The Routhian way of Attractor mechanism

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Introduction & Motivation

- The Attractor mechanism is one of the most intriguing features of extremal black hole solutions in SUGRA .

- The scalar fields of the theory depend on the radial direction only and exhibits an attractor point at the horizon (unique) of BH $\longrightarrow$ the values of the scalar fields at BH horizon are independent of their asymptotic values and completely fixed by charges of BH .

- In order to study the radial behaviour of scalar fields we need to construct an one dimensional effective Lagrangian (under the assumption of static, asymptotically flat, extremal & spherically symmetric BH) . The potential term presents in the one dimensional effective Lagrangian known as black hole potential

  : **S. Ferrara, G.W. Gibbons, & R. Kallosh ; 1997** .

Introduction & Motivation

- A. Sen introduced entropy functional $\longrightarrow$ to compute entropy of various classes of extremal black hole : **A. Sen ; 2005** .
- One dimensional effective picture is also present in entropy functional framework introduced by Sen .
- In entropy functional framework, we focus on the near horizon geometry of the extremal BH .
- Extremum value of the entropy functional gives entropy of BH .

Questions we asked

- How do we get one-dimensional effective action from the higher-dimensional seed action of the theory ??
- What is the equivalence between FGK framework & entropy functional framework ??
- Connection between black hole potential at horizon and entropy functional ??

Outline

- FGK framework and one dimensional effective action
- One dimensional effective action through Routhian formulation
- Attractor mechanism and entropy
- Entropy functional framework
- Connection between Routhian formulation and entropy functional approach
- Summary, conclusion & future directions

Black holes in ungauged $(1+3)$d SUGRA

- General action of Maxwell-Einstein-scalar

$$\begin{aligned}\mathcal{S} &= \int d^4x\sqrt{g}\Big[-\frac{R}{2} + \frac{1}{2}g^{\mu\nu}G_{a\bar{a}}(z^a,\bar{z}^{\bar{a}})\partial_\mu z^a \partial_\nu \bar{z}^{\bar{a}} \\ &\quad + \frac{1}{4}\mu_{\Lambda\Sigma}(z^a,\bar{z}^{\bar{a}})F^\Lambda_{\mu\nu}F^{\Sigma\mu\nu} \\ &\quad + \frac{1}{4}\nu_{\Lambda\Sigma}(z^a,\bar{z}^{\bar{a}})F^\Lambda_{\mu\nu}\left(\star F^{\Sigma\mu\nu}\right)\Big] ,\end{aligned} \tag{1}$$

- Extremal, static, spherically symmetric & asymptotically flat solution

$$ds^2 = -e^{2U(\tau)}dt^2 + e^{-2U(\tau)}\left(\frac{d\tau^2}{\tau^4} + \frac{1}{\tau^2}d\Omega_2^2(\theta,\phi)\right) \tag{2}$$

with $F^\Lambda_{\tau t}$, $F^\Lambda_{\theta\phi}$ and $z^a = z^a(\tau)$.

Equation of motion and black hole potential

- Equation of motion of metric function and moduli field

$$\begin{aligned}
\ddot{U}(\tau) &= e^{2U(\tau)} V_{BH} \, , \\
\ddot{z}^a(\tau) + \Gamma^a_{bc} \dot{z}^a(\tau) \dot{z}^b(\tau) &= e^{2U(\tau)} G^{a\bar{b}} \frac{\partial V_{BH}}{\partial \bar{z}^{\bar{b}}} \, , \\
\dot{U}(\tau)^2 + \frac{1}{2} G_{a\bar{a}} \dot{z}^a(\tau) \dot{\bar{z}}^{\bar{a}}(\tau) - e^{2U(\tau)} V_{BH} &= 0 \, . \qquad (3)
\end{aligned}$$

- Black hole potential (positive definite)

$$V_{BH} = -\frac{1}{2} \begin{pmatrix} p^\Lambda & q_\Lambda \end{pmatrix} \begin{pmatrix} \left(\mu + \nu\mu^{-1}\nu\right)_{\Lambda\Sigma} & \left(\nu\mu^{-1}\right)_\Lambda^{\ \Sigma} \\ \left(\mu^{-1}\nu\right)^\Lambda_{\ \Sigma} & \left(\mu^{-1}\right)^{\Lambda\Sigma} \end{pmatrix} \begin{pmatrix} p^\Sigma \\ q_\Sigma \end{pmatrix} . \qquad (4)$$

