Gravitational charges and radiation in asymptotically locally de Sitter spacetimes

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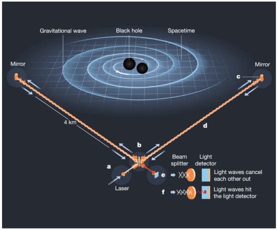
Introduction

- New era of gravitational wave astronomy **[LIGO '15]**

- We want to understand GWs using the mathematics of spacetime charges **[Pirani '57]**



- Asymptotically flat spacetimes **[Trautman '58, BMS '62, Penrose '65]** possess a positive, monotonically decreasing Bondi mass

$$\partial_u \mathcal{M}_B = -\frac{1}{32\pi} \int_{S^2_\infty} N_{AB} N^{AB} \leq 0$$

- More generally, diffeomorphism invariance $\implies$ charges must be quantities defined at spacetime boundaries

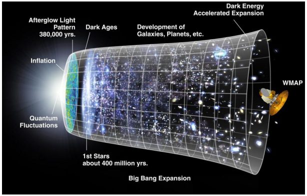
- Need to deal with divergences due to ∞-volume of spacetime

Background & motivation

- In asymptotically flat spacetime one defines charges relative to flat spacetime, including fluxes due to gravitational waves **[Arnowitt-Deser-Misner '59, Bondi-Metzner-Van der Burg '62, Sachs '62]**
- In asymptotically (locally) AdS spacetimes, holographic renormalisation **[de Haro-Skenderis-Solodukhin '00]** unambiguously treats divergences **[Papadimitriou-Skenderis '04, '05]**

This talk: focus on asymptotically (locally) de Sitter spacetimes **[Anninos et. al '10, Ashtekar et. al '15-'19, Hoque et al. '15-'25, Chruściel et. al '16, '20, Compère-Fiorucci-Ruzziconi '19, '20, Fernández-Senovilla '19-'24, Lewandowski et al. '20-'24, Bonga-Bunster-Perez '23]**

- Subtle: No global timelike Killing vector, aformentioned divergences
- Important: Inflation, GWs in cosmology

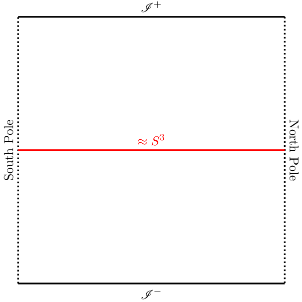


Field equations and causal structure

- Vacuum Einstein equations in the presence of a positive cosmological constant:

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu} = 0, \qquad \Lambda = \frac{3}{\ell^2} > 0$$

- Study the asymptotic structure using conformal compactification **[Penrose '63]**
- $\bar{g}_{\mu\nu} = \Omega^2 g_{\mu\nu}, \quad \Omega|_{\mathscr{I}} = 0, \quad \bar{n}_\mu = \partial_\mu \Omega|_{\mathscr{I}} \neq 0$
- In particular: $\bar{n}_\mu \bar{n}^\mu|_{\mathscr{I}} = -\ell^{-2} \implies \mathscr{I}$ is spacelike
- The conformal boundary is split into two disjoint components $\mathscr{I} = \mathscr{I}^+ \sqcup \mathscr{I}^-$



Asymptotically locally de Sitter (AldS) spacetimes

All conformally compact Einstein spacetimes ($\Lambda > 0$) can be written in a neighbourhood of $\mathscr{I}^{\pm}$ in the following form **[Starobinsky '83]**

$$ds^2 = \frac{3}{\Lambda}\left[-\frac{d\rho_{\pm}^2}{\rho_{\pm}^2} + \gamma_{ij}^{\pm} dx^i dx^j\right],$$

where

$$\gamma_{ij}^{\pm} = \frac{1}{\rho_{\pm}^2}\left(g_{(0)ij}^{\pm} + \rho^2 g_{(2)ij}^{\pm} + \rho^3 g_{(3)ij}^{\pm} + \mathcal{O}(\rho^4)\right).$$

