Comments on dynamical gauge fields in holography

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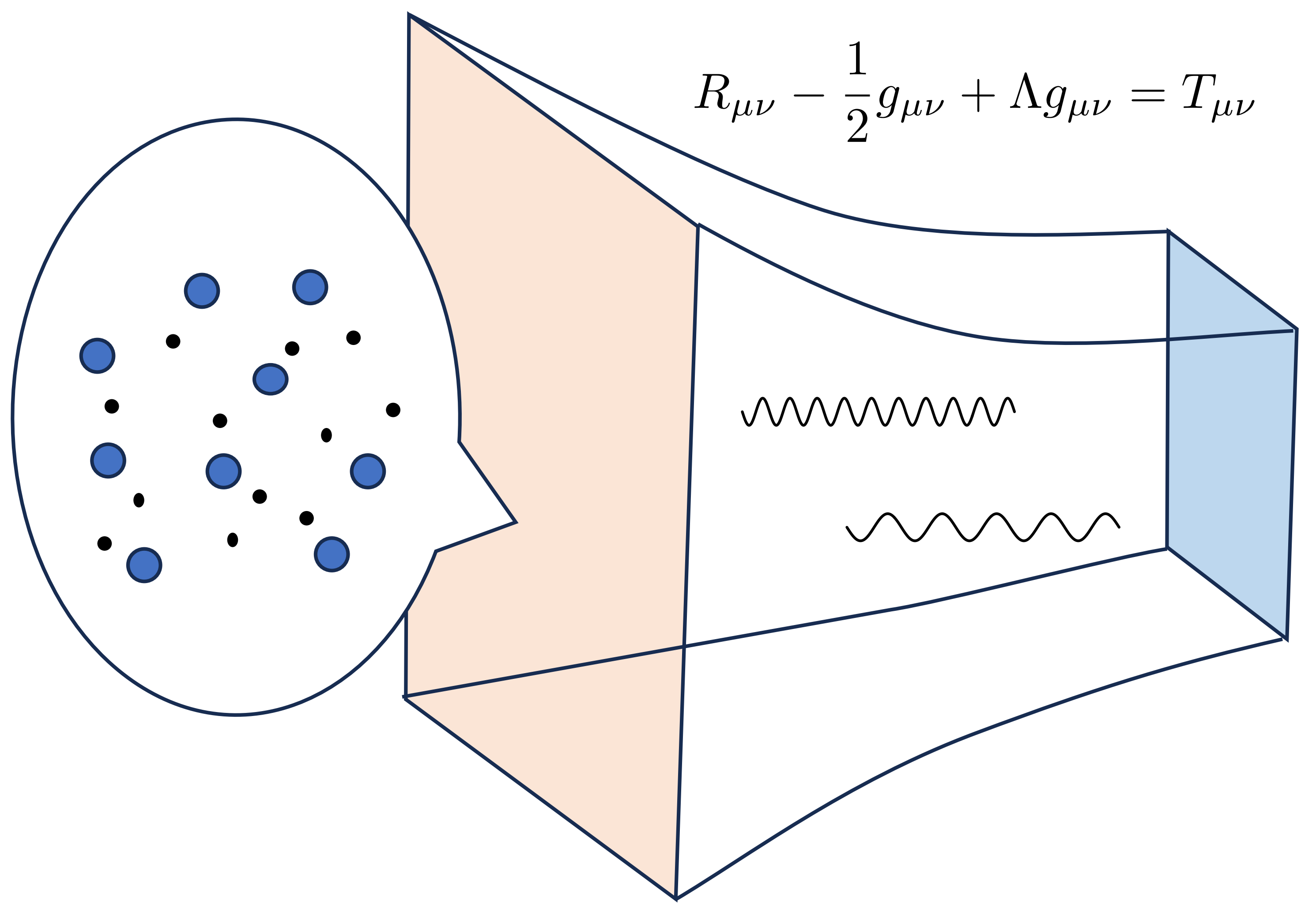
- In holography, U(1) gauge fields are usually not dynamical in the boundary theory.
- Bulk local symmetry = Boundary global symmetry
- On the other hand, the double-trace deformation is utilized to make the boundary gauge fields dynamical.

- I revisit and review this method.



Bulk	Boundary
Diffeo.	Lorentz
U(1) gauge	U(1) global

Holographic dictionary

- In asymptotic AdS, fields (e.g. scalar) behave as

$$\phi(z) = z^{d-\Delta}\phi^{(0)} + z^{\Delta}\phi^{(\nu)} + \cdots$$

The leading term is non-normalizable mode, its coefficient is source.

The sub-leading term is normalizable mode, its coefficient is VeV.

(called standard quantization)

- In some cases, both terms are normalizable.
Then, one can consider alternative quantization.

What does “normalizable” mean?

