Reconstructing the dual gravity from entanglement entropy

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Based on :

1) In preparation (collaborating with J. Huh)

2) CP, C. Hwang, K. Cho, and S. Kim, Phys. Rev. D (2022)

Motivation

AdS/CFT correspondence

Classical SUGRA on AdS space-time

⟷ 1 to 1

Super-CFT at the AdS boundary (**in a strong coupling regime**)

Due to the conformal symmetry, the IR theory is trivial.

How about a non-conformal and non-supersymmetric QFT like a condensed matter theory?

IR ← non-trivial RG flow ← UV

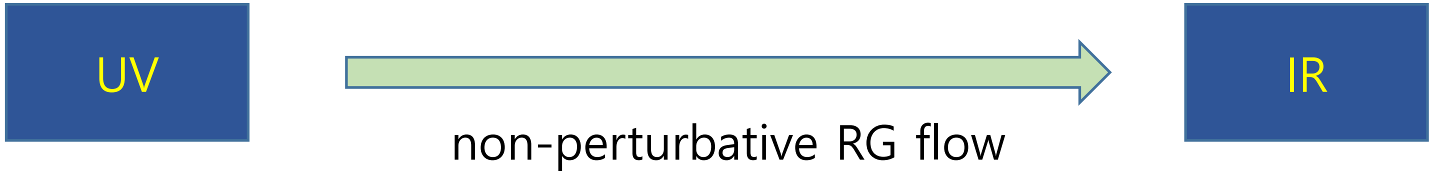
- IR physics
- QCD and CMT

- known asymptotic AdS geometries (dual QFT?)
- dual geometries?

Can we find the dual geometry of a given QFT data?

If we know the RG flow of QFT, then we can reconstruct its dual gravity.

To investigate IR (macroscopic) physics from the fundamental (microscopic) QFT, we need to figure out a non-perturbative RG flow.



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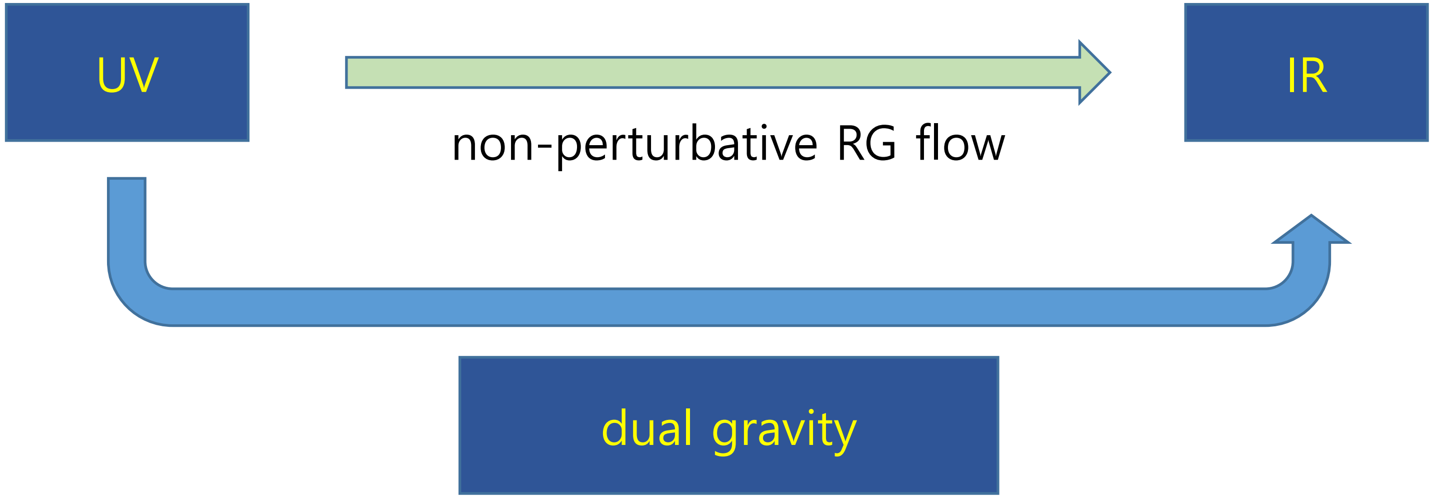

RG flow descriptions

(1) Momentum-space RG (integrate out higher frequency modes)

(i) 1PI (one particle irreducible) RG (perturbative, QFT for high energy physics)

(ii) Wilsonian Exact RG equation with a hard cutoff (non-perturbative) ⇒ holographic renormalization

(ii) Polchinski exact RG equation with a soft cutoff (non-perturbative)

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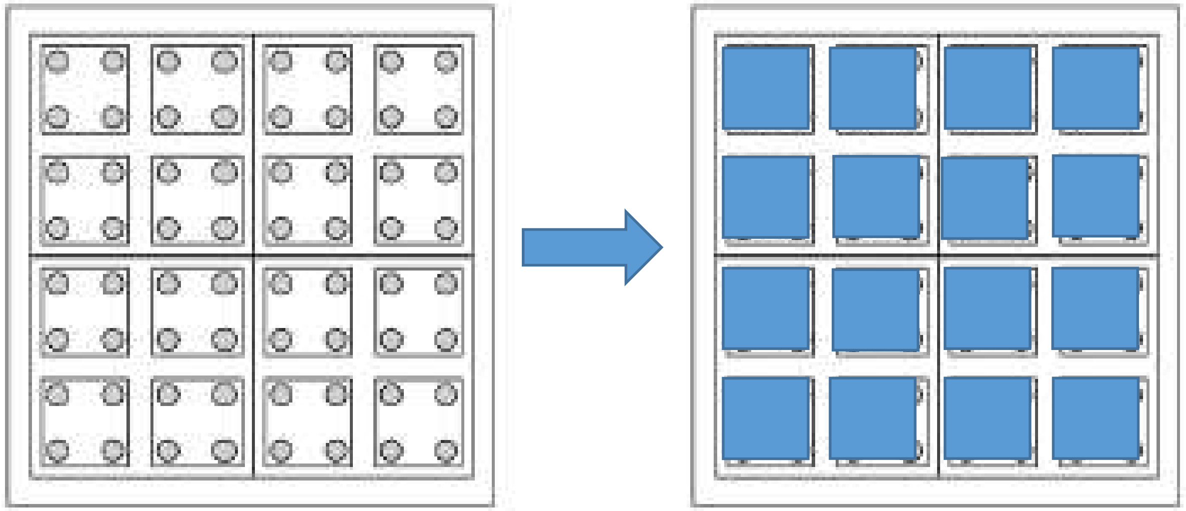
(2) Real-space RG (Migdal-Kadanoff, CMT)

Block spin renormalization

→ RG flow of Entanglement entropy



Ryu-Takayanagi conjecture

One of the most remarkable successes in the AdS/CFT correspondence is the microscopic derivation of the Bekenstein-Hawking entropy for a BPS black hole

$$S_{BH} = \frac{A}{4G}$$

This idea relates the gravitational entropy to the degeneracy of the dual quantum field theory with its microscopic description.

On the other hand, there exists a different kind of entropy called the entanglement entropy in quantum mechanical systems which measures the entanglement between quantum states.