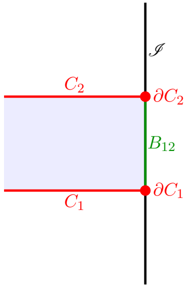
- $\mathscr{I}^{\pm}$ is located at $\rho_{\pm} = 0$
- $g_{(0)ij}^{\pm}$ is a representative of the conformal class $[g_{(0)}^{\pm}]$ at $\mathscr{I}^{\pm}$
- If $\mathscr{I}^{\pm} \approx \{S^3, \mathbb{R}^3, \mathbb{R} \times S^2\}$ and $g_{(0)ij}^{\pm}$ is conformally flat the the spacetime is asymptotically dS at $\mathscr{I}^{\pm}$, otherwise it is merely asymptotically locally dS at $\mathscr{I}^{\pm}$.
- $g_{(3)ij}^{\pm} = -\frac{2\kappa^2}{3\ell} T_{ij}^{\pm}$ where $T_{ij}^{\pm}$ is (the analytic continuation of) the energy-momentum tensor $(T_i^{\pm i} = 0, \nabla_{(0)}^{\pm}{}^{i} T_{ij}^{\pm} = 0)$ **[Skenderis '02]**.

Charges in asymptotically locally anti-de Sitter ($\Lambda < 0$)

Use the covariant phase space formalism **[Wald et al. '90-'99]** (alternatively Noether's 1st or 2nd theorem) to construct conserved charges in AlAdS spacetimes **[Papadimitriou-Skenderis '05]**

$$H_{\tilde{\xi}} = \int_{\partial C} d\tilde{\sigma}_i \tilde{T}^i_j \tilde{\xi}^j_{(0)}$$

- Slices C are partial Cauchy surfaces
- Asymptotic symmetries $\delta_{\tilde{\xi}}[\tilde{g}_{(0)}] = 0 \implies \tilde{\xi}$ asymptotic conformal Killing vectors: $\tilde{\nabla}^{(0)}_{(i} \tilde{\xi}^{(0)}_{j)} = \frac{1}{3} \tilde{g}^{(0)}_{ij} \tilde{\nabla} \cdot \tilde{\xi}^{(0)}$



Charges obey a conservation law:

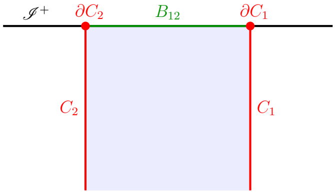
$$H_\xi \big|_{C_2} = H_\xi \big|_{C_1}$$

dS charges at $\mathscr{I}^+$

We impose the following boundary conditions at $\mathscr{I}^+$ (drop $\pm$ script)

$$\delta g_{(0)ij} = 2\left(\sigma g_{(0)ij} + \nabla_{(0)(i}\zeta_{j)}\right)$$

variational problem is well-posed ($\delta S_{\text{ren}} = 0 \Leftrightarrow \mathbf{E} = 0$)



- Asymptotic symmetries are vectors ξ which asymptote to boundary diffeomorphisms $\xi^i_{(0)}$
- Slices C are timelike hypersurfaces

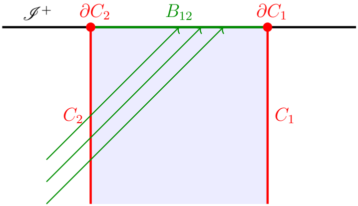
When $\xi_{(0)}$ a conformal Killing vector (CKV): $\nabla^{(0)}_{(i}\xi^{(0)}_{j)} = \frac{1}{3}g_{(0)ij}\nabla\cdot\xi^{(0)}$

$$H_\xi\big|_C = \int_{\partial C} d\sigma_i\, T^i_j \xi^j_{(0)} = Q_\xi(C)$$

- Conservation of charges: $H_\xi\big|_{C_2} - H_\xi\big|_{C_1} = 0$ - spatial conservation!

dS Fluxes at $\mathscr{I}^+$

$g_{ij}^{(0)}$ has no CKVs $\implies \nexists\, H_\xi$. Need modification **[Wald-Zoupas '99]**



- Same expression for modified charge, $\mathcal{H}_\xi$, now $\xi_{(0)} \in \text{Diff}(\mathscr{I}^+)$

$$\mathcal{H}_\xi\big|_C = \int_{\partial C} d\sigma_i \, T^i_j \xi^j_{(0)} = Q^+_\xi(C)$$

- Flux through the spacelike hypersurface $B_{12} \subset \mathscr{I}^+$

$$\mathcal{H}_\xi\big|_{C_2} - \mathcal{H}_\xi\big|_{C_1} = -\int_{B_{12}} \mathbf{F}_{\xi_{(0)}} = \frac{1}{2}\int_{B_{12}} \sqrt{g^{(0)}} T^{ij} \mathcal{L}_{\xi_{(0)}} g^{(0)}_{ij} \, d^3x$$

Piecewise CKVs interpolate the two cases discussed.