- That is in the sense of Klein-Gordon inner product,

$$(\Phi_1, \Phi_2) = \int_\Sigma \sqrt{g_\Sigma}\frac{i}{2}n^\mu(\Phi_1^*\partial_\mu\Phi_2 - \Phi_2\partial_\mu\Phi_1^*), \quad \begin{cases} \Sigma & \text{:spacelike surface} \\ n^\mu & \text{:timelike normal} \end{cases}$$

- Consider mode expansion $\Phi = e^{ik\cdot x}\phi(r)$

$$(\Phi_1, \Phi_2) = (2\pi)^{d-1}\delta^{(d-2)}(\vec{k}_2 - \vec{k}_1)(\omega_1 + \omega_2)e^{i(\omega_1-\omega_2)t}\int \mathrm{d}r\, \sqrt{g/g_{tt}^2}\phi_1^*\phi_2,$$

Strum-Liouville inner product

- In simple cases, the bulk integral becomes SL inner product

$$(\phi_1, \phi_2)_{\mathrm{SL}} = \int \mathrm{d}r \, w(r) \phi_1^*(r) \phi_2(r).$$

SL problem is given by

$$\left[\frac{\mathrm{d}}{\mathrm{d}r} \left(p(r) \frac{\mathrm{d}}{\mathrm{d}r} \right) + \lambda w(r) - q(r) \right] \phi(r) = 0, \quad \lambda : \text{eigenvalue.}$$

- We can evaluate it by

$$(\phi_m, \phi_n)_{\mathrm{SL}} = \frac{1}{\lambda_m - \lambda_n} p(r) \left(\underbrace{\phi_m^*(r) \phi_n'(r) - \phi_n(r) \phi_m'^*(r)}_{\text{Wronskian}} \right) \Big|_{\partial}$$

$$\mathcal{W}(\phi_m, \phi_n) := \phi_m^*(u) \phi_n'(u) - \phi_n(u) \phi_m^{*\prime}(u)$$

Scalar fields in AdS$_{d+1}$

- Let's consider a basic example: massive scalar in planer AdS_d+1.

$$S = \frac{1}{2}\int \mathrm{d}^d x\,\sqrt{-g}\left[-(\partial\phi)^2 - m^2\phi^2\right],$$

$$\mathrm{d}s^2 = L^2\frac{-\,\mathrm{d}t^2 + \mathrm{d}x_{d-1}^2 + \mathrm{d}u^2}{u^2}.$$

$$(L=1)$$

A massive scalar field in AdS_{d+1} obeys

$$z^2\partial_z^2\phi - (d-1)z\partial_z\phi - m^2\phi - k^2z^2\phi = 0.$$

The SL coefficients read

$$\lambda = -k^2, \quad p(z) = z^{1-d}, \quad q(z) = m^2 z^{-1-d}, \quad w(z) = z^{1-d}.$$

The asymptotic expansion is

$$\phi(z) = z^{d/2-\nu}\left(\phi_{(0)} + z^2\phi_{(1)} + z^{2\nu}\phi_{(\nu)} + \cdots\right), \quad \nu = \sqrt{\frac{d^2}{4} + m^2}.$$

Note: When $2\nu \in \mathbb{Z}$, logarithmic terms must appear in the expansion.

A regular solution with spacelike momentum

$$\phi(z) = 2^{1-\nu}\mathcal{C} z^{d/2} K_\nu(kz), \quad k = \sqrt{\eta^{ij} k_i k_j}.$$

The leading contribution from the AdS boundary becomes

$$\frac{p(z)\mathcal{W}(\phi_1^*, \phi_2)}{-k_1^2 + k_2^2} = |\mathcal{C}|^2 \left\{ \frac{z^{2-2\nu}}{2(1-\nu)} k_1^{-\nu} k_2^{-\nu} \Gamma(\nu)^2 + 2^{1-2\nu} \frac{k_1^\nu k_2^{-\nu} - k_1^{-\nu} k_2^\nu}{k_1^2 - k_2^2} \nu \Gamma(\nu) \Gamma(-\nu) + \cdots \right\}.$$

For norm ($\phi_1 = \phi_2$)

$$= |\mathcal{C}|^2 \left\{ \frac{z^{2-2\nu}}{2(1-\nu)} k^{-2\nu} \Gamma(\nu)^2 + \frac{2^{1-2\nu}}{k^2} \nu^2 \Gamma(\nu) \Gamma(-\nu) + \cdots \right\}.$$

This term diverges if $\nu > 1$.

In more general form,

$$\frac{p(z)\mathcal{W}(\phi_1^*, \phi_2)}{-k_1^2 + k_2^2} = \frac{1}{2(1-\nu)} z^{2-2\nu} \phi_1^{(0)} \phi_2^{(0)} - \frac{2\nu}{k_1^2 - k_2^2} (\phi_1^{(0)} \phi_2^{(\nu)} - \phi_2^{(0)} \phi_1^{(\nu)}) + \cdots$$

This term diverges if $\nu > 1$. $\nu = \sqrt{\frac{d^2}{4} + m^2}$

Recall $\phi(z) = z^{d/2-\nu} \left(\phi^{(0)} + z^2 \phi^{(1)} + z^{2\nu} \phi^{(\nu)} + \cdots \right).$

In this sense, $\phi^{(0)}$ represents non-normalizable mode.

(The contribution from another side (horizon) is finite.)

