\left(V_{Mp}\frac{\delta L_{\text{matter}}}{\delta \bar{V}_M{}^{\bar{q}}} - \bar{V}_{M\bar{q}}\frac{\delta L_{\text{matter}}}{\delta V_M{}^p}\right) = -2V_{Mp}\bar{V}_{N\bar{q}}\frac{\delta L_{\text{matter}}}{\delta \mathcal{H}_{MN}}\,, \qquad T_{(0)} := e^{2d} \times \frac{\delta\left(e^{-2d}L_{\text{matter}}\right)}{\delta d}$$

– The 'General Covariance' of the action,

$$0 = \int_\Sigma e^{-2d}\left[\tfrac{1}{\kappa}\xi^N \mathcal{D}^M\Big\{4V_{[M}{}^p\bar{V}_{N]}{}^{\bar{q}}(S_{p\bar{q}} - \kappa K_{p\bar{q}}) - \tfrac{1}{2}\mathcal{J}_{MN}(S_{(0)} - \kappa T_{(0)})\Big\} + \hat{\mathcal{L}}_\xi \Upsilon\,\frac{\delta L_{\text{matter}}}{\delta \Upsilon}\right]$$

then guides us to identify the Einstein curvature, w/ S. Rey, W. Rim, Y. Sakatani 2015

$$G_{MN} := 4V_{[M}{}^p\bar{V}_{N]}{}^{\bar{q}}S_{p\bar{q}} - \tfrac{1}{2}\mathcal{J}_{MN}S_{(0)}\,, \qquad \nabla_M G^{MN} = 0 \qquad \text{(off-shell)}$$

and the Energy-Momentum tensor,

$$T_{MN} := 4V_{[M}{}^p\bar{V}_{N]}{}^{\bar{q}}K_{p\bar{q}} - \tfrac{1}{2}\mathcal{J}_{MN}T_{(0)}\,, \qquad \nabla_M T^{MN} = 0 \qquad \text{(on-shell)}$$

► **Equating them, we finally obtain the Einstein equation of DFT, or EDFEs:** $G_{MN} = \kappa T_{MN}$

II. Non-Riemannian Geometry

Question: Is DFT a mere O(D,D)-symmetric reformulation of SUGRA?

– The answer would be (and had been in literature) yes, if we assume

$$\mathcal{H}_{MN}=\begin{pmatrix} g^{-1} & -g^{-1}B \\ Bg^{-1} & g-Bg^{-1}B \end{pmatrix}, \qquad e^{-2d}=\sqrt{|g|}e^{-2\phi}$$

Giveon, Rabinovici, Veneziano '89, Duff '90

Upon this parametrisation, the pure DFT action produces SUGRA (bosonic part):

$$\int \mathrm{d}^D x\ e^{-2d} S_{(0)} = \int \mathrm{d}^D x\ \sqrt{-g}e^{-2\phi}\left(R+4\partial_\mu\phi\partial^\mu\phi-\tfrac{1}{12}H_{\lambda\mu\nu}H^{\lambda\mu\nu}\right)$$

and the EDFE, $G_{MN}=T_{MN}$, reduces to

$$\begin{array}{rcll} R_{\mu\nu}+2\bigtriangledown_\mu(\partial_\nu\phi)-\tfrac{1}{4}H_{\mu\rho\sigma}H_\nu{}^{\rho\sigma} &=& K_{(\mu\nu)} & \Longleftarrow\ \delta g_{\mu\nu} \\ \tfrac{1}{2}e^{2\phi}\bigtriangledown^\rho\left(e^{-2\phi}H_{\rho\mu\nu}\right) &=& K_{[\mu\nu]} & \Longleftarrow\ \delta B_{\mu\nu} \\ R+4\Box\phi-4\partial_\mu\phi\partial^\mu\phi-\tfrac{1}{12}H_{\lambda\mu\nu}H^{\lambda\mu\nu} &=& T_{(0)} & \Longleftarrow\ \delta d \end{array}$$

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Question: Is DFT a mere O(D, D)-symmetric reformulation of SUGRA?