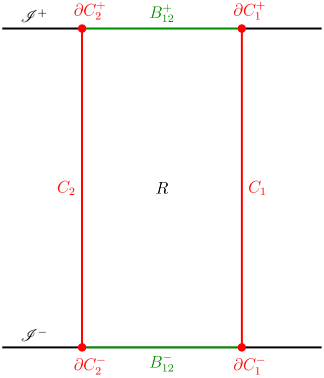
Temporally conserved quantities [AP-Skenderis-Taylor '21, '25]

- Locally, analytic continuation allows us to use techniques from AdS
- Globally, AldS are different. We need to include the past contribution of the timelike slice C

A first simple application is to take $\partial C^- \subset \mathscr{I}^-$ and consider ξ an asymptotic conformal Killing vector of both $\mathscr{I}^\pm$

- $H_\xi\big|_C = Q^+_\xi(C) - Q^-_\xi(C)$
- Spatially conserved charges:
  $H_\xi\big|_{C_2} = H_\xi\big|_{C_1}$

  $$\Delta Q^+_\xi(C_2, C_1) = \Delta Q^-_\xi(C_2, C_1)$$
- ΔQ_ξ provides a notion of a temporally conserved quantity
- In fact: $\Delta Q^+_\xi = \Delta Q^-_\xi = 0$ (spatial conservation at each end)



Two-ended fluxes

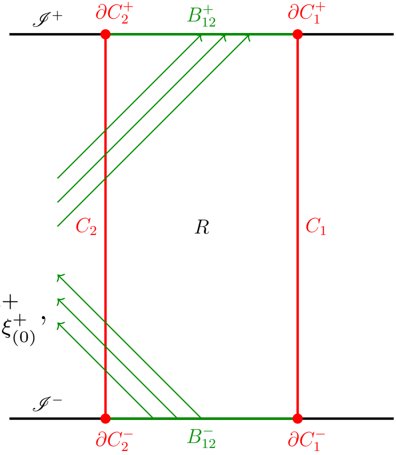
Criterion: An AldS spacetime may have non-trivial gravitational radiation at $\mathscr{I}^{\pm}$ when $\nexists$ a solution to $\nabla^{\pm}_{(0)(i}\xi^{\pm}_{(0)j)} = \frac{1}{3} g^{\pm}_{(0)ij} \nabla^{\pm}_{(0)} \cdot \xi^{\pm}_{(0)}$

- Fluxes at $\mathscr{I}^{+}$ **[Anninos et al. '11, Compère et al. '20, Kolanowski-Lewandowski '21]** previously established
- Flux through the (two component) spacelike hypersurface B_{12}

$$\mathcal{H}_{\xi}|_{C_2} - \mathcal{H}_{\xi}|_{C_1} = \int_{B^{-}_{12}} \mathbf{F}^{-}_{\xi^{-}_{(0)}} - \int_{B^{+}_{12}} \mathbf{F}^{+}_{\xi^{+}_{(0)}},$$

where

$$\mathbf{F}^{\pm}_{\xi^{\pm}_{(0)}} = -T^{ij}_{\pm} \nabla^{\pm}_{(0)(i}\xi^{\pm}_{(0)j)} \boldsymbol{\epsilon}^{\pm}_{(0)}$$



Asymptotically locally dS example - Robinson-Trautman

- [Robinson-Trautman '60] metrics admit a geodesic null congruence with zero twist and shear and non-vanishing divergence
- Recent interest [de Freitas-Reall '14, Bakas-Skenderis '14, Skenderis-Withers '16, Adami et al. '24, Arenas et al. '25] (hydro expansion, AdS grav. waves)

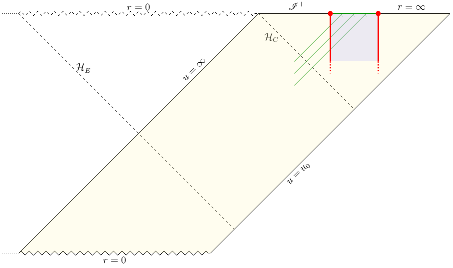
$$ds^2 = -\left(r\partial_u\Phi - \Delta\Phi - \frac{2m}{r} - \frac{\Lambda}{3}r^2\right)du^2 - 2dudr + 2r^2e^{\Phi}dzd\bar{z},$$

where $\Delta = e^{-\Phi}\partial_z\partial_{\bar{z}} = \frac{1}{2}\nabla^2$.