Mass window

The condition for the convergence:

$$\nu = \sqrt{\frac{d^2}{4} + m^2L^2} < 1$$

Scaling dimension: $\Delta = \frac{d}{2} + \sqrt{\frac{d^2}{4} + m^2L^2}.$

Reality condition = Breitenlohner-Freedman bound

$$\frac{d^2}{4} + m^2L^2 \geq 0$$

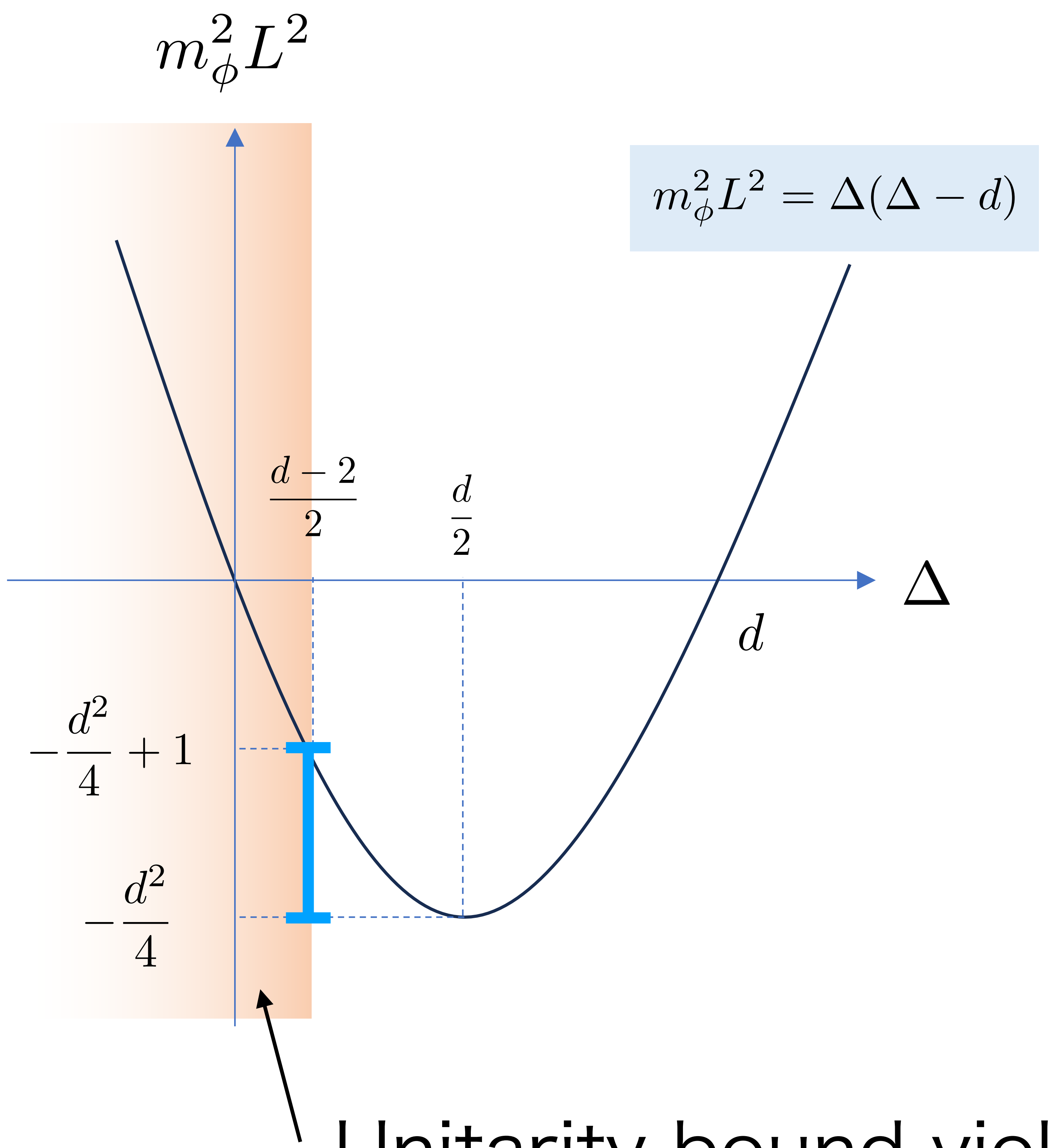
$$-\frac{d^2}{4} \leq m^2L^2 < -\frac{d^2}{4} + 1$$

