New thermodynamic phases and probes in Lyapunov exponents of EEH-AdS black holes

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1. EEH-AdS Black Hole Properties
2. RN-AdS thermodynamics
3. EEH-AdS thermodynamics
4. Probes in Lyapunov exponents
5. Summary

The four dimensional Einstein-Euler-Heisenberg AdS (EEH-AdS) black hole is formed by

$$ds^2 = -h(r)dt^2 + \frac{1}{h(r)}dr^2 + r^2 d\Omega_2^2,$$

with the blackening factor

$$h(r) = 1 - \frac{M}{r} + \frac{r^2}{l^2} + \frac{q^2}{r^2} - \lambda\frac{q^4}{20r^6},$$

$$M = \frac{r_+}{2} + \frac{q^2}{2r_+} - \frac{\lambda q^4}{40r_+^5} + \frac{r_+^3}{2l^2}.$$

Hawking temperature takes

$$T = \frac{3r_+}{4\pi l^2} + \frac{1}{4\pi r_+} - \frac{q^2}{4\pi r_+^3} + \frac{\lambda q^4}{16\pi r_+^7},$$

the entropy takes

$$S = \pi r_+^2,$$

the potential Φ takes

$$\Phi = \frac{q}{r_+}\left(1 - \lambda \frac{q^2}{10 r_+^4}\right),$$

and the charge are $Q = q$.

With $\lambda = 0$, the EEH-AdS BH reduces to RN-AdS BH, and we investigate the RN-AdS thermodynamics first[1].
The correspoding thermodynamic parameters take

$$T = \frac{3r_+}{4\pi l^2} + \frac{1}{4\pi r_+} - \frac{q^2}{4\pi r_+^3} + \frac{\lambda q^4}{16\pi r_+^7},$$

$$\Phi = \frac{q}{r_+}.$$

[1] Based on arXiv:hep-th/9902170

Fixed charge case

For fixed charge case, the canonical thermodynamic ensemble is supposed.
The free energy taking the form

$$F = E - TS$$

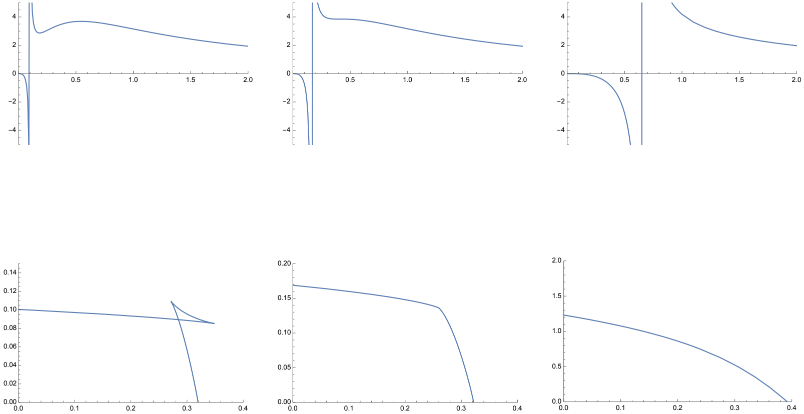
The q formed inverse temperature is

$$\beta = \frac{4\pi l^2 r_+^3}{3r_+^4 + l^2 r_+^2 - q^2 l^2}.$$

Satisfying

$$\frac{\partial \beta}{\partial r_+} = 0 = \frac{\partial^2 \beta}{\partial r_+^2},$$

there are critical values $q = q_{\text{crit}}$ and $r_+ = r_{+\text{crit}}$.