– Yet, DFT works perfectly fine, with any DFT-metric that satisfies the defining properties:

$$\mathcal{H}_{MN} = \mathcal{H}_{NM}\,, \qquad \mathcal{H}_{M}{}^{K}\mathcal{H}_{N}{}^{L}\mathcal{J}_{KL} = \mathcal{J}_{MN}\,.$$

And the previous parametrisation is not the most general solution to them.

Hence the answer to the question is **No.**

– In fact, the most or perfectly symmetric vacua of DFT are given by

$$\mathcal{H}_{MN} = \pm\mathcal{J}_{MN} = \begin{pmatrix} \mathbf{0} & \pm\mathbf{1} \\ \pm\mathbf{1} & \mathbf{0} \end{pmatrix}$$

which do not admit any Riemannian interpretation, *c.f.* $\mathcal{H}_{MN} = \begin{pmatrix} g^{-1} & -g^{-1}B \\ Bg^{-1} & g - Bg^{-1}B \end{pmatrix}$

and thus, non-Riemannian, *i.e.* $\nexists\, g_{\mu\nu}$.

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Riemannian Spacetime Emergence

- Analysing DFT Killing equation, w/ C. Blair and G. Oling 2020

$$\hat{\mathcal{L}}_\xi \mathcal{H}_{MN} = \xi^L \partial_L \mathcal{H}_{MN} + 2\partial_{[M}\xi_{L]}\mathcal{H}^L{}_N + 2\partial_{[N}\xi_{L]}\mathcal{H}_M{}^L = 8\bar{P}_{(M}{}^{[K}P_{N)}{}^{L]}\nabla_K \xi_L = 0$$

  one can address the notion of isometries in DFT.

- Especially when $\mathcal{H}_{MN} = \pm\mathcal{J}_{MN}$, any ξ^L sets $\hat{\mathcal{L}}_\xi \mathcal{H}_{MN} = 0$, implying ∞-dimensional isometries.

  Further, it can be shown that the non-Riemannian geometry of $\mathcal{H}_{MN} = \pm\mathcal{J}_{MN}$ prohibits any infinitesimal variation:

$$\mathcal{H}^2 = 1 \quad \Longrightarrow \quad \delta\mathcal{H}\mathcal{H} + \mathcal{H}\delta\mathcal{H} = 0$$

  "*Riemannian spacetime emerges after SSB of* $\mathbf{O}(D,D)$*, identifying* $\{g, B\}$ *as Nambu–Goldstone boson moduli.*" *Berman, Blair and Otsuki 2019*

Non-Riemannian Geometry in a Nutshell

- DFT provides a universal framework for (Riemannian) SUGRA as well as non-Riemannian Gravities: *e.g.* non-relativistic Newton–Cartan, ultra-relativistic Carroll, and fracton physics.

- Strings and particles become chiral and immobile in non-Riemannian (sub)space.

- DFT enlarges the concept of spacetime geometries, redefines the notion of spacetime singularity, and provides novel string vacua.

  - **First example** w/ Kanghoon Lee 2013
  - **Non-Relativistic String** w/ Sung Moon Ko, Charles Melby-Thompson, Rene Meyer 2015
  - **Classification** w/ Kevin Morand 2017
  - **Moduli-free Kaluza–Klein reduction** w/ Kyoungho Cho and Kevin Morand 2018
  - **-Dynamics through EDFE** w/ Kyoungho Cho 2019
  - **Quantum Consistency on Worldsheet** w/ Shigeki Sugimoto 2020 PRL
  - **∞-dimensional Isometries** w/ Chris Blair and Gerben Oling 2020
  - **Some Riemannian Singularities $=$ Non-Riemannian Regularity** w/ Kevin Morand and Miok Park 2021 PRL
  - **Fracton Physics** w/ Stephen Angus and Minkyoo Kim 2021