Ryu and Takayanagi proposed the formula following the black hole entropy

$$S = \frac{\text{Area of } \gamma}{4G}$$

The goal of this work is to figure out the entanglement entropy in the strong coupling regime following the AdS/CFT correspondence.

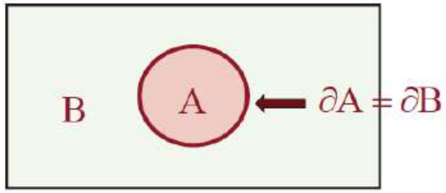
Review of the holographic entanglement entropy

The entanglement entropy measures

how closely and quantumly a given wave function is entangled.

Definition of EE (entanglement entropy)

- Divide a quantum system into two parts, A and B.



$$H_{tot} = H_A \otimes H_B \quad .$$

- Reduced density matrix of the subsystem A : $\rho_B = \mathrm{Tr}\,_A \rho_{tot}$

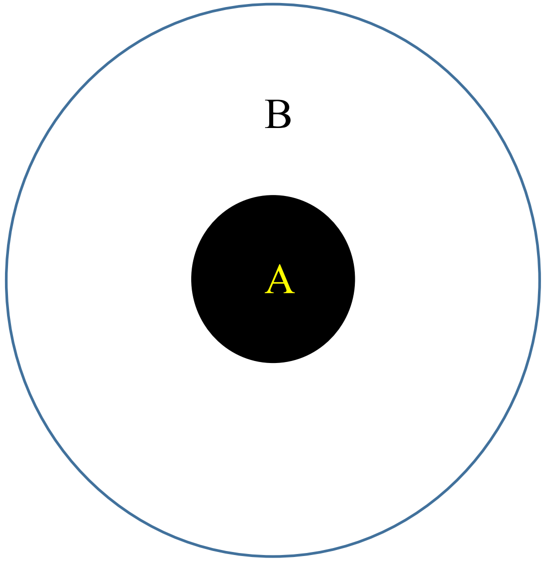
- The entanglement entropy (EE)

$$S_B = -\mathrm{Tr}\,_B \rho_B \log \rho_B$$

which is proportional to the area of the entangling surface (∂A)

S_B describes the quantum entanglement detected by an observer who is only accessible to the subsystem B and can not receive any signal from A.

This is similar to the Bekenstein-Hawking entropy of the black hole.
Since an observer sitting in the outside of the horizon, B, can not receive any information from A, we can regard A as a black hole and the boundary of A as the black hole horizon.



1. The area law of the entanglement entropy is also similar to that of the black hole entropy

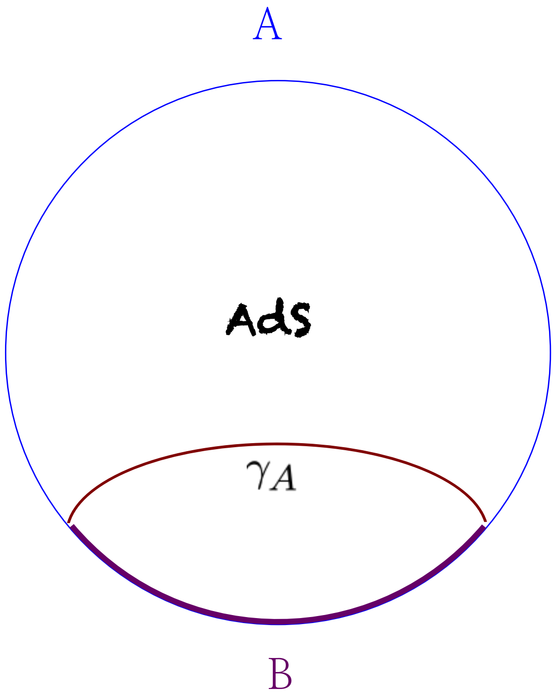
2. The entanglement entropy is utilized to figure out the black hole entropy

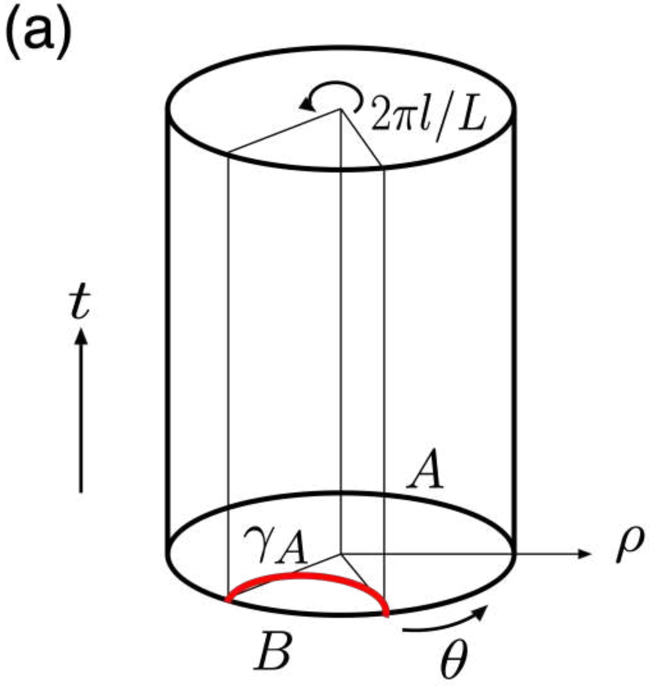
Due to the similarity to the black hole,

Ryu and Takayanagi [2006] proposed the holographic entanglement entropy (hEE) following the AdS/CFT correspondence

the EE of a d-dimensional CFT can be evaluated by the area of the minimal surface in the d+1-dim dual AdS gravity

$$S_E = \frac{Area(\gamma_A)}{4G}$$





2-dim. CFT result [Calabrese-Cardy, 2004]

It is known that the entanglement entropy of the 2-dim CFT is given by

$$S_E = \frac{c}{3} \log \left(\frac{L}{\pi\epsilon} \sin \frac{\pi l}{L} \right) \approx \frac{c}{3} \log \frac{l}{\epsilon}$$

where l and L are the length of the subsystem A and the total system and ϵ is a UV cutoff (lattice spacing) and c is the central charge of the CFT.

Away from criticality (fixed point), the entanglement entropy is replaced by

$$S_E = \frac{c}{6} \mathcal{A} \log \frac{\xi}{\epsilon}$$

where ξ is the correlation length.

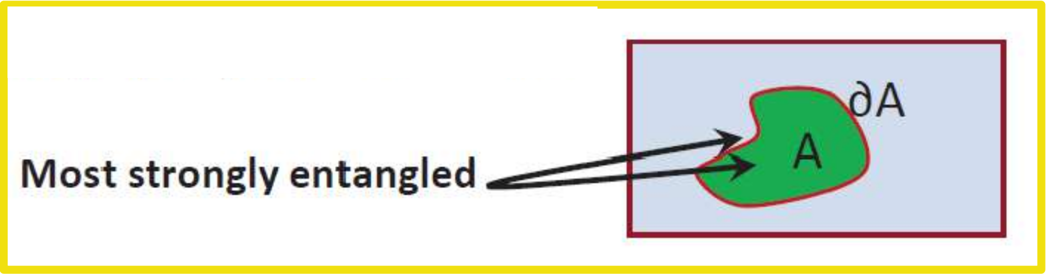
This is due to the infinite conformal symmetry and modular invariance of a 2-dim. CFT defined on the torus.