4 2 0 −2 −4
0.5 1.0 1.5 2.0
4 2 0 −2 −4
0.5 1.0 1.5 2.0
4 2 0 −2 −4
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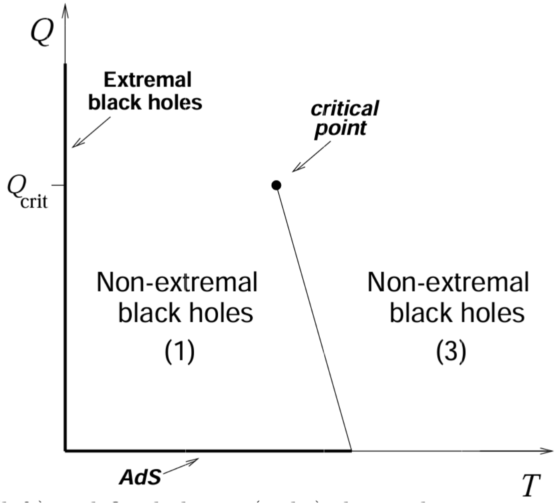
0.14 0.12 0.10 0.08 0.06 0.04 0.02 0.00
0.0 0.1 0.2 0.3 0.4
0.20 0.15 0.10 0.05 0.00
0.0 0.1 0.2 0.3 0.4
2.0 1.5 1.0 0.5 0.0
0.0 0.1 0.2 0.3 0.4


Q
Extremal black holes
critical point
Q_crit
Non-extremal black holes (1)
Non-extremal black holes (3)
AdS
T

Fixed potential case

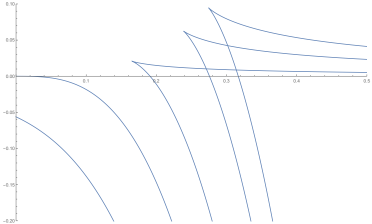
For fixed potential case, the grand canonical thermodynamic ensemble is supposed. The grand potential takes the form

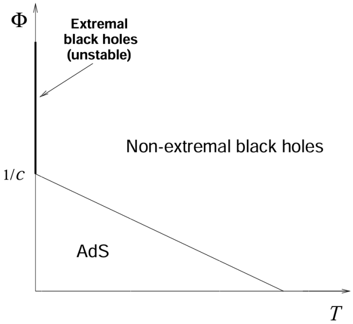
$$W = M - TS - \Phi Q.$$

The inverse temperature can be transformed in

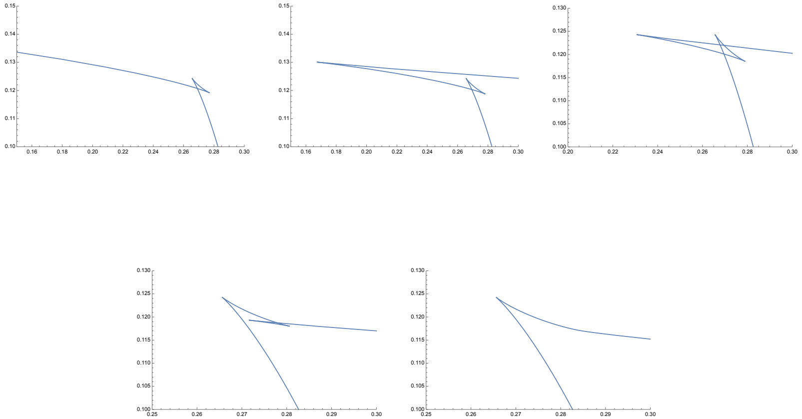
$$\beta = \frac{4\pi l^2 r_+}{3r_+^2 + l^2 - l^2\Phi^2} = \frac{4\pi l^2 r_+}{3r_+^2 + l^2(1-\Phi^2)}.$$

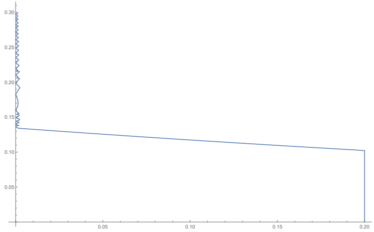
There exists a critical value of the potential $\Phi_c = 1$ that divides the inverse temperature into different behaviors. For $\Phi > \Phi_c$, β diverges at the point when $r_+ = \frac{1}{3} l^2(\Phi^2 - \Phi_c^2)$, while for $\Phi < \Phi_c$,β goes smoothly towards zero as $r_+ \to 0$, which means that the temperature of the black hole can not reach to zero, and so that there's no extremal black holes then.