Classification of DFT geometries

w/ Kevin Morand 2017

The most general parametrisations of the DFT-metric, $\mathcal{H}_{MN}=\mathcal{H}_{NM}$, $\mathcal{H}_M{}^K\mathcal{H}_N{}^L\mathcal{J}_{KL}=\mathcal{J}_{MN}$, can be classified by two non-negative integers, $(n,\bar{n})$, $0\leq n+\bar{n}\leq D$:

$$\mathcal{H}_{MN}=\begin{pmatrix} H^{\mu\nu} & -H^{\mu\sigma}B_{\sigma\lambda}+Y_i^\mu X_\lambda^i-\bar{Y}_{\bar{\imath}}^\mu\bar{X}_\lambda^{\bar{\imath}} \\ B_{\kappa\rho}H^{\rho\nu}+X_\kappa^iY_i^\nu-\bar{X}_\kappa^{\bar{\imath}}\bar{Y}_{\bar{\imath}}^\nu & K_{\kappa\lambda}-B_{\kappa\rho}H^{\rho\sigma}B_{\sigma\lambda}+2X^i_{(\kappa}B_{\lambda)\rho}Y_i^\rho-2\bar{X}^{\bar{\imath}}_{(\kappa}B_{\lambda)\rho}\bar{Y}_{\bar{\imath}}^\rho \end{pmatrix}$$

$$=\begin{pmatrix} 1 & 0 \\ B & 1 \end{pmatrix}\begin{pmatrix} H & Y_i(X^i)^T-\bar{Y}_{\bar{\imath}}(\bar{X}^{\bar{\imath}})^T \\ X^i(Y_i)^T-\bar{X}^{\bar{\imath}}(\bar{Y}_{\bar{\imath}})^T & K \end{pmatrix}\begin{pmatrix} 1 & -B \\ 0 & 1 \end{pmatrix}$$

where

$$H^{\mu\nu}=H^{\nu\mu}\,,\qquad K_{\mu\nu}=K_{\nu\mu}\,,\qquad B_{\mu\nu}=-B_{\nu\mu}$$

$$H^{\mu\nu}X_\nu^i=0=H^{\mu\nu}\bar{X}_\nu^{\bar{\imath}}\,,\qquad K_{\mu\nu}Y_j^\nu=0=K_{\mu\nu}\bar{Y}_{\bar{\jmath}}^\nu \qquad : \quad i,j=1,2,\cdots,n;\quad \bar{\imath},\bar{\jmath}=1,2,\cdots,\bar{n}$$

$$H^{\mu\rho}K_{\rho\nu}+Y_i^\mu X_\nu^i+\bar{Y}_{\bar{\imath}}^\mu\bar{X}_\nu^{\bar{\imath}}=\delta^\mu{}_\nu \qquad : \quad \text{completeness relation}$$

- It follows that $Y_i^\mu X_\mu^j=\delta_i{}^j$, $\bar{Y}_{\bar{\imath}}^\mu\bar{X}_\mu^{\bar{\jmath}}=\delta_{\bar{\imath}}{}^{\bar{\jmath}}$, $Y_i^\mu\bar{X}_\mu^{\bar{\jmath}}=0=\bar{Y}_{\bar{\imath}}^\mu X_\mu^j$, and $\mathcal{H}_M{}^M=2(n-\bar{n})$.
- Obviously, only $(0,0)$ is Riemannian but all others are non-Riemannian.
- Underlying coset is $\frac{\mathbf{O}(D,D)}{\mathbf{O}(t+n,s+n)\times\mathbf{O}(s+\bar{n},t+\bar{n})}$ with dimensions $D^2-(n-\bar{n})^2$.

III. Phenomenological Implication

The Fate of all Physical Theories is to be Tested; DFT is No Exception

– Solar System Test: Equation of State Matters

– Cosmological Test: Alternative to de Sitter

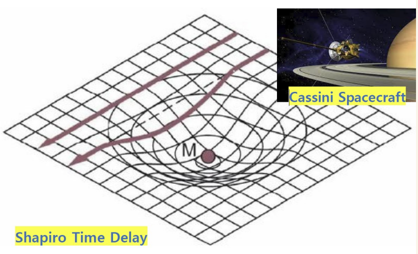
2202.07413 w/ Kang-Sin Choi PRL

2308.07149 w/ Hocheol Lee, Liliana Velasco-Sevilla, and Lu Yin

Solar System Test: Parametrised Post Newtonian (PPN) formalism

– Two dimensionless PPN parameters $\beta_{PPN}, \gamma_{PPN}$ *à la* Eddington-Robertson-Schiff are defined in an asymptotically flat isotropic coordinate system: with $r = \sqrt{x^i x^j \delta_{ij}}$,

