*Hocheol Lee & Bogeun Gwak, arXiv:2511.13086 (2025)

Unbounded Radius of Innermost Stable Circular Orbit in Higher-Dimensional Black Holes

Hocheol Lee, Bogeun Gwak

Shanghai-APCTP-Sogang-GIST workshop on Gravity, Astroparticle, and Cosmology,

Dec. 28, 2025

Hocheol Lee

Dongguk University

dongguk UNIVERSITY

Overview of Our Results

- Based on Hocheol Lee & Bogen Gwak, arxiv:2511.13086
- Consider the D-dimensional Einstein-Hilbert Action & Anisotropic Matter
- Assume Spherically Symmetric D-dimensional Metric
- Calculate the Innermost Stable Circular Orbit

Overview of Our Results

Radial Pressure	Dimensions	$\mathcal{A}(\mathbf{r})$	$\mathbf{r}_{\mathbf{ISCO}}$
$p_{\mathrm{r}} > 0$	$D = 4$	≥ 0	$r_{\mathrm{ISCO}} \leq 6M$
	$5 \leq D \leq 7$	≥ 0	Unbounded
	$D \geq 8$	≥ 0	Unbounded
		< 0	Non-existent
$p_{\mathrm{r}} \leq 0$	$D = 4$	≥ 0	$r_{\mathrm{ISCO}} \leq 6M$
	$D \geq 5$	≥ 0	Unbounded

Review: Innermost Stable Circular Orbit

**Wikipedia: Innermost Stable Circular Orbit*

- **Innermost Stable Circular Orbit (ISCO) is the smallest marginally stable circular orbit in in which a test particle can stably orbit a massive object in General Relativity.**

$$\dot{x}^2 = -1 \quad \Rightarrow \quad \dot{r}^2 + V_{eff} = 0$$

1) Circular Motion:

$$\dot{r} = 0, \quad \ddot{r} = 0 \quad \Rightarrow \quad V_{eff} = 0, \quad V'_{eff} = 0$$

$$V''_{eff} > 0 \quad \text{(Stable)}$$

2) $V''_{eff} = 0$ **(Marginal, ISCO)**

$$V''_{eff} < 0 \quad \text{(Unstable)}$$

Review: Innermost Stable Circular Orbit

$$V_{eff}(r_{ISCO}) = V'_{eff}(r_{ISCO}) = V''_{eff}(r_{ISCO}) = 0$$

Review: Schwarzshcild Black Hole (Vacuum)

$$V_{eff}(r_{ISCO}) = -\frac{E^2}{2} + \frac{1}{2}\left(1 + \frac{L^2}{{r_{ISCO}}^2}\right)\left(1 - \frac{2M}{r_{ISCO}}\right)$$

$$E^2 = \frac{(r_{ISCO} - 2M)^2}{r_{ISCO}(r_{ISCO} - 3M)} \quad L^2 = \frac{r_{ISCO}^2 M}{r_{ISCO} - 3M}$$

$$V''_{eff}(r_{ISCO}) = 0 \quad \Rightarrow \quad r_{ISCO} = 6M$$

Review: Anisotropic Fluid

*Y. Cen & Y. Song PLB 866 (2025)

$$r_{ISCO} \leq 6M_{TOTAL}$$

$$M_{TOTAL} = \frac{r_h}{2} + 4\pi \int_{r_h}^{\infty} r^2 \rho(r) dr$$

Action & Metric

Action: $$S = \int d^D x \sqrt{-g}\left(\frac{R}{2\kappa} + \mathcal{L}_M\right)$$

Metric: $$ds^2 = -e^{2g(r)} f(r) dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega_{\mathrm{D}-2}^2$$

$$d\Omega_{\mathrm{D}-2}^2 = d\theta_1^2 + sin^2\theta_1 d\theta_2^2 + \cdots + sin^2\theta_1 \cdots sin^2\theta_{D-3} d\theta_{D-2}^2$$