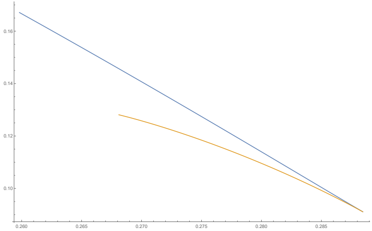

0.10
0.05
0.00
-0.05
-0.10
-0.15
-0.20
0.1
0.2
0.3
0.4
0.5

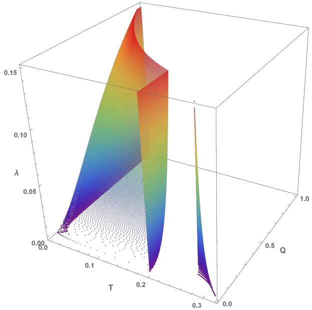

Φ
Extremal
black holes
(unstable)
Non-extremal black holes
1/c
AdS
T

Fixed charge case




0.30
0.25
0.20
0.15
0.10
0.05
0.05
0.10
0.15
0.20


0.16
0.14
0.12
0.10
0.260
0.265
0.270
0.275
0.280
0.285


0.15
0.10
λ
0.05
0.00
0.0
0.1
T
0.2
0.3
0.0
0.5
Q
1.0

Fixed potential case

$$\text{Problem:} \quad \Phi = \frac{q}{r_+}\left(1 - \lambda \frac{q^2}{10 r_+^4}\right)$$

One fixed potential value $< -- >$ Three possible charge value

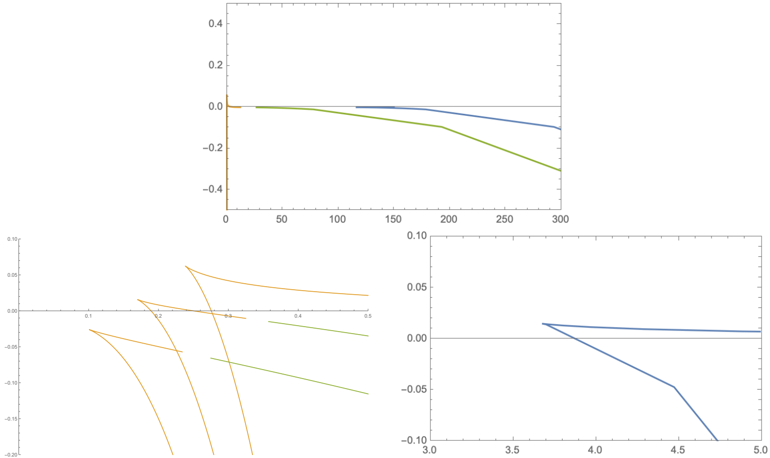
$$\begin{aligned}
q_{s.1} =& \frac{2 \times 5^{2/3} r_+^4}{3^{1/3} A_Q} - \frac{5^{1/3} A_Q}{3^{2/3} \lambda}, \\
q_{s.2} =& \frac{5^{2/3}(1 + \mathrm{i}\sqrt{3}) r_+^4}{3^{1/3} A_Q} - \frac{5^{1/3}(1 - \mathrm{i}\sqrt{3}) A_Q}{2 \times 3^{2/3} \lambda}, \\
q_{s.3} =& \frac{5^{2/3}(1 - \mathrm{i}\sqrt{3}) r_+^4}{3^{1/3} A_Q} - \frac{5^{1/3}(1 + \mathrm{i}\sqrt{3}) A_Q}{2 \times 3^{2/3} \lambda}; \\
A_Q =& (9\Phi_0 r_+^5 \lambda + \sqrt{3} r_+^5 \lambda^2 \sqrt{B_Q})^{1/3}, B_Q = -40 r_+^2/\lambda + 27\Phi_0^2,
\end{aligned}$$