FGK framework and one-dimensional effective action

- V_{BH} is the functional of the moduli scalars $z^a, \bar{z}^{\bar{a}}$.
- The last equation is a constraint, can be derived from Einstein equation of the $4d$ action .
- EOM of the metric function $U(\tau)$ and the moduli fields $z^a, \bar{z}^{\bar{a}}$ can be read from the one-dimensional effective action

$$\mathfrak{S}_{eff} \quad = \int d\tau \Big(\dot{U}(\tau)^2 + \frac{1}{2} G_{a\bar{a}} \dot{z}^a(\tau) \dot{\bar{z}}^{\bar{a}}(\tau) + e^{2U(\tau)} V_{BH} \Big) \ , \qquad (5)$$

the role of the time is played by the radial coordinate τ .

S. Ferrara, G.W. Gibbons, & R. Kallosh ; 1997

4d action to one-dimensional effective action : Naïve approach

- Field strength and dual field strength in terms of potentials :

$$F^{\Lambda}_{\tau t} = \partial_{\tau}\psi^{\Lambda} \; ; \; \mathcal{G}_{\Lambda \tau t} = \partial_{\tau}\chi_{\Lambda} \; ,$$

$$\mathcal{G}_{\Lambda\mu\nu} = -\epsilon_{\mu\nu\rho\sigma}\frac{\partial \mathcal{S}}{\partial F^{\Lambda}_{\rho\sigma}} = -\mu_{\Lambda\Sigma}\big(\star F\big)^{\Sigma}_{\mu\nu} + \nu_{\Sigma\Lambda}F^{\Sigma}_{\mu\nu} \; . \tag{6}$$

$\psi^{\Lambda} \rightarrow$ electric potential, $\chi_{\Lambda} \rightarrow$ magnetic potential .

- Integrating angular parts, the one-dimensional effective Lagrangian

$$\mathcal{L}_{eff} \quad = \dot{U}(\tau)^2 + \tfrac{1}{2}G_{a\bar{a}}\dot{z}^a(\tau)\dot{\bar{z}}^{\bar{a}}(\tau) + e^{2U(\tau)}\tilde{V}_{BH} \; . \tag{7}$$

Naïve approach : Wrong expression of black hole potential

- The expression of black hole potential

$$\tilde{V}_{BH} = \frac{1}{4} \begin{pmatrix} \tilde{p}^{\Lambda} & \tilde{q}_{\Lambda} \end{pmatrix} \begin{pmatrix} -\left(\mu + \nu\mu^{-1}\nu\right)_{\Lambda\Sigma} & -\left(\nu\mu^{-1}\right)_{\Lambda}^{\ \Sigma} \\ \left(\mu^{-1}\nu\right)^{\Lambda}_{\ \Sigma} & \left(\mu^{-1}\right)^{\Lambda\Sigma} \end{pmatrix} \begin{pmatrix} \tilde{p}^{\Sigma} \\ \tilde{q}_{\Sigma} \end{pmatrix} . \quad (8)$$

with $\tilde{p}^{\Lambda} = \frac{\partial \mathcal{L}}{\partial(\partial_\tau \chi_\Lambda)}$; $\tilde{q}_{\Lambda} = \frac{\partial \mathcal{L}}{\partial(\partial_\tau \psi^\Lambda)}$.

- V_{BH} and $\tilde{V}_{BH}$ differs by signs in front of the vector coupling terms in diagonal along with off diagonal entities and $\tilde{V}_{BH}$ is NOT positive definite .

- The above approach leads to wrong expression of black hole potential, hence discarded !

Routhian formulation

- Field strength $F^{\Lambda}_{\tau t} = \partial_\tau \psi^\Lambda$ behaves like momentum, not the $F^{\Lambda}_{\theta\phi}$ component .
- Write down the seed four dimensional Lagrangian in terms of the radial derivative of the potential ψ^Λ as $\partial_\tau \psi^\Lambda$.
- Take the Legendre transformation of the 4d Lagrangian with respect to the all momentum like variables of it : $\{\dot{U}, \dot{z}^a, \dot{\bar{z}}^{\bar{a}}, \dot{\psi}^\Lambda\} \rightarrow$ these are the very ones who couple (in the 4d Lagrangian) to $\partial_\tau \psi^\Lambda$.
- Routhian

$$\mathcal{R}_{eff} = \dot{U}\frac{\partial \mathcal{L}}{\partial \dot{U}} + \dot{z}^a \frac{\partial \mathcal{L}}{\partial \dot{z}^a} + \dot{\bar{z}}^{\bar{a}} \frac{\partial \mathcal{L}}{\partial \dot{\bar{z}}^{\bar{a}}} + \partial_\tau \psi^\Lambda \frac{\partial \mathcal{L}}{\partial(\partial_\tau \psi^\Lambda)} - \mathcal{L} \ . \tag{9}$$