Asymptotic Flatness: $$\lim_{r\to\infty} f(r) = 1 \qquad \lim_{r\to\infty} g(r) = 0$$

Energy-Momentum Tensor

$$\left(T^{\mu}{}_{\nu}\right) = diag\left(-\rho(r), p_r(r), \underbrace{p_t(r), \cdots p_t(r)}_{\text{D-2 Terms}}\right)$$

p_r : Radial Pressure

p_t : Transverse Pressure

Einstein Equations

$$tt: \quad 0 = f' - \frac{(D-3)(1-f)}{r} + \frac{2\kappa r\rho}{D-2}$$

$$rr: \quad 0 = g' - \frac{\kappa r(\rho + p_r)}{(D-2)f}$$

$$f'' = -\frac{(D-3)(rf' - f + 1)}{r^2} - \frac{2\kappa(r\rho' + \rho)}{D-2}$$

$$g'' = -\frac{\kappa\{r(\rho + p_r)f' - [r(\rho' + p_r') + \rho + p_r]f\}}{(D-2)f^2}$$

Einstein Equations

$$f(r_h) = 0 \qquad f'(r_h) \geq 0$$

$$0 = f' - \frac{(D-3)(1-f)}{r} + \frac{2\kappa r\rho}{D-2}$$

$$0 = g' - \frac{\kappa r(\rho + p_r)}{(D-2)f}$$

$$\rho(r_h) \leq \frac{(D-3)(D-2)}{2\kappa r_h^2} \qquad \rho(r_h) = -p_r(r_h)$$

Einstein Equations

$$f(r) \equiv 1 - \frac{2M(r)}{r^{D-3}}$$

$$M(r) = \frac{r_h^{D-3}}{2} + \frac{\kappa}{D-2}\int_{r_h}^{r} s^{D-2}\rho(s)ds$$

Finiteness: $\lim_{r\to\infty} r^{D-1}\rho(r) = 0$

Conservation Law

$$\nabla_\mu \mathrm{T}^{\mu}{}_{\mathrm{r}} = 0:$$

$$p_r' = \frac{2T + (D-1)\rho - (D+1)p_r}{2r} - \left(\frac{D-3}{2r} + \frac{\kappa r p_r}{D-2}\right)\frac{\rho + p_r}{f}$$

$$T \equiv T^{\mu}{}_{\mu}$$

Energy-Momentum Conditions

1) **Weak Energy Condition**

$$\rho \geq 0 \qquad \rho + p_r \geq 0 \qquad \rho + p_t \geq 0$$

2) **Non-Positive Trace:** $T \leq 0$

3) **Radial & Tangential Pressure**

$$p_t \geq 0 \qquad p_t \geq |p_r|$$

➡ **Dominant Energy Condition & Additional Conditions**

$$\rho \geq |p_r| \qquad \rho \geq |p_t| \qquad \rho \geq p_r + (D-2)p_t \geq 0$$

Current Status

- D-Dimensional Einstein-Hilbert Action

- D-Dimensional Sphericallly Symmetric Metric

- Anisotropic Fluid

- Dominant Energy Condition & Additional Conditions

Timelike Geodesic Equation

D-Velocity: $u = \{\dot{t}, \dot{r}, \dot{\theta}_1, \cdots, \dot{\theta}_{D-2}\}$

Polar Angles: (Equatorial Plane) $\theta_1 = \theta_2 = \cdots = \theta_{D-1} = \frac{\pi}{2}$

Azimuthal Angle: $\phi \equiv \theta_{D-2}$

$$E \equiv -\xi^{(t)}_{\mu} u^{\mu} = e^{2g} f \dot{t} \qquad L \equiv \xi^{(\phi)}_{\mu} u^{\mu} = r^2 \dot{\phi}$$

$$u^2 = -1 \quad \Rightarrow \quad \frac{1}{2}\dot{r}^2 + \frac{f}{2}\left(-\frac{E^2 e^{-2g}}{f} + \frac{L^2}{r^2} + 1\right) = 0$$