(Bulk) mass window for alternative quantization

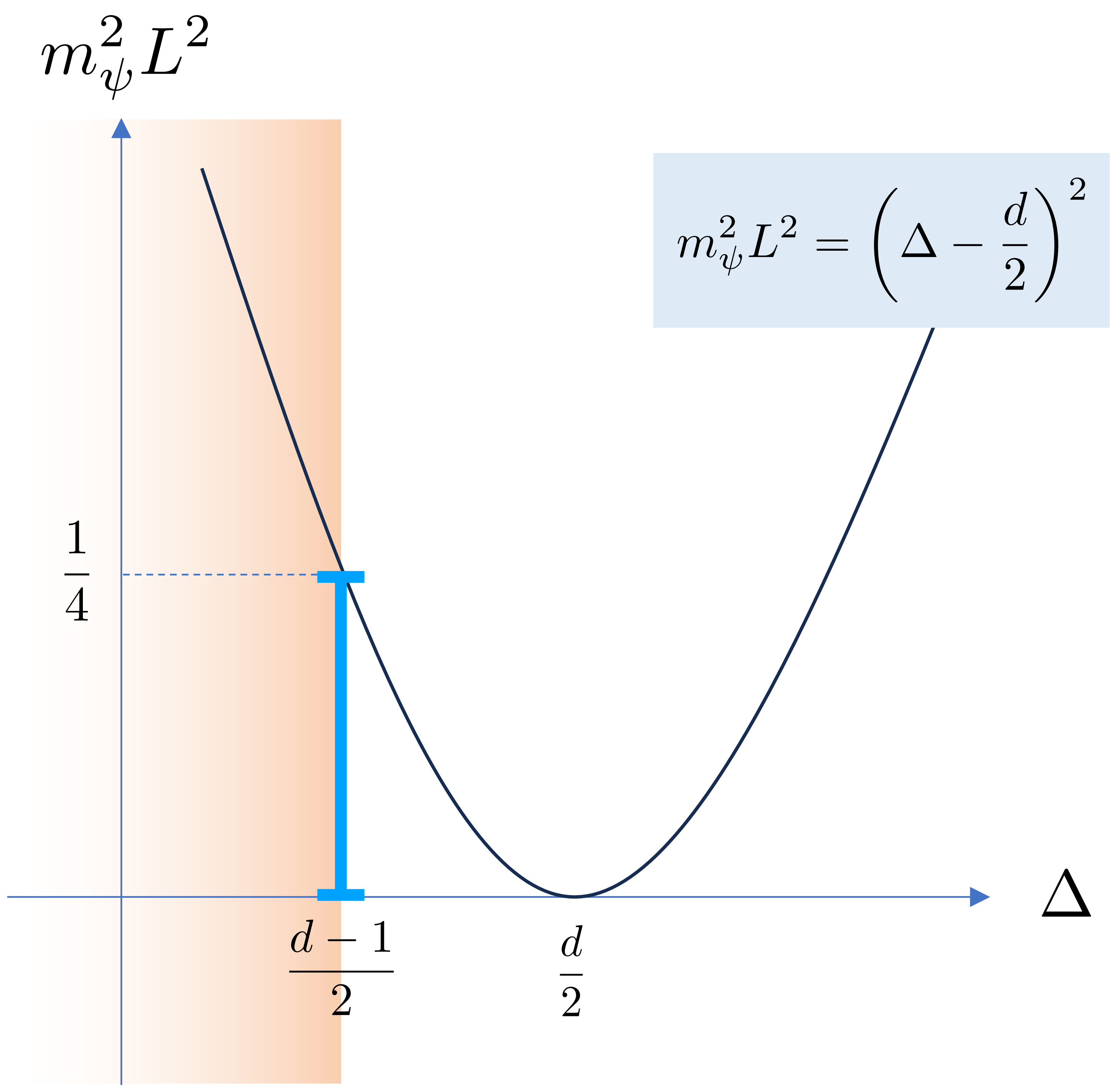
~ unitarity bound in CFT

Mass window for various spin

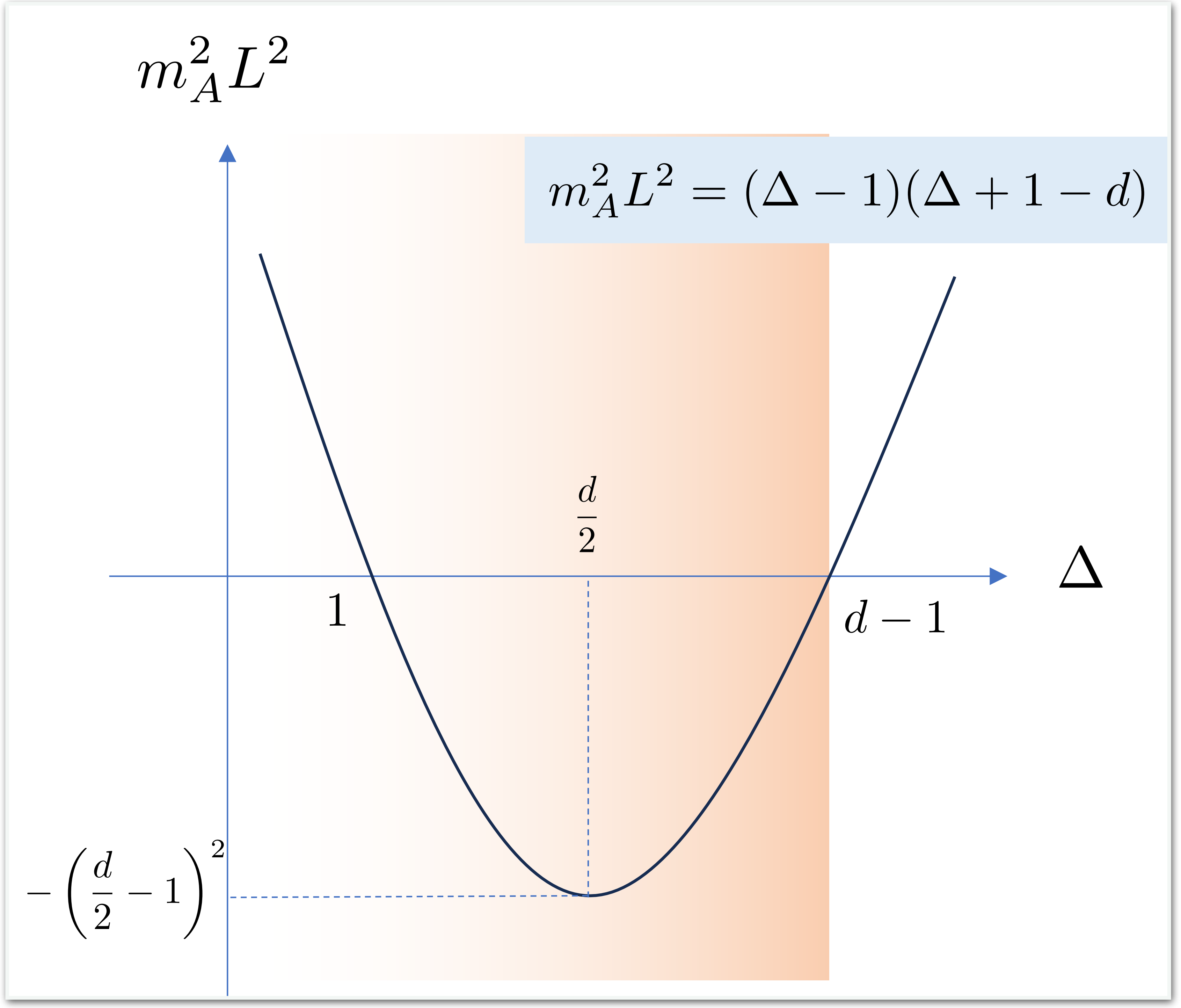
Scalar



Spinor



Vector



Unitarity bound violated

Note: The bound depends on spin.

No range allows alternative quantization

Notes

- Holographic models require holographic renormalization by adding coutertems.
- In the case of non-normalizable modes exist, the counterterms must include kinetic terms.
- If one considers the KG inner product of this kinetic term, it precisely cancel the divergence of bulk inner product.
- However, such a prescription leads to ghosts.

Andrade, Marolf (2011) [1105.6337]

Andrade, Marolf (2011) [1112.3085]

Gauge fields in AdS5

Bulk action: $$S_{\text{bulk}} = \int \mathrm{d}^5x \sqrt{-g} \left(-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right) ,$$

Counterterms: $$S_{\text{ct}} = \int_\epsilon \mathrm{d}^4x \sqrt{-\gamma} \left(-\frac{1}{4} \log\left(\frac{\epsilon}{u_{\text{ct}}} \right) F_{mn} F^{mn} \right) .$$