General properties of the entanglement entropy

1) Area law of the entanglement entropy

The leading term of the entanglement entropy is provided by the short distance interaction between two subsystems near the boundary. In the continuum limit, this term causes a UV divergence and its coefficient is proportional to the area of the entangling surface ∂A (*UV cutoff sensitive, regularization scheme dependent*).

$$S_E \sim \frac{Area(\partial A)}{\epsilon^{d-1}} + \text{ subleading finite terms}$$



2) Subleading finite terms

There exists the terms not relying on a UV cutoff, which can provide an important physical information associated with the long range correlations.

In general, the entanglement entropy crucially depends on the shape and size of the entangling surface.

(i) for d=odd

$$A = \Omega_{d-2} \left[\frac{1}{d-2} \left(\frac{l}{\epsilon} \right)^{d-2} + F + \mathcal{O}\left(\frac{\epsilon}{l} \right) \right]$$

- No logarithmic term

- There exists a constant term, F, which is identified with a free energy of the 3-dimesional dual CFT for d=3.

- For d=3,

F is the exact same as the free energy of 3-dim. CFT which has been checked by the comparison with the localization result.

(ii) for d=even

$$A = \Omega_{d-2}\left[\frac{1}{d-2}\left(\frac{l}{\epsilon}\right)^{d-2} + a' \log\left(\frac{l}{\epsilon}\right) + \mathcal{O}(1)\right]$$

with

$$a' = (-)^{d/2-1}\frac{(d-3)!!}{(d-2)!!}$$

- There exists a universal logarithmic term. Its coefficient is universal in that it is independent of the regularization scheme.

- The coefficient of the logarithmic term is independent of the entangling surface area, which is related to the a-type anomaly.

- Weyl anomaly of 4-dim. CFT,

$$\langle T^{\alpha}_{\alpha}\rangle = -\frac{c}{8\pi} W_{\mu\nu\rho\sigma}W^{\mu\nu\rho\sigma} + \frac{a}{8\pi}\tilde{R}_{\mu\nu\rho\sigma}\tilde{R}^{\mu\nu\rho\sigma}$$

with

$$\begin{aligned} W_{\mu\nu\rho\sigma}W^{\mu\nu\rho\sigma} &= R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 2R_{\mu\nu}R^{\mu\nu} + \frac{1}{3}R^2, \\ \tilde{R}_{\mu\nu\rho\sigma}\tilde{R}^{\mu\nu\rho\sigma} &= R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 4R_{\mu\nu}R^{\mu\nu} + R^2. \end{aligned}$$

As a consequence, *the logarithmic term is related to the anomaly and crucially depends on the dimension and shape of the entangling surface.*

c-theorem by Zamoldchikov

When a 2-dim. CFT is deformed by a relevant operator, it flows to a new IR fixed point.

In this case, the central charge, which describes degrees of freedom of a system, monotonically decreases along the RG flow.

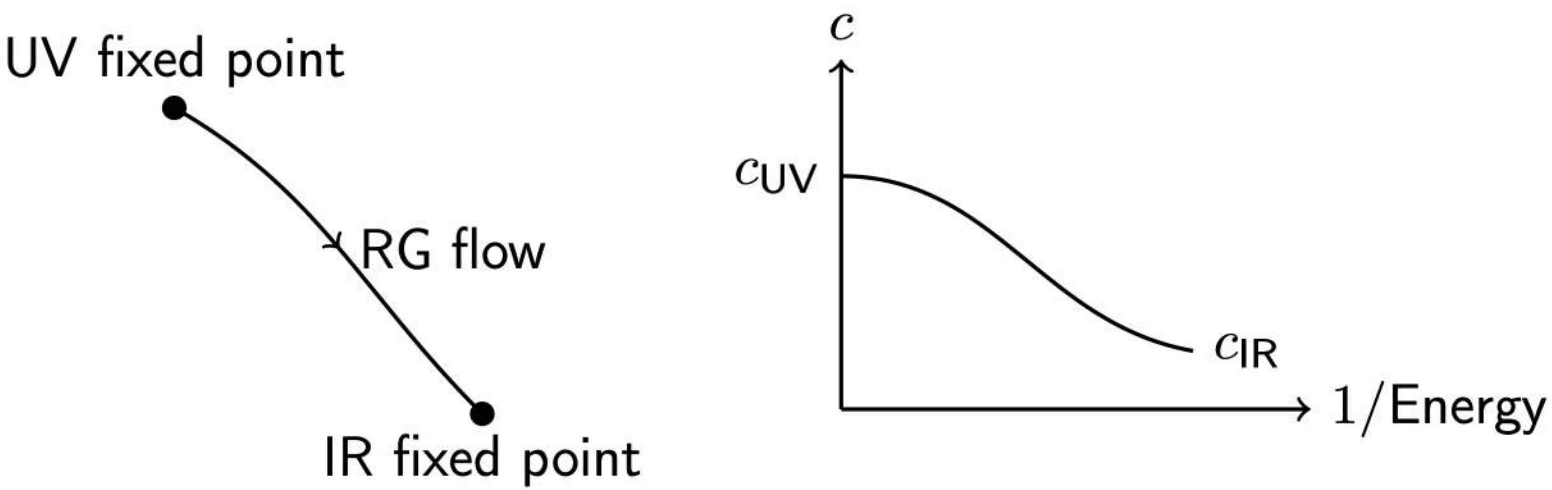
In higher dimensional theory, is there a theorem similar to the C-theorem?

- For d=4, there exists two central charges, a and c. It has been believed that the a-type anomaly satisfies the c-theorem (***a-theorem***).

- For d=3, it has been conjectured that the free energy monotonically decreases along the RG flow (***F-theorem***).

F-theorem in 3-dim. CFT [Jafferis-Klebanov-Pufu-Safdi 2011, Myers-Sinha 2010]

RG flow under a relevant deformation



$$F_{\rm UV}(\mathbb{S}^3) \geq F_{\rm IR}(\mathbb{S}^3) \ , \quad F = -\log Z(\mathbb{S}^3)$$

HEE with a spherical entangling surface in a 3-dim. CFT

$$S_E = \alpha \frac{l}{\epsilon} - F(\mathbf{S}^3)$$

For the entanglement entropy

we can also derive the similar structure, where the quantum entanglement transfers into a thermal quantity with a small quantum corrections.