Effective Potential

$$V_{eff}(r) = -\frac{E^2}{2} e^{-2g(r)} + \frac{1}{2}\left(1 + \frac{L^2}{r^2}\right) f(r)$$

$$V_{eff} = V'_{eff} = 0: \quad E^2 = \left.\frac{2e^{2g} f^2}{2f - r(f' + 2fg')}\right|_{r=r_{ISCO}} \qquad L^2 = \left.\frac{r^3(f' + 2fg')}{2f - r(f' + 2fg')}\right|_{r=r_{ISCO}}$$

$$E^2 \geq 0$$
$$L^2 \geq 0$$

$$0 < 1 + \frac{2\kappa r^2 p_r}{(D-3)(D-2\,)}$$

Second Derivative of Effective Potential

$$0 = \frac{2\kappa^2 r^4 p_r(\rho - 3p_r)}{(D-2)^2} + \frac{\kappa r^2\{2fT + [(D-1)f + D - 3]\rho + [(5D-11)f - 7(D-3)]p_r\}}{D-2}$$
$$\left. -(D-3)(D-1)f^2 + (3D-7)(D-3)f - 2(D-3)^2 \right|_{r=r_{ISCO}}$$

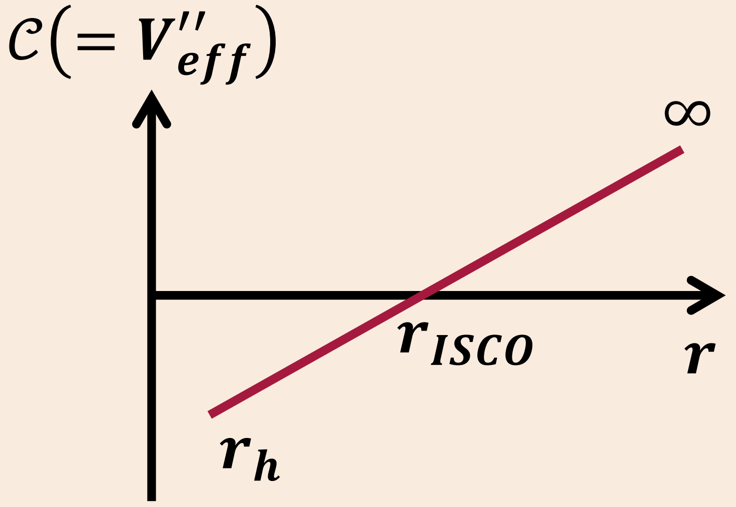
$$V''_{eff}(r) \equiv \mathcal{C}(r) = \frac{2\kappa^2 r^4 p_r(\rho - 3p_r)}{(D-2)^2} + \frac{\kappa r^2\{2fT + [(D-1)f + D - 3]\rho + [(5D-11)f - 7(D-3)]p_r\}}{D-2}$$
$$-(D-3)(D-1)f^2 + (3D-7)(D-3)f - 2(D-3)^2$$

$$\mathcal{C}(r) \equiv \mathcal{A}(r) + \mathcal{B}(r)$$

Second Derivative of Effective Potential

$$\mathcal{C}(r_h) = -\frac{2\left[2\kappa r^2\rho - (D-3)(D-2)\right]^2}{(D-2)^2} \leq 0$$

$$\mathcal{C}(r \to \infty) \sim \frac{2\kappa r\left[(D-3)\rho - (D-6)p_r + (D-2)p_t\right]}{D-2} \geq 0$$



$$\delta \equiv 1 + \frac{2\kappa r^2 p_r}{(D-3)(D-2)} > 0 \qquad \sigma \equiv p_r + p_t \geq 0$$