Asymptotic expansion: $$A_m = \alpha_m \left(1 + \frac{k^2}{2} u^2 \log \frac{u}{u_{\text{ex}}} \right) + \beta_m u^2 + \cdots .$$

Variation of the renormalized on-shell action:

$$\delta(S_{\text{bulk}} + S_{\text{ct}}) = \int \frac{\mathrm{d}^4k}{(2\pi)^4} \underbrace{\left[2\beta^m(k) + k^2 \left(\frac{1}{2} + \log \frac{u_{\text{ct}}}{u_{\text{ex}}} \right) \alpha^m(k) \right]}_{\text{Response}} \underbrace{\delta\alpha_m(-k)}_{\text{Source}}$$

The double-trace deformations

For this, we additionally consider

E.g., Mauri, Stoof [1811.11795]

$$S_{\text{fin}} = \int \mathrm{d}^4x \left[-\frac{1}{4\lambda} f^{mn} f_{mn} + \alpha_m J^m_{\text{ext}} \right], \quad f_{mn} := \partial_m \alpha_n - \partial_n \alpha_m .$$

Total variation:

$$\delta(S_{\text{ren}} + S_{\text{fin}}) = \int \mathrm{d}^4x \left[\underbrace{\left(\frac{1}{\lambda} \nabla_m f^{mn} + \Pi^n + J^n_{\text{ext}} \right)}_{=0 \ \text{(imposed)}} \delta\alpha_m + \underline{\alpha_m}\, \underline{\delta J^m_{\text{ext}}} \right].$$