For the three-dimensional AdS (BTZ) black hole

$$ds^2 = -\frac{R^2}{z^2}f(z)dt^2 + \frac{R^2}{z^2 f(z)}dz^2 + \frac{R^2}{z^2}dx^2, \qquad \text{with} \qquad f(z) = 1 - \frac{z^2}{z_h^2}$$

thermodynamic quantities are given by

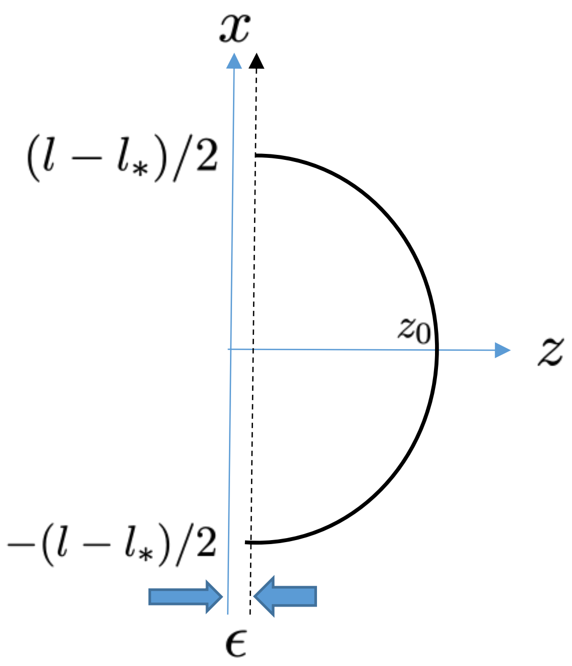
$$\begin{aligned} T_H &= \frac{1}{2\pi}\frac{1}{z_h}, \\ S_{th} &= \frac{1}{4G}\frac{l}{z_h}, \\ E &= \frac{1}{16\pi G}\frac{l}{z_h^2}. \end{aligned}$$

and satisfy the first law of thermodynamics $dE = T_H dS_{th}$

In the holographic context

the entanglement entropy can be evaluated as the area of the minimal surface extended in the dual geometry.

$$A = \int_0^{l/2} dx \, \frac{R}{z}\sqrt{1+\frac{z'^2}{f}}$$



Then, the subsystem size and the entanglement entropy can be rewritten in terms of the turning point

$$\begin{aligned} z_0 &= z_h \tanh\left(\frac{l}{2z_h}\right), \\ S_E &= \frac{1}{2G}\log\frac{2z_0}{\epsilon} - \frac{1}{4G}\log\left(1-\frac{z_0^2}{z_h^2}\right). \end{aligned}$$

This is an exact and analytic result.

When $z_h \to \infty$,

$$z_0 = l/2$$

$$S_E^0 = \frac{1}{2G}\log\frac{l}{\epsilon}.$$ (ground state entanglement entropy, UV divergence)

RG flow of the entanglement entropy

Thermodynamics-like law of the entanglement entropy in the UV limit

$$T_E \; S_E = E \qquad \text{with} \qquad T_E = \frac{6}{\pi l}.$$

- This relation is defined in the UV region with neglecting higher order corrections.

- It reproduces the linearized Einstein equation of the dual geometry.

- It is not valid in the IR region.

In order to go beyond the linearized lever and to describe the RG flow correctly, we need generalized concepts involving all higher order corrections.

We define a generalized thermodynamics-like law and generalized entanglement temperature involving all higher order correction and satisfying in the entire region

$$\bar{T}_E \; \bar{S}_E = \bar{E}$$

Define a renormalized entanglement entropy (subtracting the ground state EE)

$$\bar{S}_E \equiv S_E - S_E^0.$$

Then, the exact renormalized EE and a generalized entanglement temperature

$$\bar{S}_E = \frac{1}{2G} \log\left(\frac{2z_h}{l} \sinh\left(\frac{l}{2z_h}\right)\right)$$

$$\frac{1}{\bar{T}_E} \equiv \frac{1}{2}\frac{\bar{S}_E}{\bar{E}} = \frac{4\pi z_h^2}{l} \log\left(\frac{2z_h}{l} \sinh\left(\frac{l}{2z_h}\right)\right)$$

In the UV region ($l/z_h \ll 1$),

$$\bar{S}_E = \frac{1}{48G}\frac{l^2}{z_h^2}\left(1 - \frac{l^2}{120 z_h^2} + \cdots\right)$$

$$\frac{1}{\bar{T}_E} = \frac{1}{T_E}\left(1 - \frac{l^2}{120 z_h^2} + \cdots\right)$$ with the previously defined entanglement temperature $T_E = \frac{6}{\pi l}$.

Ignoring l^2 order corrections, they are reduced to the known results.

Note that the generalized entanglement temperature was defined to satisfy the thermodynamics-like law exactly with involving all higher order correction. Therefore, we can apply the thermodynamics-like law to the IR entanglement entropy.

In the IR region ($z_0 \approx z_h$),

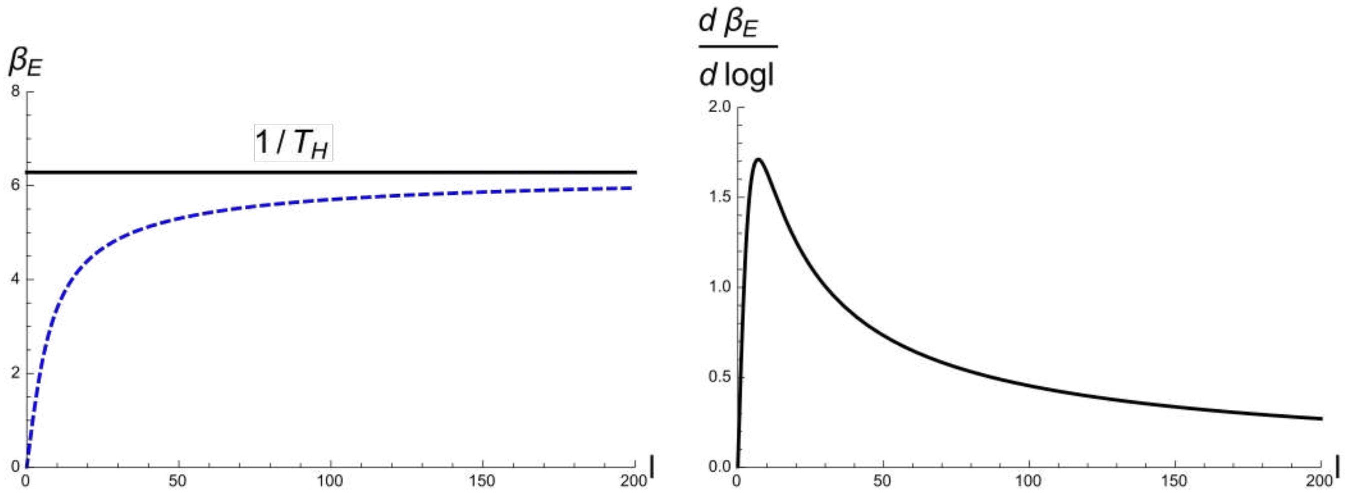
Reexpressing it in terms of the black hole entropy involved in the volume l , we reach to the similar result obtained from the black hole and CFT calculations

$$\bar{S}_E = S_{th} - \frac{1}{2G} \log S_{th} + \mathcal{O}(1)$$

Since $S_{th} \to \infty$ in the IR limit, the IR entanglement entropy reduces to the thermal entropy with small quantum corrections. Also, we can see that the generalized entanglement temperature reduces to the real temperature.

$$\beta_E = 2\pi z_h + \frac{4\pi z_h^2}{l} \log\left(\frac{z_h}{l}\right) + \cdots ,$$

$$l\frac{d\beta_E}{dl} = 2\pi z_h \left[\coth\frac{l}{2z_h} - \frac{2z_h}{l}\left\{1 + \log\left(\frac{2z_h}{l}\sinh\frac{l}{2z_h}\right)\right\}\right].$$



(a) (b)