$$ds^2 = -\left(1 - \frac{2MG_N}{r} + \frac{2\beta_{PPN}(MG_N)^2}{r^2} + \cdots\right)dt^2 + \left(1 + \frac{2\gamma_{PPN} MG_N}{r} + \cdots\right) dx^i dx^j \delta_{ij}$$



- **Observational values** Will 2014
  - Shapiro Time Delay:

$$\gamma_{PPN} - 1 = (2.1 \pm 2.3) \times 10^{-5}$$

  - Perihelion shifts of Mercury:

$$\beta_{PPN} - 1 = (-4.1 \pm 7.8) \times 10^{-5}$$

  - Earth Gravity:

$$4\beta_{PPN} - \gamma_{PPN} - 3 = (4.44 \pm 4.5) \times 10^{-4}$$

  - Galactic size scale: $\gamma_{PPN} = 0.98 \pm 0.07$

GR predicts $\beta_{PPN} = \gamma_{PPN} = 1$

– In GR, the geometry of a spherical object, or "star", is in general

$$ds^2 = -e^{-2\Delta(r)}\left(1 - \frac{2G_N M(r)}{r}\right)dt^2 + \frac{dr^2}{1 - \frac{2G_N M(r)}{r}} + r^2 d\Omega^2\,,$$

where r denotes areal radius and

$$M(r) := -\int_0^r dr'\, 4\pi r'^2\, T_t{}^t(r')\,, \qquad \Delta(r) := 4\pi G_N \int_r^\infty dr'\, \frac{\{T_r{}^r(r') - T_t{}^t(r')\}r'}{1 - \frac{2G_N M(r')}{r'}}\,.$$

– Outside the star $r > r_\star$ (star radius), $T_{\mu\nu} = 0$ hence $\Delta(r) = 0$. The outer geometry is given by Schwarzschild metric having the only one parameter $M = M(r_\star)$: Birkhoff's theorem

– Mapped to the isotropic coordinate system, one gets rather exactly $\beta_{PPN} = \gamma_{PPN} = 1$.

This has been viewed as the "success" of GR.

Stringy Spherical Vacuum

Burgess-Myers-Quevedo 1994

– The spherical vacuum solution to $G_{AB} = 0$ in DFT has three "free" parameters $\{a, b, h\}$,

$$e^{2\phi} = \gamma_+ \left(\frac{4r-\sqrt{a^2+b^2}}{4r+\sqrt{a^2+b^2}}\right)^{\frac{2b}{\sqrt{a^2+b^2}}} + \gamma_- \left(\frac{4r+\sqrt{a^2+b^2}}{4r-\sqrt{a^2+b^2}}\right)^{\frac{2b}{\sqrt{a^2+b^2}}} ,$$

$$H_{(3)} = h\,\mathrm{d}t \wedge \mathrm{d}\varphi \wedge \mathrm{d}\cos\vartheta\,, \qquad \mathrm{d}s^2 = g_{tt}(r)\,\mathrm{d}t^2 + g_{rr}(r)\left[\mathrm{d}r^2 + r^2\left(\mathrm{d}\vartheta^2 + \sin^2\vartheta\mathrm{d}\varphi^2\right)\right] ,$$

where $\gamma_\pm = \frac{1}{2}(1 \pm \sqrt{1 - h^2/b^2})$, $g_{tt}(r) = -e^{2\phi(r)}\left(\frac{4r-\sqrt{a^2+b^2}}{4r+\sqrt{a^2+b^2}}\right)^{\frac{2a}{\sqrt{a^2+b^2}}}$ and

$$g_{rr}(r) = e^{2\phi(r)}\left(\frac{4r+\sqrt{a^2+b^2}}{4r-\sqrt{a^2+b^2}}\right)^{\frac{2a}{\sqrt{a^2+b^2}}}\left(1 - \frac{a^2+b^2}{16r^2}\right)^2 .$$