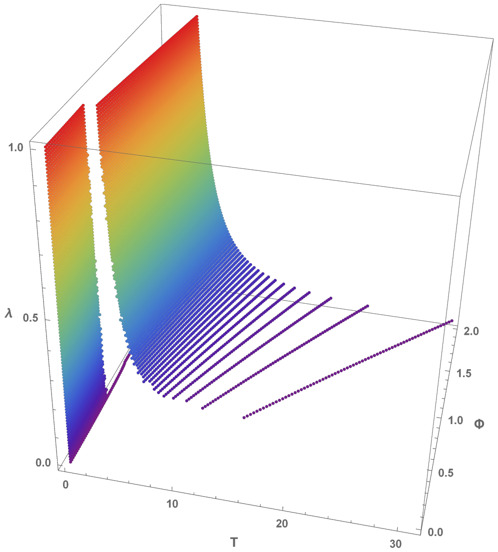
The extremal limit reads $h'(r_+) = 0$, with the solution

$$q = q_{ex.\pm} = \pm\sqrt{2}\frac{r_+^2}{\lambda}\sqrt{1 - \sqrt{1 - \frac{\lambda^2}{r_+^2} - 3\frac{\lambda}{l^2}}}$$

(non-physical solutions have been omitted).

For $1 - \frac{\lambda^2}{r_+^2} - 3\frac{\lambda}{l^2} > 0$, or just $r_+^2 > \frac{\lambda^2 l^2}{l^2 - 3\lambda}$, there will exist extremal black holes, while the charge of non-extremal black holes should obey $|q| < |q_{ex}|$. For $1 - \frac{\lambda^2}{r_+^2} - 3\frac{\lambda}{l^2} < 0$, or just $r_+^2 < \frac{\lambda^2 l^2}{l^2 - 3\lambda}$, there will be no extremal black holes, in another words, all the black holes are non-extremal and the value of charge can contain larger rigime. What's more, once if $l^2 - 3\lambda < 0$, or $\lambda > l^2/3$, there will always exist extremal black holes and always a limit in q.


0.4
0.2
0.0
−0.2
−0.4
0
50
100
150
200
250
300
0.10
0.05
0.00
−0.05
−0.10
3.0
3.5
4.0
4.5
5.0


1.0
λ
0.5
0.0
0
10
20
30
T
0.0
0.5
1.0
1.5
2.0
Φ

Lyapunov exponents

A static, spherically symmetric black hole, $ds^2 = -h(r)\,dt^2 + \frac{dr^2}{h(r)} + r^2 d\Omega^2$, its corresponding Lagrangian for a test particle on the equatorial plane ($\theta = \pi/2$) is

$$\mathcal{L} = \frac{1}{2} g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu = \frac{1}{2}\left[-h(r)\dot{t}^2 + \frac{\dot{r}^2}{h(r)} + r^2\dot{\phi}^2\right].$$

The associated conserved quantities are $E = h(r)\dot{t}$, $L = r^2\dot{\phi}$. The radial equation then becomes

$$\dot{r}^2 + \mathcal{V}_{\text{eff}}(r) = 0, \quad \mathcal{V}_{\text{eff}}(r) = -E^2 + h(r)\left(\frac{L^2}{r^2} - \eta\right),$$

where $\eta = 0$ corresponds to null geodesics and $\eta = 1$ corresponds to timelike geodesics.