VeV Source

⬇

New external source: $J^n_{\text{ext}} = -\frac{1}{\lambda} \partial_m f^{mn} - \Pi^n.$

$$\Pi^n := \frac{\delta S_{\text{ren}}}{\delta \alpha_n}.$$

We can rewrite it in terms of the gaige invariants:

$$J^t_{\mathrm{ext}} = \frac{-ik_x}{k_x^2-\omega^2}\left(2Z_1^{(S)} - (k_x^2-\omega^2)Z_1^{(L)}\left(\frac{1}{\lambda} - \log\frac{u_{\mathrm{ct}}}{u_{\mathrm{ex}}} - \frac{1}{2}\right)\right),$$

$$J^x_{\mathrm{ext}} = \frac{-i\omega}{k_x^2-\omega^2}\left(2Z_1^{(S)} - (k_x^2-\omega^2)Z_1^{(L)}\left(\frac{1}{\lambda} - \log\frac{u_{\mathrm{ct}}}{u_{\mathrm{ex}}} - \frac{1}{2}\right)\right),$$

$$J^y_{\mathrm{ext}} = \frac{1}{-i\omega}\left(2Z_2^{(S)} - (k_x^2-\omega^2)Z_2^{(L)}\left(\frac{1}{\lambda} - \log\frac{u_{\mathrm{ct}}}{u_{\mathrm{ex}}} - \frac{1}{2}\right)\right),$$

where

$$Z_1 = i\omega A_x(u) + ik_x A_t(u), \quad Z_2 = i\omega A_y(u).$$

By solving $J_{\mathrm{ext}} = 0$, we obtain

$$\frac{1}{\lambda} = \frac{2}{k_x^2-\omega^2}\frac{Z_\alpha^{(S)}}{Z_\alpha^{(L)}} + \frac{1}{2} + \log\frac{u_{\mathrm{ct}}}{u_{\mathrm{ex}}}.$$

AdS5

The solution in AdS5 is

$$Z_\alpha = \mathcal{C} \times \begin{cases} uH_1^{(1)}(u\sqrt{\omega^2 - k_x^2}) & \omega^2 - k_x^2 > 0,\ \omega \geq 0, \\ uH_1^{(2)}(u\sqrt{\omega^2 - k_x^2}) & \omega^2 - k_x^2 > 0,\ \omega < 0, \\ uK_1(u\sqrt{k_x^2 - \omega^2}) & \omega^2 - k_x^2 \leq 0. \end{cases}$$

For spacelike momentum, we obtain

$$\boxed{\frac{1}{\lambda} = \gamma_{\mathrm{E}} + \log u_{\mathrm{ct}} + \frac{1}{2}\log\frac{k^2}{2}.}$$

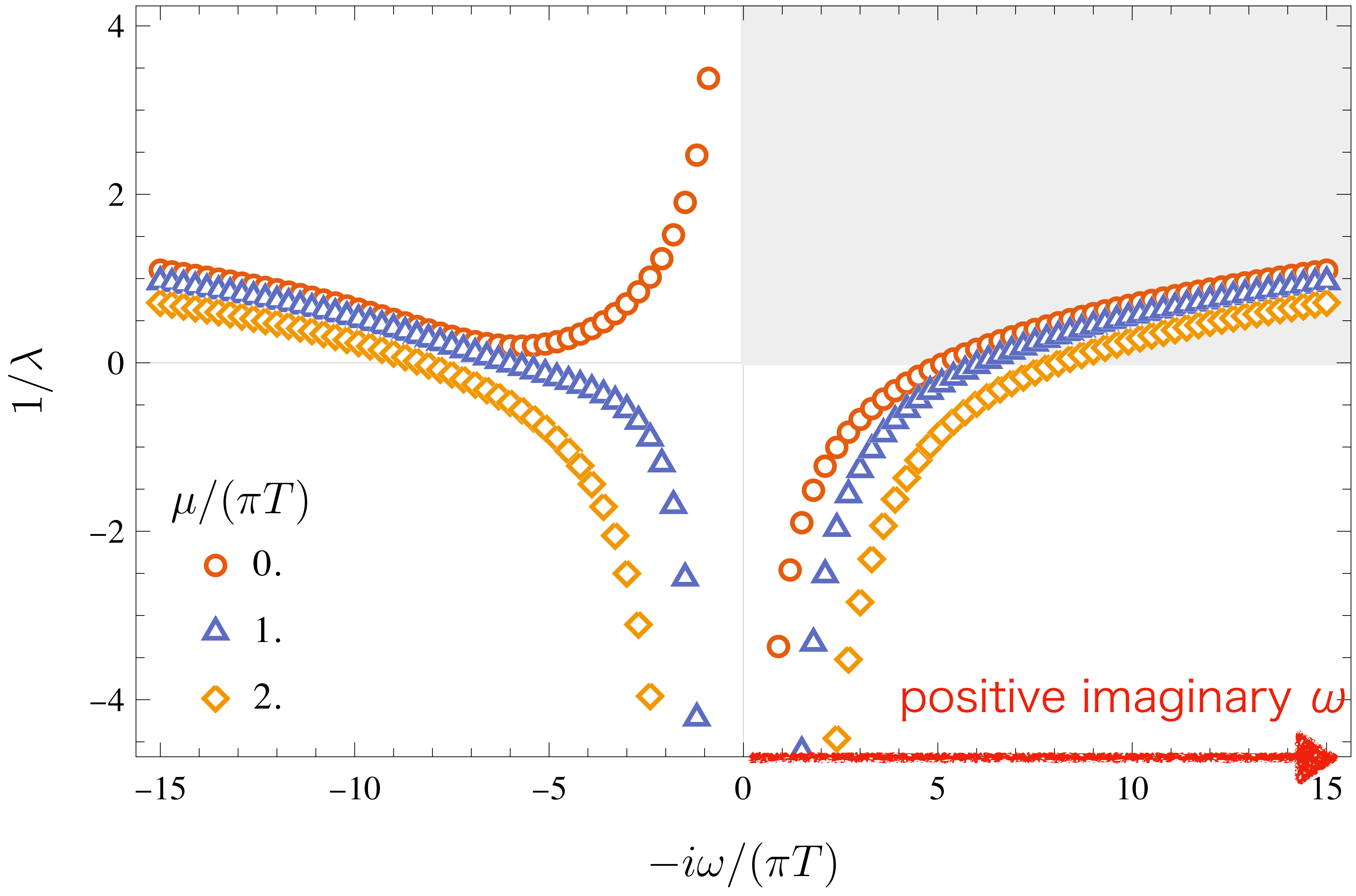
For given k, we find real-valued λ.