– One can read off the mass and the two PPN parameters,

$$MG_N = \tfrac{1}{2}\left(a + b\sqrt{1 - h^2/b^2}\right), \quad (\beta_{PPN}-1)(MG_N)^2 = \frac{h^2}{4}\,, \quad (\gamma_{PPN}-1)MG_N = -b\sqrt{1-\frac{h^2}{b^2}}\,,$$

and further take $\{MG_N, \beta_{PPN}, \gamma_{PPN}\}$ as alternative three parameters, such that

$$\phi \simeq \frac{(\gamma_{PPN}-1)MG_N}{2r} + \frac{(\beta_{PPN}-1)(MG_N)^2}{r^2}\,, \qquad H_{(3)} = \pm 2\sqrt{\beta_{PPN}-1}\,MG_N\,\mathrm{d}t \wedge \mathrm{d}\varphi \wedge \mathrm{d}\cos\vartheta$$

Namely, the deviations $\gamma_{PPN}-1$ and $\sqrt{\beta_{PPN}-1}$ correspond to the dilaton and H-flux *charges*.

Stringy Star has $\beta_{PPN} = 1$ due to weak energy condition

– In a similar fashion to GR, the vacuum solution in the previous page can be identified as the outer geometry of a stringy star (non-singular), while it becomes possible to relate the three parameters to the stress-energy tensor of the star. [Angus-Cho-JHP 2018]

It turns out that, by assuming weak energy condition for positive mass,

$$-K_t{}^t > 0 \qquad\qquad MG_N = \frac{1}{4\pi}\int_{star} \mathrm{d}^3x\, e^{-2d}\left(-K_t{}^t\right),$$

one can show the electric H-flux must be trivial, $h = 0$, which implies

$$\beta_{PPN} = 1$$

PPN parameter γ_{PPN} is an equation-of-state parameter:

– On the other hand, γ_{PPN} can be identified as a generalized equation-of-state parameter which should be subject to the experimental bound:

$$|\gamma_{PPN} - 1| \simeq \left| \frac{\int_{SUN} \mathrm{d}^3x \;\; e^{-2d}\left(K_\mu{}^\mu - T_{(0)}\right)}{\int_{SUN} \mathrm{d}^3x \;\; e^{-2d}\left(-K_t{}^t\right)} \right| \lesssim 10^{-5}$$

Thus, for DFT to pass the solar system test, the matter forming the sun needs to satisfy

$$\left| K_\mu{}^\mu - T_{(0)} \right| << \left| K_t{}^t \right|$$

Failure or NOT? $\Rightarrow$ the choice of right degrees-of-freedom Weinberg

– If a star were modeled as an ideal gas of particles, we have

$$\gamma_{PPN} \simeq 3p/\rho = \langle (v/c)^2 \rangle \,.$$

To be consistent with the observation, the constituting particles should be ultrarelativistic ($v/c \sim 1$) rather than “pressureless dusts”.

– The pressure outside an atom may be negligible, but this is also true for the energy density.

Both ρ and p should be confined inside baryons.

Recent experiment reveals high pressure $p \sim \rho$ inside proton. Burkert-Elouadrhiri-Girod 2018 Nature

– Instead, chiral effective theory of nuclear physics,

$$S_{eff.} = -\int d^4x\, e^{-2d} g^{\mu\nu} \partial_\mu \Phi^I \partial_\nu \Phi^J \mathcal{G}_{IJ}(\Phi)$$

sets $K_\mu{}^\mu = T_{(0)}$ and thus rather precisely $\gamma_{PPN} = 1$.