Regardless of the dimensionality and microscopic detail of the dual field theory, the IR entanglement entropy reduces to

$$\bar{S}_E = S_{th} + S_{correction}$$

S_{th} : Universal

$S_{correction}$: depending on the dual theory

For a two-dimensional scale invariant theory

$$S_{correction} \sim -\log S_{th}$$

Intriguingly, the universality of the IR entanglement entropy proposed from the holography

$$\bar{S}_E \approx S_{th}$$

occurs in the real space renormalization group flow of the lattice theory (Ising model). This follows the volume law

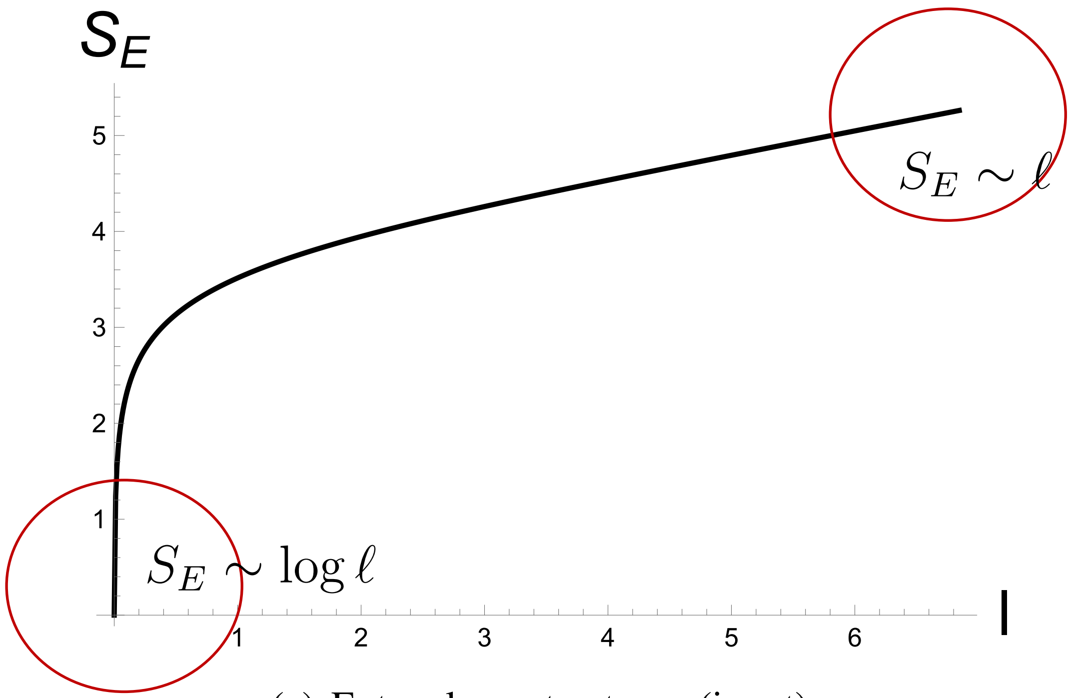
$$S_{th} \sim l$$

Reconstruction of the dual gravity from the entanglement entropy

When the entanglement entropy of a 2-dim system consisting of two kinds of matter, radiation and massive particles,

(1) can we reconstruct the dual gravity of this system?

(2) Can we read other physical properties of this system?



(a) Entanglement entropy (input)

1. Logarithmic behavior in the UV limit ($S_E \sim \log \ell$): asymptotic AdS space

2. Volume law in the IR limit ($S_E \sim \ell$): thermal system

3. A general form of the dual gravity (black hole geometry)

$$ds^2 = \frac{R^2}{z^2}\left(-f(z)dt^2 + \frac{1}{f(z)}dz^2 + dx^2\right)$$

Using the above metric ansatz, the corresponding entanglement entropy is determined by

$$S_E = \frac{R^{d-1}\, V_{d-2}}{4G} \int_{-\ell/2}^{\ell/2} dx \frac{\sqrt{z'^2 + f}}{z^{d-1}\sqrt{f}}$$

In this case, the subsystem size and entanglement entropy are characterized by a turning point z_t

$$\ell(z_t) = \int_{\epsilon}^{z_t} dz \frac{2z^{d-1}}{\sqrt{f(z)}\,\sqrt{z_t^{2(d-1)} - z^{2(d-1)}}},$$

$$S_E(z_t) = \frac{R^{d-1}\, V_{d-2}}{2G} \int_{\epsilon}^{z_t} dz\, \frac{z_t^{d-1}}{z^{d-1}\sqrt{f(z)}\,\sqrt{z_t^{2(d-1)} - z^{2(d-1)}}},$$

Solving these integral equations, we find z_t and $f(z_t)$, which determines the black hole geometry.

Parametrizing the turning point as $z_i = \epsilon + i\,\delta z$ with $\delta z = \dfrac{z_h - \epsilon}{N}$

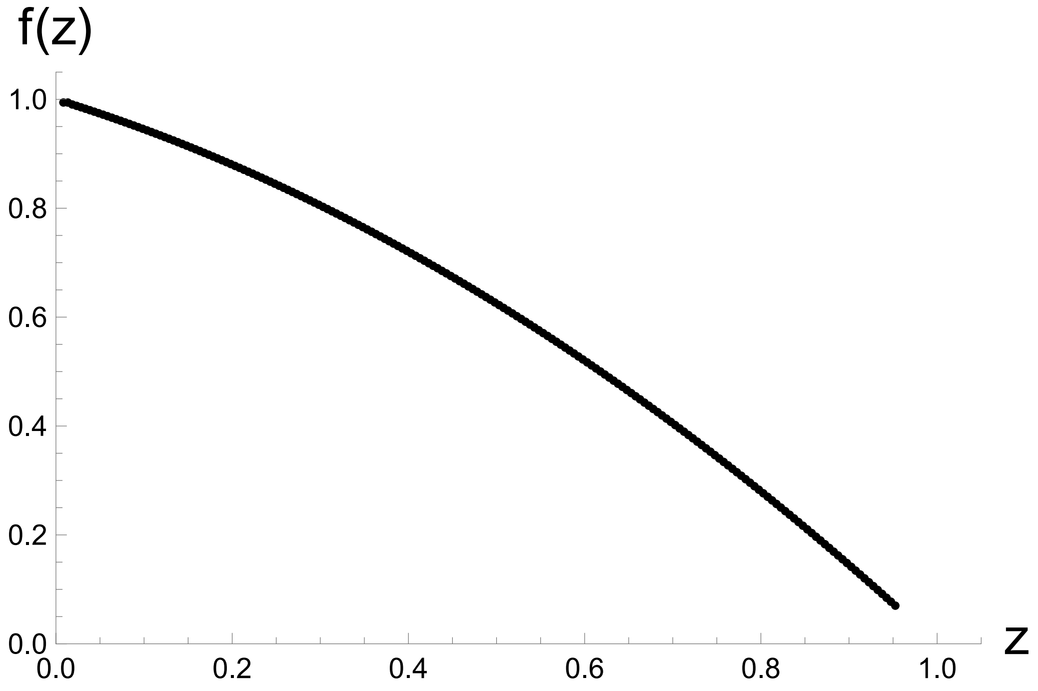
At given z_i, we find $f(z_i)$ satisfying the input entanglement entropy $S_E(\ell)$

$$\ell_n = \sum_{i=1}^{n} \int_{z_{i-1}}^{z_i} dz \frac{2z}{\sqrt{\bar{f}(z_i)}\,\sqrt{z_n^2 - z^2}},$$

$$S_n = \sum_{i=1}^{n} \frac{R}{2G} \int_{z_{i-1}}^{z_i} dz\, \frac{z_n}{z\sqrt{\bar{f}(z_i)}\,\sqrt{z_n^2 - z^2}},$$