Result: Positive Radial Pressure ($p_r > 0$)

$$A(r) \geq \frac{4\kappa^2 r^4 p_r\{(D-4)(D-1)\rho + \left[(D-3)^2+2\right]p_r + (D-2)(D-1)p_t\}}{(D-3)(D-2)^2(D-1)}$$

$$+\frac{2\kappa r^2\{(D-4)(D-1)\rho + \left[(D-1)^2-6\right]p_r + (D-2)(D-1)p_t\}}{(D-2)(D-1)}$$

$$= \frac{2(D-4)\kappa r^2\delta}{D-2} + \frac{4\kappa^2 r^4 p_r\{\left[(D-3)^2+2\right]\sigma + 3(D-3)p\_t\}}{(D-3)(D-2)^2(D-1)}$$

$$+\frac{2(D-4)\kappa r^2\delta}{D-2} + \frac{4\kappa^2 r^4 p_r\{\left[(D-3)^2+2\right]\sigma + 3(D-3)p\_t\}}{(D-3)(D-2)^2(D-1)}$$

$$\geq \frac{2\kappa r^2[-(D-7)p_r + (D-2)(D-1)(p_t - p_r)]}{(D-2)(D-1)}$$

$$\geq -\frac{2(D-7)\kappa r^2 p_r}{(D-2)(D-1)}$$

Result: Positive Radial Pressure ($p_r > 0$)

$$A(r) \geq -\frac{2(D-7)\kappa r^2 p_r}{(D-2)(D-1)}$$

$$0 = C(r_{ISCO}) = A(r_{ISCO}) + B(r_{ISCO})$$

$D \leq 7$:

$$A(r) \geq 0 \qquad B(r_{ISCO}) \leq 0: \qquad \frac{2M(r_{ISCO})}{r_{ISCO}^{D-3}} \geq -\frac{D-5}{D-1}$$

$$D = 4: \qquad r_{ISCO} \leq 6M(r_{ISCO}) \leq 6M(r \to \infty)$$

$$5 \leq D \leq 7: \qquad r_{ISCO} > 0$$

$D \geq 8$:

$$A(r) \geq 0: \qquad r_{ISCO} > 0$$

$$A(r) < 0: \qquad r_{ISCO} < -\frac{2(D-1)M(r_{ISCO})}{D-5} < 0$$

Result: Non-Positive Radial Pressure ($p_r \leq 0$)

$$A(r) \geq \frac{4\kappa^2 r^4 p_r\{(D-4)\rho + Dp_r + (D-2)p_t\}}{(D-3)(D-2)^2} + \frac{2\kappa r^2\{(D-4)\rho + (D-6)p_r + (D-2)p_t\}}{D-2}$$

$$= \frac{2\kappa r^2 \delta}{D-2}[(D-4)\rho + (D-2)\sigma] + 2(D-3)(D-3-\delta)(1-\delta)$$

$$\geq 0$$

$$B(r_{ISCO}) \leq 0: \quad \frac{2M(r_{ISCO})}{r_{ISCO}^{D-3}} \geq -\frac{D-5}{D-1}$$

$$D = 4: \quad r_{ISCO} \leq 6M(r \to \infty)$$

$$D \geq 5: \quad r_{ISCO} > 0$$

Overview of Our Results

Radial Pressure	Dimensions	$\mathcal{A}(\mathbf{r})$	$\mathbf{r}_{\text{ISCO}}$
$p_{\mathrm{r}} > 0$	$D = 4$	≥ 0	$r_{\text{ISCO}} \leq 6M$
	$5 \leq D \leq 7$	≥ 0	Unbounded
	$D \geq 8$	≥ 0	Unbounded
		< 0	Non-existent
$p_{\mathrm{r}} \leq 0$	$D = 4$	≥ 0	$r_{\text{ISCO}} \leq 6M$
	$D \geq 5$	≥ 0	Unbounded

Summary & Conclusion

The radius of innermost circular orbit is unbounded or non-existent for $D \geq 5$.

Thank you!

The radius of innermost circular orbit is unbounded or non-existent for $D \geq 5$.