Routhian formulation $\rightarrow$ correct V_{BH}

- Routhian density

$$\mathcal{R} = \dot{U}(\tau)^2 + \frac{1}{2} G_{a\bar{a}} \dot{z}^a(\tau) \dot{\bar{z}}^{\bar{a}}(\tau) + e^{2U(\tau)} V_{BH} \ . \tag{10}$$

with the correct positive definite expression of V_{BH}

$$V_{BH} = -\frac{1}{2} \begin{pmatrix} p^\Lambda & q_\Lambda \end{pmatrix} \begin{pmatrix} \left(\mu + \nu\mu^{-1}\nu\right)_{\Lambda\Sigma} & \left(\nu\mu^{-1}\right)_\Lambda^{\ \Sigma} \\ \left(\mu^{-1}\nu\right)^\Lambda_{\ \Sigma} & \left(\mu^{-1}\right)^{\Lambda\Sigma} \end{pmatrix} \begin{pmatrix} p^\Sigma \\ q_\Sigma \end{pmatrix} , \tag{11}$$

with $p^\Lambda = \frac{\partial \mathcal{R}}{\partial(\partial_\tau \chi_\Lambda)}$; $q_\Lambda = \frac{\partial \mathcal{R}}{\partial(\partial_\tau \psi^\Lambda)}$ and $V_{BH} > 0$.

A. Chattopadhyay, A. Marrani & SRC ; To appear

Routhian formulation gives one dimensional effective action

- One can view Routhian as an effective Lagrangian where the radial coordinate τ plays as time .

- The effective one dimensional action of the BH described by FGK is actually our Routhian of the seed four-dimensional action . This puts FGK one-dimensional effective Lagrangian on solid foundations, in the context of classical mechanics .

- The Routhians with respect to other momentum like quantities do NOT provide correct V_{BH} . Our choice/prescription is the only one giving rise to a positive definite black hole potential .

A. Chattopadhyay, A. Marrani & SRC ; To appear

Hamiltonian constraint

- The Routhian does not have any explicit τ dependence $\rightarrow$ reparametrisation invariance .

- τ is an arbitrary parameter. Therefore, the corresponding Hamiltonian must generate reparametrisation rather than time evolution .

- Unlike standard mechanics the system does not evolve in a traditional time-dependent manner, but constrained to satisfy the equations at all points along $\tau \rightarrow$ Hamiltonian to be zero .

- Hamiltonian constraint :

$$\mathcal{H} = p_U \dot{U} + p_a \dot{z}^a + p_{\bar{a}} \dot{\bar{z}}^{\bar{a}} - \mathcal{R} = 0 = \dot{U}^2 + \frac{1}{2} G_{a\bar{a}} \dot{z}^a \dot{\bar{z}}^{\bar{a}} - e^{2U} V_{BH} \,. \quad (12)$$

Equation of motion of Moduli fields

- EOM of scalar field reads from :

$$\partial_\tau\left(\frac{\partial\mathcal{R}}{\partial\left(\partial_\tau\bar{z}^{\bar{a}}\right)}\right)-\frac{\partial\mathcal{R}}{\partial\bar{z}^{\bar{a}}}=0\ . \tag{13}$$

- Considering moduli space as a complex Kähler manifold

$$G_{a\bar{b}}=\partial_{z^a}\partial_{\bar{z}^{\bar{b}}}K\ . \tag{14}$$

$K\rightarrow$ Kähler potential .

- EOM

$$\partial_\tau^2 z^a+\Gamma^a_{bc}\partial_\tau z^b\partial_\tau z^c=e^{2U}G^{a\bar{b}}\frac{\partial V_{BH}}{\partial\bar{z}^{\bar{b}}}\ ;\ \Gamma^a_{bc}=G^{a\bar{d}}\partial_{\bar{z}^{\bar{d}}}\partial_{z^b}\partial_{z^c}K\ . \tag{15}$$

Scalar fields at the horizon

- At the horizon (resides at $\tau \sim +\infty$) the metric function behaves

$$e^{-2U}|_{\tau\sim\infty} = \frac{A}{4\pi}\tau^2 \ . \tag{16}$$

  A is the horizon area .