As a result, we obtain the blackening factor $\bar{f}(\bar{z}_i)$ at $\bar{z}_i$

$$\bar{f}(\bar{z}_i) = \frac{f(z_{i-1}) + f(z_i)}{2} \quad \text{at} \quad \bar{z}_i = \frac{z_{i-1} + z_i}{2},$$

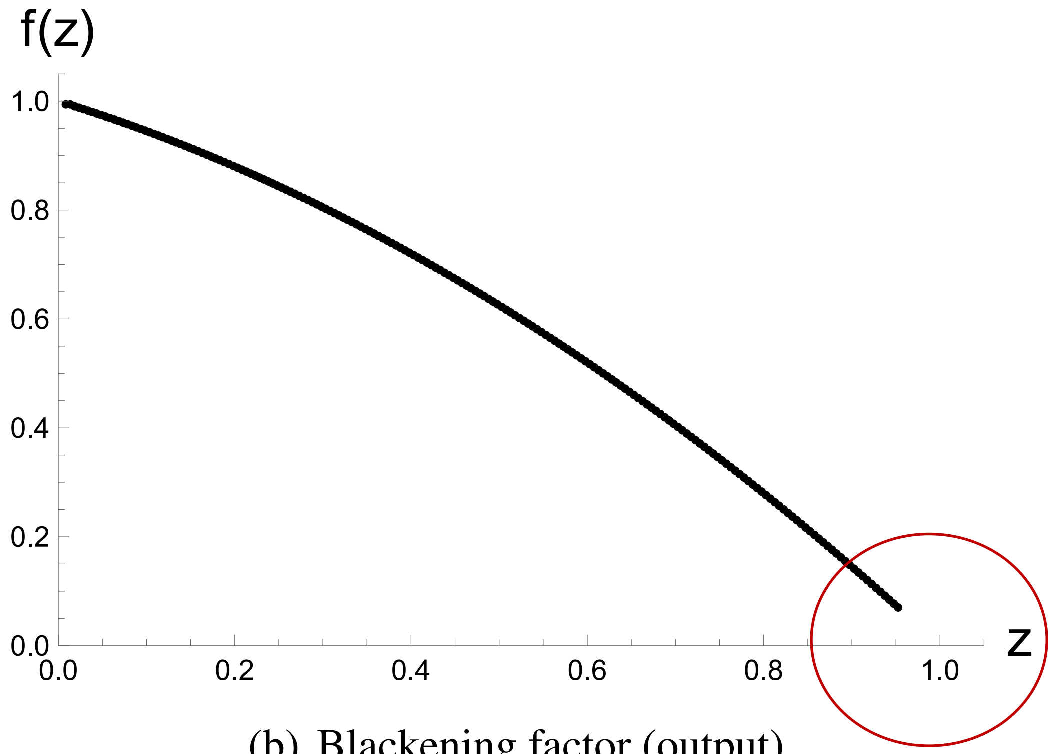


(b) Blackening factor (output)

After extrapolation, we determine the horizon and Hawking temperature

$$z_h \approx 0.99986$$

$$T_H = -\frac{f'(z_h)}{4\pi} \approx 0.12054$$



(b) Blackening factor (output)

To understand other physical properties, we guess the analytic form of the blackening factor

$$f(z) = 1 - \rho_0\, z - M\, z^2,$$

M : number density of massless radiation

ρ_0 : number density of massive particles

Then, the massive particle's density is determined from the above numerical data

$$\rho_0 = \frac{2}{\bar{z}_h} - 4\pi T_H \approx 0.48553$$

Finally, the following blackening factor is reconstructed from the entanglement entropy data

$$f(z) = 1 - 0.48553\, z - 0.51468\, z^2.$$

From the black hole's thermodynamics

Internal energy: $U_{th} = \frac{\pi T_H^2}{4} \frac{RV}{G} \approx 0.01141 \frac{RV}{G}$

Pressure: $P_{th} = \frac{2\pi T_H^2 + \rho_0 T_H}{8} \frac{R}{G} \approx 0.01873 \frac{R}{G}.$

Equation of state: $w = 1 + \frac{\rho_0}{2\pi T_H} \approx 1.64107,$

Specific heat: $c_V = \frac{\pi T_H}{2} \frac{RV}{G} \approx 0.18934 \frac{RV}{G} > 0.$

	Derived value	True value	Error
U_{th}	0.01141	0.01119	1.97 %
P_{th}	0.01873	0.01865	0.43 %
w	1.64107	1.66667	1.54 %
c_V	0.18934	0.18750	0.98 %

with $R = V = G = 1$

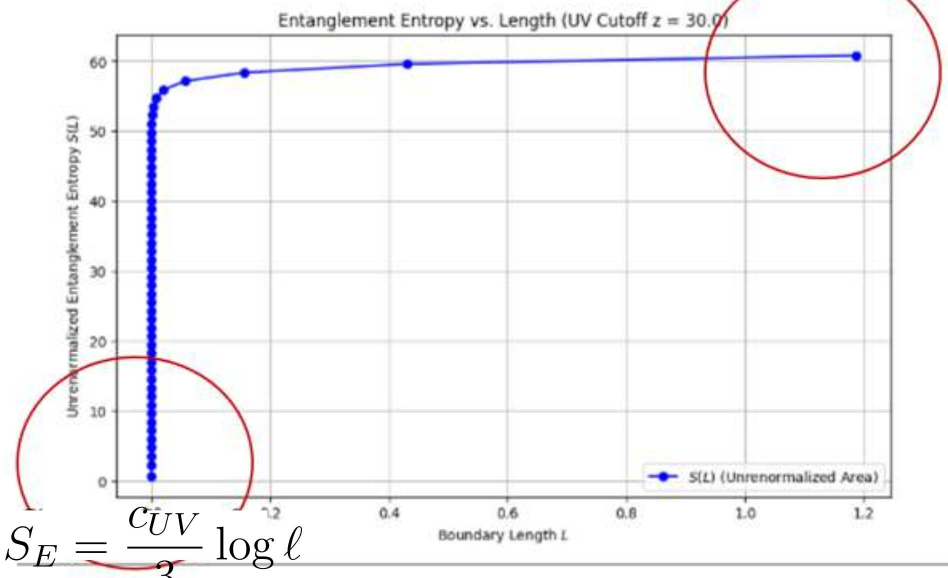
- True values of M and ρ_0 are $M = q_0 = 1/2$ which are utilized to derive the input data.

Reconstruction of the dual gravity theory

Assuming that the following entanglement entropy is given, then what is the dual geometry and gravity theory?