– Applied to QCD, the condition boils down to the gluon and quark condensates:

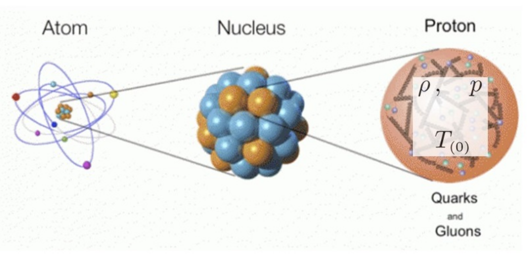
$$\gamma_{PPN} - 1 \;\simeq\; \frac{\int_{star} d^3x \,\left[e^{-2d} \mathrm{Tr}(B^2 - E^2) - m\bar{\psi}\psi \right]}{\int_{star} d^3x \,\left[e^{-2d} \mathrm{Tr}(E^2) + i\bar{\psi}\gamma^t D_t \psi \right]}$$

which may vanish, as the electric and magnetic fields may cancel each other, while the quarks get negligible. Barate *et al.* 1998; Del Debbio-Zwicky, Hyun Kyu Lee, Mannque Rho 2022.

Solar System Test: Gravitational Probe into the Interior of Hadrons

– To summarize, DFT sets $\beta_{PPN} = 1$ and lets γ_{PPN} be the equation-of-state parameters.

The observations $\gamma_{PPN} \simeq 1$ may hint at the equation of state inside baryons.



Cosmological Test: Exact Vacuum Solution alternative to de Sitter

– In GR, de Sitter is the simplest cosmological solution: $\Omega_\Lambda = 0.73$ for ΛCDM.
Yet, the Hubble tension is getting worse by James Webb telescope: 67 vs. 73 km/s/Mpc.
Besides, there is swampland no-go argument for the existence of de Sitter. Vafa *et al.*

– What would be the cosmological vacuum solution to EDFE?
The answer is traceable to the work (1994) by Copeland, Lahiri, and Wands.

Here we elaborate their solution further to feature three free parameters, $\{H_0,\, \mathfrak{h},\, l \equiv 1/\sqrt{-k}\,\}$ as for an open Universe which turns out to fit observational data.
Dilaton ϕ which does not run away because $k<0$,

$$e^{2\phi(\eta)} = \frac{1-\sqrt{1-\frac{1}{12}(\mathfrak{h}l\,\sinh\zeta)^2}}{2}\left[\frac{\tanh\left(\frac{\eta}{l}+\frac{\zeta}{2}\right)}{\tanh\frac{\zeta}{2}}\right]^{\sqrt{3}} + \frac{1+\sqrt{1-\frac{1}{12}(\mathfrak{h}l\,\sinh\zeta)^2}}{2}\left[\frac{\tanh\left(\frac{\eta}{l}+\frac{\zeta}{2}\right)}{\tanh\frac{\zeta}{2}}\right]^{-\sqrt{3}}$$

Magnetic H-flux and FLRW metric (homogeneous & isotropic),

$$H_{(3)} = \frac{\mathfrak{h}\,r^2\sin\vartheta}{\sqrt{1+r^2/l^2}}\,\mathrm{d}r\wedge\mathrm{d}\vartheta\wedge\mathrm{d}\varphi\,, \qquad \mathrm{d}s^2 = a^2(\eta)\left[-\mathrm{d}\eta^2+\frac{\mathrm{d}r^2}{1+r^2/l^2}+r^2\left(\mathrm{d}\vartheta^2+\sin^2\vartheta\mathrm{d}\varphi^2\right)\right]$$

with the scale factor and the Hubble constant,

$$a^2(\eta) = e^{2\phi(\eta)}\,\frac{\sinh\left(2\eta/l+\zeta\right)}{\sinh\zeta}\,, \qquad H_0 = \frac{1}{2l\sinh\zeta}\left[2\cosh\zeta-\sqrt{12-(\mathfrak{h}l\,\sinh\zeta)^2}\right].$$

Bayesian Inference of Observational Data

– **Type Ia Supernovae by Pantheon+:** Distance Modulus $\mu(z)$ & Luminosity Distance $d_L(z)$,

$$\mu(z) = 5\,\mathrm{Log}_{10}\left[\frac{d_L(z)}{10\,\mathrm{pc}}\right], \qquad d_L(z) = \frac{1+z}{\sqrt{-k}}\sinh\left[\sqrt{-k}\int_0^z \frac{\mathrm{d}z'}{H(z')}\right]$$