- Considering a coordinate transformation $\partial_\tau \to \frac{1}{\tau}\partial_\omega$, EOM of moduli field becomes

$$\partial_\omega^2 z^a + \Gamma^a_{bc}\partial_\omega z^b \partial_\omega z^c = G^{a\bar{b}}\frac{\partial V_{BH}}{\partial \bar{z}^{\bar{b}}} \ . \tag{17}$$

- The new coordinate ω can now be considered as the physical distance from the horizon in the units of r_h , $ds^2 = r_h^2 d\omega^2$.

Attractor Mechanism

- With this notion of physicality, one can expect that at the horizon the moduli field along all its derivatives in this coordinate system should tend to some finite value. However, If we consider the first derivative of z^a takes some non-zero value at $\omega \to \infty$, that would inevitably lead to $z^a \to \infty$ as one approaches the horizon and hence the scalar field diverges .

- In order to avoid the divergence, at the horizon one should have $z^a = z_h^a =$ constant which leads to say any order derivative of z^a at horizon vanishes .

Attractor Mechanism Contd.

- Constant value of moduli fields at the horizon implies

$$\frac{\partial V_{BH}}{\partial \bar{z}^{\bar{b}}}\Big|_{z^a=z^a_h} = 0 \ . \tag{18}$$

- The value of z^a_h that solves the above equation is independent of its initial condition $\Rightarrow$ the horizon behaves an attractor point .

- z^a_h is fixed by the critical point of V_{BH} .

S. Ferrara, G.W. Gibbons & R. Kallosh ; 1997
R. Kallosh, N. Sivanandam & M. Soroush ; 2006

Entropy

- To compute entropy we focus on the near horizon region .
- Horizon is a attractor point for z^a with the metric function $e^{-2U}|_{\tau\sim\infty} = \frac{A}{4\pi}\tau^2$.
- Considering Hamiltonian constraint ($\mathcal{H} = 0$) at the horizon

$$\dot{U}^2 = e^{2U} V_{BH} \; , \tag{19}$$

leads the entropy

$$S_{BH} = \frac{A}{4\pi} = \pi V_{BH}|_{horizon} \; . \tag{20}$$

- S_{BH} is fixed by the critical point of V_{BH} and matches with *Bekenstein-Hawking* entropy .

Sen prescription

- Sen prescription: compute entropy from $4d$ action with static, spherically symmetric solution .

- Focusing on the near horizon geometry and upon changing the coordinate $\tau \to 1/r$ the geometry becomes

$$ds^2 = -\frac{1}{v} r^2 dt^2 + v \left[\frac{dr^2}{r^2} + d\Omega_2^2(\theta, \phi) \right] \; ; \; v = \frac{A}{4\pi} = \frac{S}{\pi} \; . \tag{21}$$

- At near horizon, $z^a = \text{const}$, $F^\Lambda_{rt} = e^\Lambda$, $F^\Lambda_{\theta\phi} = -p^\Lambda \sin\theta$.

Sen Prescription Contd.

- Defining the functional $f(z^a, e^\Lambda, p^\Lambda)$ as the evaluation of the $\mathcal{L}$ over angular coordinates on the near horizon

$$f(z^a, e^\Lambda, p^\Lambda) = \int d\theta d\phi \, \sqrt{-g}\mathcal{L} = -2\pi \, v \, \mu_{\Lambda\Sigma} \, e^\Lambda e^\Sigma + \frac{2\pi}{v}\mu_{\Lambda\Sigma} \, p^\Lambda p^\Sigma - 4\pi \, \nu_{\Lambda\Sigma} \, e^\Lambda p^\Sigma \, . \tag{22}$$

- The scalar field equations can be read from $\frac{\partial f}{\partial z^a} = 0$ and magnetic charge is given by $q_\Lambda = \frac{1}{4\pi}\frac{\partial f}{\partial e^\Lambda} \rightarrow$ the actual electric charge appears as conjugate variable of e^Λ (thus, only after dualization) .

Entropy functional and entropy

- Sen entropy functional

$$\mathcal{E}(z^a, e^\Lambda, p^\Lambda, q_\Lambda) = \frac{v}{4}\left(4\pi e^\Lambda q_\Lambda - f(z^a, e^\Lambda, p^\Lambda)\right), \tag{23}$$

- At the extrema of this function where

$$\begin{aligned} \frac{\partial \mathcal{E}}{\partial z^a} = 0 &\implies z^a = \text{constant}, \\ \frac{\partial \mathcal{E}}{\partial e^\Lambda} = 0 &\implies 4\pi q_\Lambda = \frac{\partial f}{\partial e^\Lambda} . \end{aligned} \tag{24}$$

- Wald entropy of BH is given by the extremum of $\mathcal{E}$.