- The given entanglement entropy data shows that a UV CFT deforms and then leads to another IR CFT.

$$S = S_{CFT} + \int d^2x \, \lambda \, O$$



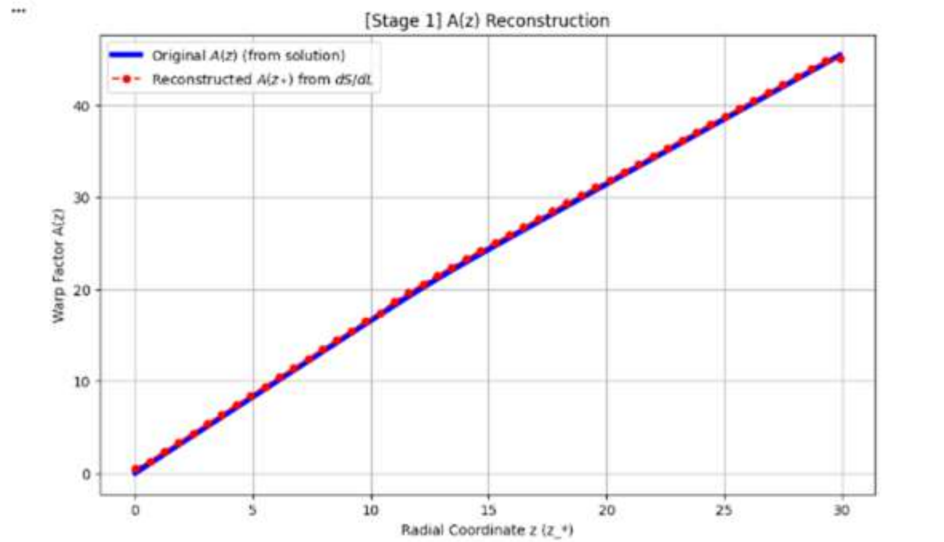
$$S_E = \frac{c_{IR}}{3} \log \ell$$

$$c_{UV} > c_{IR}$$

$$S_E = \frac{c_{UV}}{3} \log \ell$$

Repeating the previous holographic reconstruction, we obtain the following dual geometry, where we use the following metric ansatz

$$ds^2 = dy^2 + e^{2A(y) - 2A(y_\Lambda)} \delta_{\mu\nu} \, dx^\mu dx^\nu$$



When the UV CFT is deformed by one scalar operator, we expect that the dual gravity theory is given by

$$S = -\frac{1}{2\kappa^2}\int_{\mathcal{M}} d^{d+1}x\sqrt{G}\left(\mathcal{R} - 2\Lambda_{d+1} - \frac{1}{2}G^{MN}\partial_M\phi\partial_N\phi - \frac{V(\phi)}{R^2}\right)$$

whose equations of motion are

$$0 = 2d(d-1)\dot{A}^2 - \dot{\phi}^2 + 4\Lambda_{d+1} + \frac{2V}{R^2},$$

$$0 = 4(d-1)\ddot{A} + 2d(d-1)\dot{A}^2 + \dot{\phi}^2 + 4\Lambda_{d+1} + \frac{2V}{R^2},$$

$$0 = \ddot{\phi} + d\dot{A}\dot{\phi} - \frac{1}{R^2}\frac{\partial V}{\partial \phi}.$$

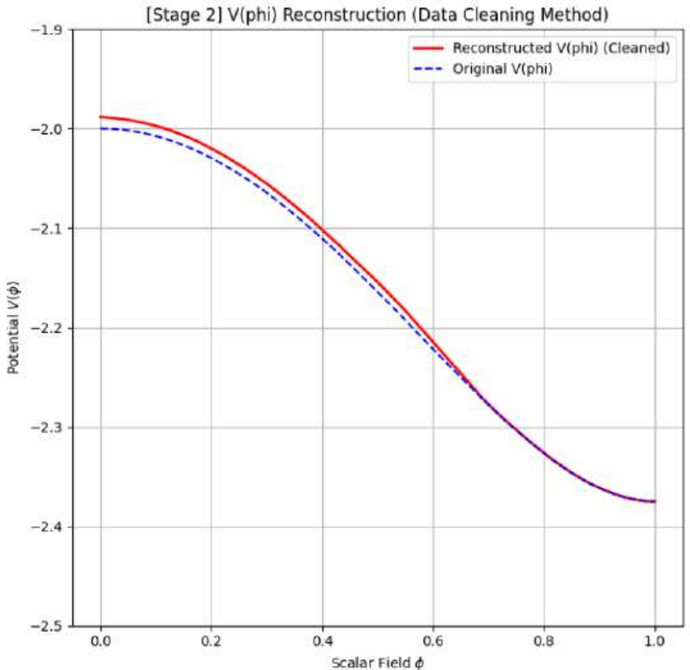
For d=2

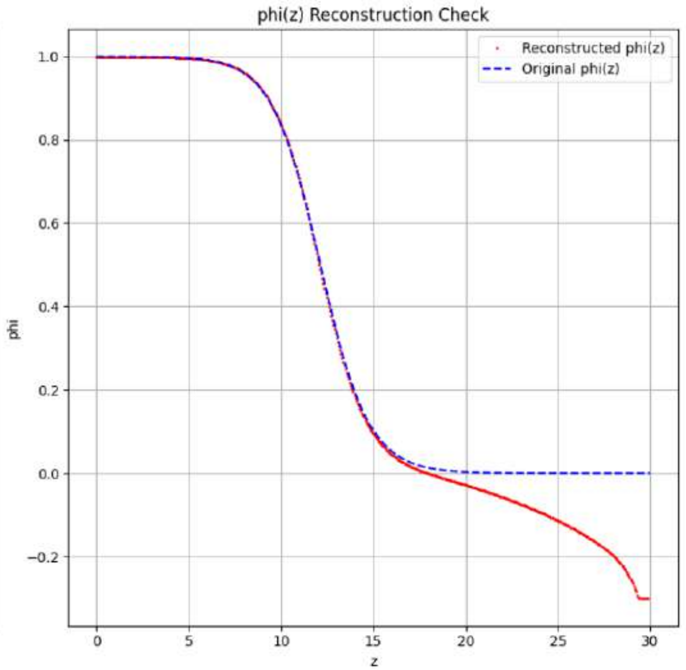
$$\phi(y) = \phi_\infty \pm \sqrt{2}\int dy\,\sqrt{|A''(y)|},$$

$$V(y) = 2 - R^2\left(2A'(y)^2 + A''(y)\right).$$

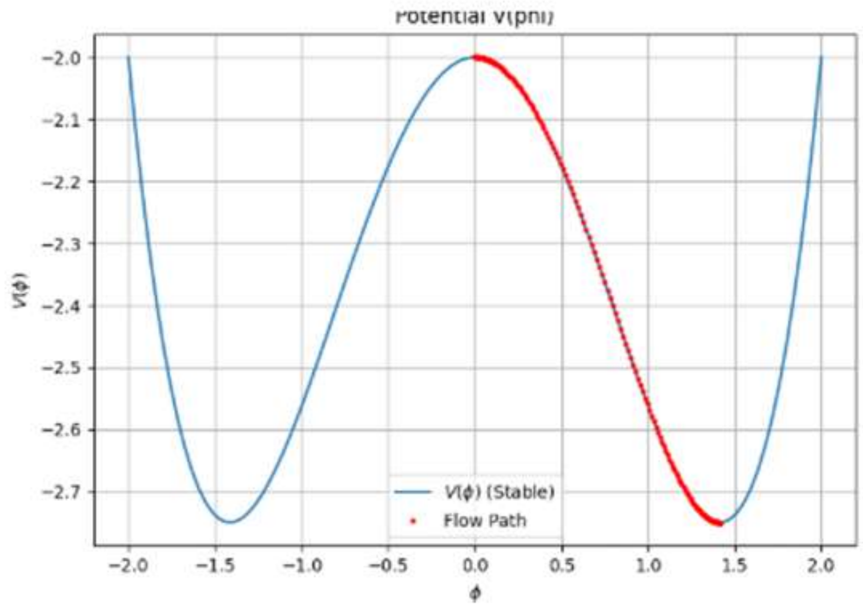
with an appropriate initial condition of $\phi(y)$