Implying tachyonic mode under some values of real λ.

Similar issues:

Hofman, Iqbal [1707.08577]

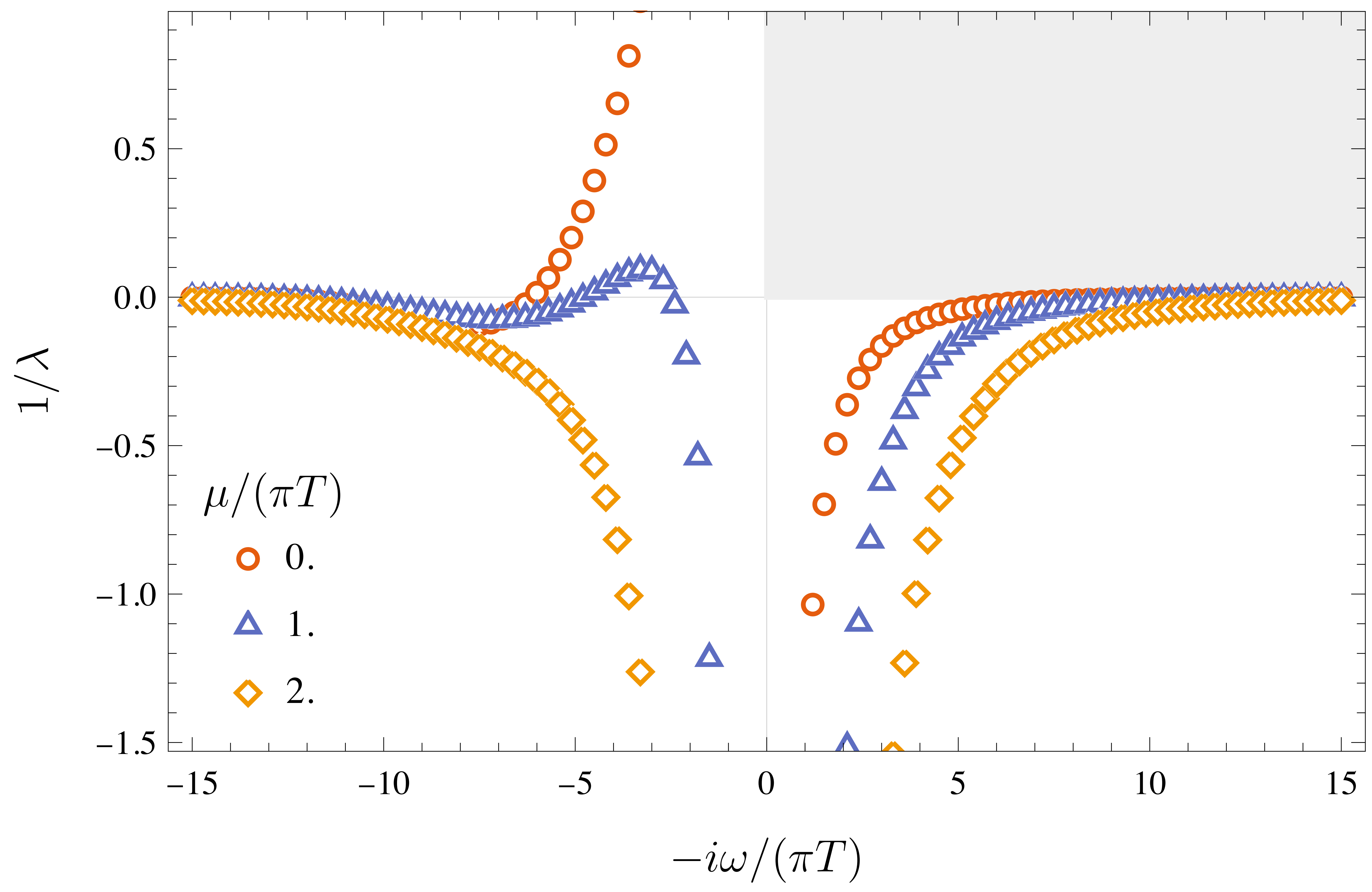
RNAdS5



The same problem exists in the case of RNAdS5.