A. Sen ; 2005

Entropy functional, black hole potential and entropy

- Using the relation

$$S_W = \mathcal{E}_{extremum} = \pi \, V_{BH}\big|_{horizon} \, . \tag{25}$$

$\rightarrow$ relates the entropy function ($\mathcal{E}$) with the black hole potential V_{BH} along with the Wald entropy (S_W) of the black hole at their respected extrema.

Entropy functional and Routhian

- Wald entropy can be read as Legendre transformation of $f(z^a, e^\Lambda, p^\Lambda)$, leads to say that the Routhian formulation ($\mathcal{R}$) can relate with $\mathcal{E}$ formulation .

- Instead of $\{p^\Lambda, q_\Lambda\}$ write down $\mathcal{R}$ in terms of $\{e^\Lambda, p^\Lambda\}$

$$\mathcal{R} = \dot{U}^2 + \frac{1}{2} G_{a\bar{a}} \dot{z}^a \dot{\bar{z}}^{\bar{a}} - \frac{1}{2} \left(e^{-2U} \mu_{\Lambda\Sigma} e^\Lambda e^\Sigma + e^{2U} \mu_{\Lambda\Sigma} p^\Lambda p^\Sigma \right) \ . \quad (26)$$

- Make a coordinate transformation $\tau = 1/r$ so that

$$\mathfrak{S} = 4\pi \int \mathcal{R}(\tau) d\tau = 4\pi \int \mathfrak{R}(r) dr \ ; \ \mathfrak{R}(r) = -\frac{\mathcal{R}(r)}{r^2} \quad (27)$$

Entropy functional and Routhian Contd.

- At near horizon

$$\mathfrak{R}_{NH} = -r^2 U'^2 - \frac{r^2}{2} G_{a\bar{a}} z'^a \dot{\bar{z}}'^{\bar{a}} + \frac{1}{2}\left(v\ \mu_{\Lambda\Sigma} e^\Lambda e^\Sigma + \frac{1}{v}\mu_{\Lambda\Sigma} p^\Lambda p^\Sigma \right) . \tag{28}$$

$'$ denotes derivative with respect to r coordinate .

- At the horizon $(r = 0)$

$$\mathfrak{R}_{Horizon} = \frac{1}{2}\left(v \mu_{\Lambda\Sigma} e^\Lambda e^\Sigma + \frac{1}{v}\mu_{\Lambda\Sigma} p^\Lambda p^\Sigma \right) . \tag{29}$$

- Defining $\mathfrak{R}$ we already integrated out the 4π factor arising from integrating angular parts .

Routhian-Black hole potential-Entropy function-Wald Entropy

- At horizon

$$-\, v\pi\ \mathfrak{R}_{Horizon} = \pi V_{BH}\big|_{Horizon}\ . \tag{30}$$

- All together we have

$$-\, v\pi\ \mathfrak{R}_{Horizon} = \pi V_{BH}\big|_{Horizon} = \mathcal{E}_{extremum} = S_W \tag{31}$$

$\rightarrow$ a close logical circle with Routhian, the black hole potential, the entropy functional and the Wald entropy .

- The entropy computation through entropy function formulation is equivalent to Routhian framework .

A. Chattopadhyay, A. Marrani & SRC ; To appear

Summary and Conclusions

- We considered extremal BH in Maxwell-Einstein-scalar theory in bosonic sector of 4d ungauged SUGRA .
- We showed that one-dimensional effective action is a very specific Routhian of the original 4d theory .
- The Horizon behaves the Attractor and the value of the moduli scalar at horizon is fixed by the critical point of the black hole potential .
- The entropy of BH is given by extremum value of the entropy function which is same as the horizon value of black hole potential (up to a multiplicative factor) .
- Extremum value of the entropy function, horizon value of black hole potential and the Routhian at horizon all are equivalent and provide the correct expression of entropy of BH (in the case of two-derivative theory) .

Future Directions & Ongoing works

- Rotating Attractor with near horizon geometry $AdS_2 \times S^1$.
- Higher derivative corrections .
- Topological solutions in ungauged SUGRA .
- Attractors in gauged SUGRA solutions with Abelian gauging (Fayet-Iliopoulos, STU-model) and non-Abelian gauging .
- Attractors in 5d black rings .

Key References

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Thank You !