Using these equations of motion, we also find the scalar field profile and the scalar potential





The true scalar potential, which we exploited to derive the entanglement entropy, is

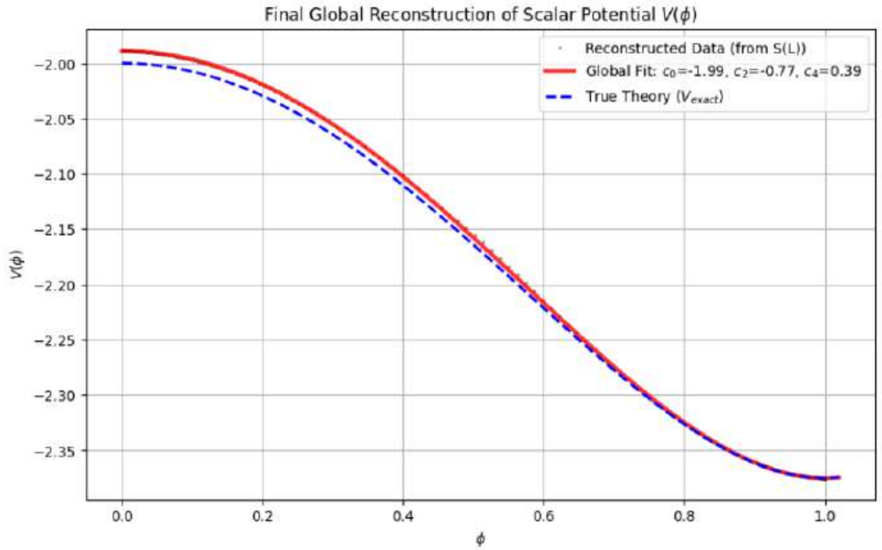


$$R_{UV} = 1,\, m_\phi = \sqrt{3}/2\,,\;\; \lambda = 3$$

[최종 라그랑지안 파라미터 역추적]

모델: V(phi) = c0 + c2*phi^2 + c4*phi^4

Parameter	Reconstructed	True Theory	Error
c0 (Vac)	-1.9889	-2.0000	0.56%
c2 (Mass)	-0.7714	-0.7500	2.85%
c4 (Int)	0.3852	0.3750	2.71%



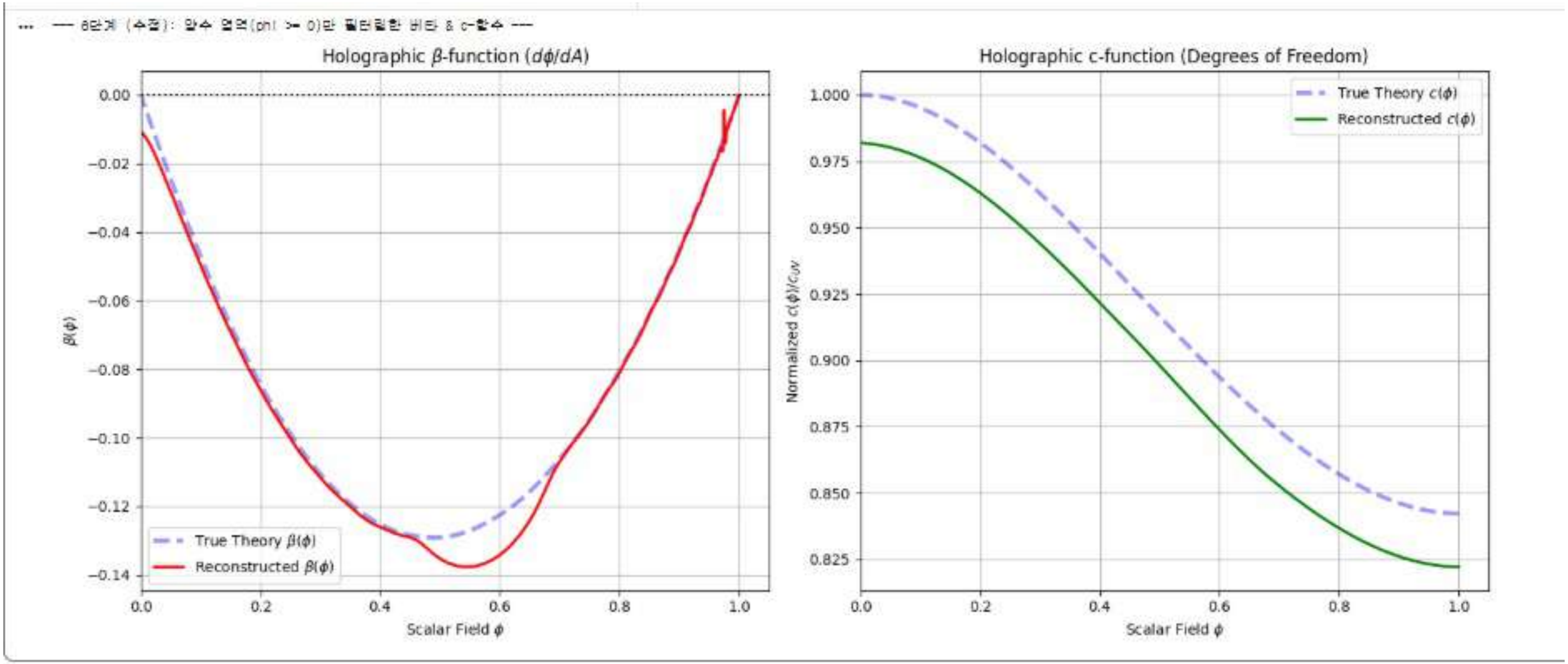
$$S = \frac{1}{16\pi G}\int d^3X\sqrt{-g}\left(\mathcal{R} - \frac{1}{2}\partial_M\phi\partial^M\phi - \frac{V(\phi)}{R_{UV}^2}\right)$$

with

$$V(\phi) = 2R_{UV}^2\Lambda_{UV} + \frac{M_\phi^2}{2}\phi^2 + \frac{\lambda}{4}\phi^4 = 2R_{UV}^2\Lambda_{UV} + \frac{\lambda}{4}\phi^2\left(\phi^2 - 2\frac{m_\phi^2}{\lambda}\right)$$

where $M_\phi^2 = -m_\phi^2 < 0$, $\lambda > 0$ and $\Lambda_{UV} = -1/R_{UV}^2$ with a UV AdS radius R_{UV}.

Beta-function (left) and c-function (right)



Conclusion

1. We reconstruct the black hole geometry from the entanglement entropy data of a thermal system.

2. We also find the dual gravity theory of a deformed CFT, which leads to the given entanglement entropy data.

3. In Future work, we try to reconstruct the dual gravity theory of the known CMT, like a transverse field Isim model.

Thank you!