The constant u_{ct} gives just a vertical offset of the plot.

Ad-hoc fix?



If we set $u_{\rm ct} = 2e^{-\gamma_{\rm E}}/|k|$,

we can remove the unstable modes.

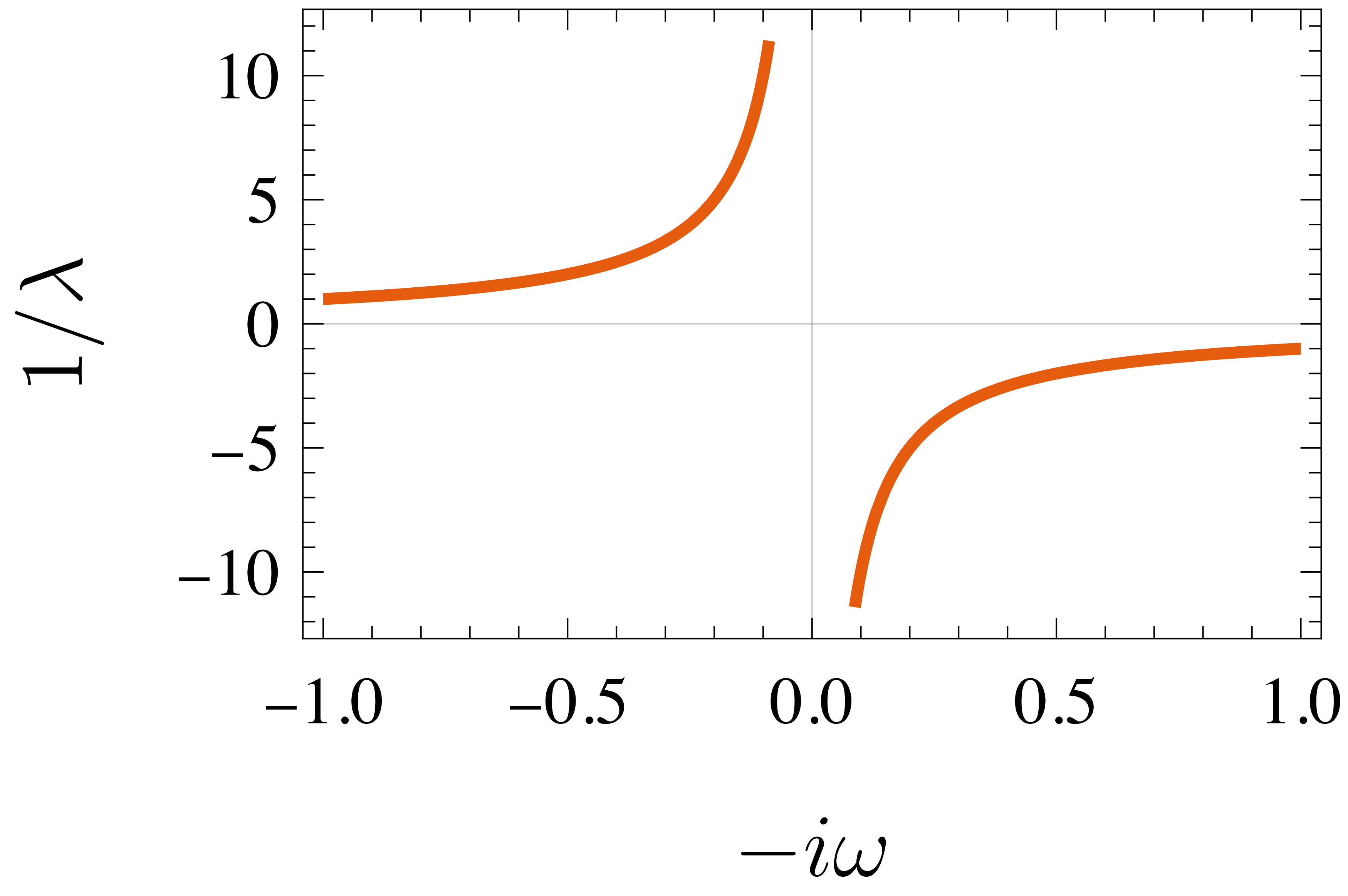
- k-dependent scale in counterterms?
- Still unstable for finite momentum.

Gauge fields in AdS4

SAdS4

In Schwarzschild AdS4, one can obtain

$$\frac{1}{\lambda} = \frac{1}{i\omega}$$



So, it is stable for positive λ.

But, it is independent of T.

Is the zero-T limit safe?

Summary

- The dual-operators of the gauge fields always violate the unitarity bound.
- As a result, the leading term is always non-normalizable.
- For AdS5, the logarithmic divergence appears.
- In practice, we observe tachyonic modes, whose frequencies depend on the arbitrary cut-off scale.

- After all, is this pathological?
  Is there a way to fix these problem?