⇒ 1583 data points over $0.01 \leq z \leq 2.26$ Riess *et al.* 2021

– **Quasar Absorption Spectrum:** Temporal Variation of the Fine Structure Constant,

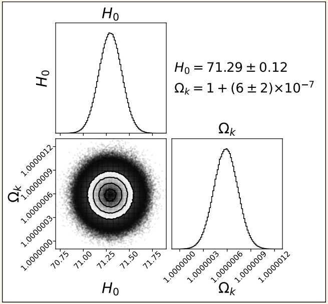
$$\frac{e^{-2\phi(t)}}{\alpha}F_{\mu\nu}F^{\mu\nu} = \frac{1}{\alpha_{eff.}(t)}F_{\mu\nu}F^{\mu\nu}$$

⇒ 199 data points over $0.22 \leq z \leq 7.06$

King *et al.* 2012; Wilczynska *et al.* 2015 & 2020; Martins *et al.* 2017

– We perform analyses of **Bayesian Inference (BI)** against these observational data.
We use Markov Chain Monte Carlo (MCMC) ensemble sampler called ‘emcee’.
With 100 walkers, we run the samplers on a supercomputer (KiSTi) for 10^6 steps.

Two Parameter Fitting by the Exact Vacuum (trivial H-flux)

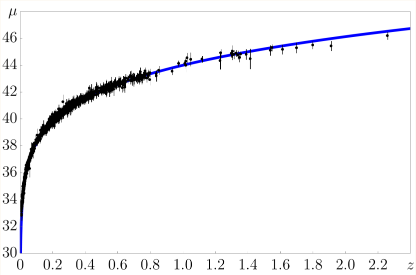


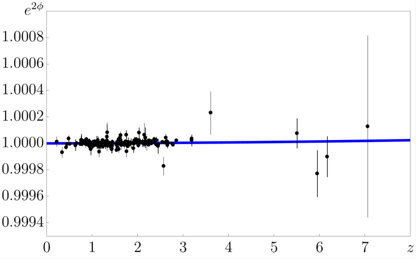
– BI: very well converged, $\Omega_k = 1/(lH_0)^2$

– Distance Modulus μ : Complete agreement with the type Ia supernova data.

– Suppressed time-evolution of $e^{2\phi}$ or the fine-structure constant: Consistency with the quasar data.

$*$ Admirable agreement, without DE or DM.





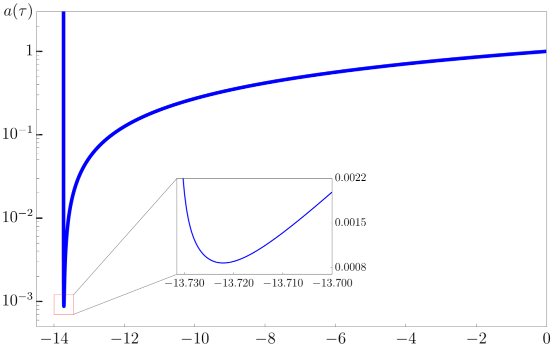
Extrapolations to Future and Past

- The exact vacuum solution predicts that, at future infinity the dilaton converges to constant, and the Universe expands forever as $a(\eta) \propto e^{\eta/l}$ such that

$$\lim_{\eta\to\infty} \Omega_k = 1$$

which agrees with our BI fitting. Thus, there is No Coincidence Problem in our scenario.

- Extrapolated to the past, the Universe bounces about 13.72 gigayears ago which is intriguingly close to the "age" of the flat Universe estimated in ΛCDM.



Conclusion

- ⋆ **GR, including Einstein equation, has been successfully doubled:**

$$G_{AB} = T_{AB} \qquad \text{where} \quad A, B \ \text{ are } \ \mathbf{O}(D, D) \ \text{ indices.}$$

- ⋆ **While the theory yields sharp $\mathbf{O}(D,D)$-symmetric predictions, it has not been excluded by observations and awaits further verification.**

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Thank you