Circular orbits $r = r_0$ are determined by

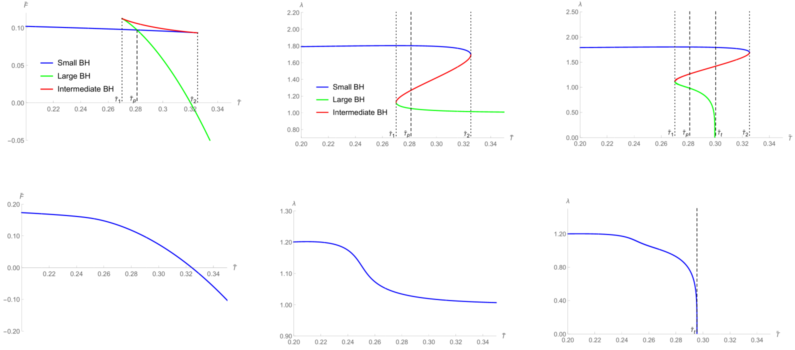
$$\mathcal{V}_{\text{eff}}(r_0) = 0, \qquad \mathcal{V}'_{\text{eff}}(r_0) = 0.$$

The instability of the circular orbit is characterized by the Lyapunov exponent

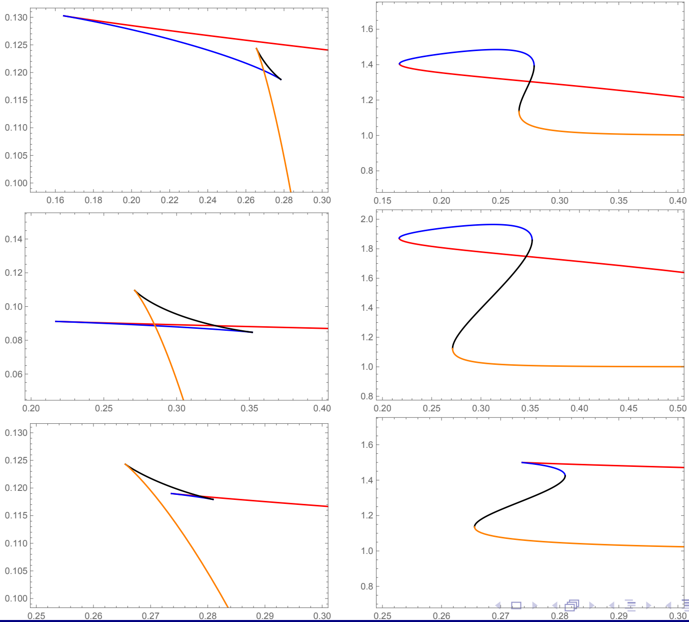
$$\lambda_L(\eta) = \sqrt{-\frac{\mathcal{V}''_{\text{eff}}(r_0;\eta)}{2\,\dot{t}^{\,2}(r_0;\eta)}},$$

which provides a unified description for both photon orbits ($\eta = 0$) and massive particle orbits ($\eta = 1$).

Probing phase structure with Lyapunov exponents [2]



[2] arXiv:2205.02122


0.130
0.125
0.120
0.115
0.110
0.105
0.100
0.16 0.18 0.20 0.22 0.24 0.26 0.28 0.30
1.6
1.4
1.2
1.0
0.8
0.15 0.20 0.25 0.30 0.35 0.40
0.14
0.12
0.10
0.08
0.06
0.20 0.25 0.30 0.35 0.40
2.0
1.8
1.6
1.4
1.2
1.0
0.8
0.20 0.25 0.30 0.35 0.40 0.45 0.50
0.130
0.125
0.120
0.115
0.110
0.105
0.100
0.25 0.26 0.27 0.28 0.29 0.30
1.6
1.4
1.2
1.0
0.8
0.25 0.26 0.27 0.28 0.29 0.30

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We illustrate the thermodynamics of the non-linear charged AdS black hole within the framework of Einstein-Euler-Heisenberg (EEH) gravity.
During different thermodynamic ensembles, specifically the canonical thermodynamic ensemble with fixed charge and the grand canonical thermodynamic ensemble with fixed potential, new phases emerge.
We draw the phase diagrams and are still trying to probe the new phases by Lyapunov exponents.

Thank you!