- Einstein equations reduce to the RT equation:

$$3m\partial_u\Phi + \Delta\Delta\Phi = 0$$

- Insensitive of Λ! Suggests that RTdS has Bondi mass loss as in AF case [Chrusciel '91]



RTdS monotonic charges

- Asymptotic charges need Starobinsky/Fefferman-Graham expansion **[Bakas-Skenderis '14]** ($\hat{\Phi} = \Phi|_{\mathscr{I}^+}$)

$$g_{(0)} = dt^2 + \frac{6}{\Lambda} e^{\hat{\Phi}} dz d\bar{z} \implies [g_{(0)}] \neq [\delta_{ij}]$$

- Generically no conformal Killing $\xi_{(0)}$, need to define $\mathcal{H}_\xi$
- In particular, choosing

$$\xi_{(0)} = e^{\frac{1}{2}(\hat{\Phi}-\hat{\Phi}_0)}\partial_t \implies \mathcal{H}_\xi = \frac{2m}{\kappa^2}\int_{\partial C} e^{\frac{3}{2}(\hat{\Phi}-\hat{\Phi}_0)}\, d\Omega = \mathcal{M}_B$$

gives the "Bondi mass", $\mathcal{M}_B$, with negative flux:

$$\mathcal{M}_B|_{\partial C_2} - \mathcal{M}_B|_{\partial C_1} \leq 0$$

- Another monotonic charge is the Calabi functional **[Calabi '82]**:

$$\xi_{(0)} = (\hat{\Delta}\hat{\Phi})^2\partial_t \implies \mathcal{H}_\xi = \frac{m}{2\kappa^2}\int_{\partial C}(R_2)^2\, d\Omega = \mathcal{C}\,, \quad \mathcal{C}|_{\partial C_2} - \mathcal{C}|_{\partial C_1} \leq 0$$

see e.g. **[Chrusciel '91]** for nice proofs of the monotonicity properties

RTdS conserved charges

Despite not admitting conformal Killing vectors $\xi_{(0)}$, RTdS spacetimes do possess several conserved charges:

- Cross-section area: $\xi_{(0)} = \partial_t \implies \mathcal{H}_\xi = \frac{2m}{\kappa^2} \int_{\partial C} d\Omega = \frac{2m}{\kappa^2} A_2$
- Euler characteristic: $\xi_{(0)} = -\hat{\Delta}\hat{\Phi}\partial_t \implies \mathcal{H}_\xi = \frac{2m}{\kappa^2} \int_{\partial C} R_2 \, d\Omega = \frac{m}{G}\chi$
- Lorentz charges: $\xi_{(0)} = g(z)\partial_z + h(\bar{z})\partial_{\bar{z}} \implies H_\xi = 0$ (vanishing "angular momentum" type charges)

Radially invariant charges not associated with CKVs satisfy

$$\mathbf{F}_\xi \neq 0 \,, \qquad \int_{B_{12}} \mathbf{F}_\xi = 0 \,.$$

All of the charges above are conserved by virtue of

$$\iota_{\partial_t} \mathbf{F}_\xi = d_2 \mathbf{a} \,, \tag{1}$$

i.e. the flux-form is exact over the compact directions. Finding vectors which solve (1) appears to be a strategy to construct such charges

Conclusions and Future directions

Conclusions:

- Computed Hamiltonians in AldS spacetimes associated with timelike, two-ended hypersurfaces C
- For a spacetime admitting (conformal) boundary isometries, the differences between charges $\Delta Q_\xi^\pm$ are temporally conserved quantities
- In the absence of (conformal) boundary isometries the charges admit a flux due to gravitational radiation e.g. Robinson-Trautman-dS

Future directions:

- Want a physically motivated example with interior flux - bulk gravitational waves
- Need to fix the ambiguities in the correction to the charges
- Past ends of particular interest: Cross sections of $\mathcal{H}_{\text{Cos}}$, connection with entropy **[Wald '93]**, near horizon symmetries **[Grumiller et al. '19, Donnay-Giribet '19]** + extension to the static patch **[Ashtekar-Bahrami '19]**