# Thermodynamic phase structures and Quasi-normal mode of Nonlinear charged black hole

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Shanghai University

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**Based on arxiv:2508.00404, in cooperation with Zi-Yu Hou and Xian-Hui Ge.**

1. Background and Motivation
2. Black Hole Model and Thermodynamics
3. Quasi-Normal Modes (QNMs)
4. BH Thermodynamics and QNMs
5. Summary and Discussion

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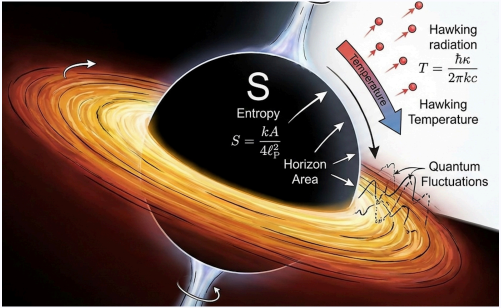
# Black Hole Thermodynamics



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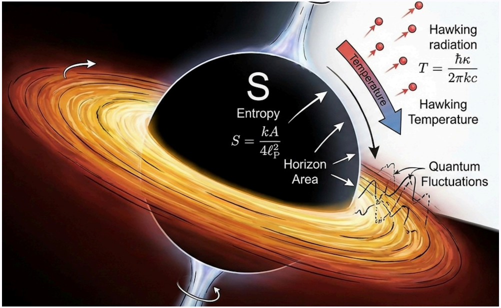


- The four laws of black hole mechanics;
- The thermodynamic phase transition of black holes;
- Black hole information;
- Quantum chaos;
- ...

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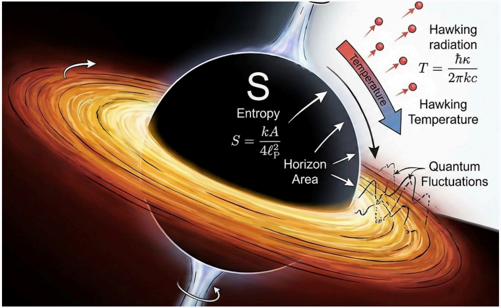


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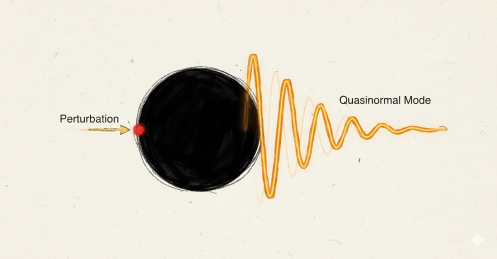
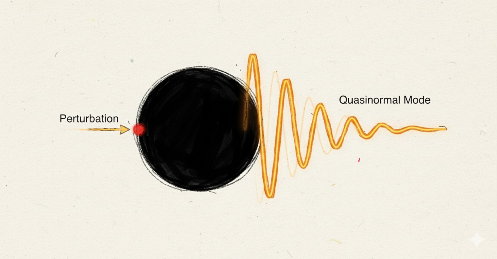
Black hole Thermodynamics & Dynamics

# Quasi-Normal Modes




**Perturbation Dynamics:** $\delta\psi \sim e^{-i\omega t}$

# Quasi-Normal Modes

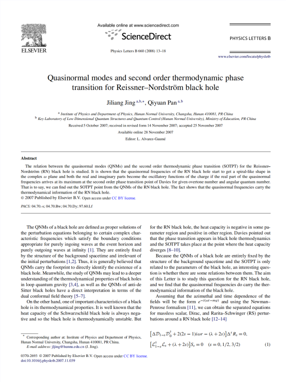



**Perturbation Dynamics:** $\delta\psi \sim e^{-i\omega t}$

- The "Fingerprin" of black holes;
- The stability of black holes;
- Related to GWs and theoretical observable effects;
- ...

## Black Hole Thermodynamics and Phase Transition and QNMs of RN Black Holes

*J.L. Jing, Q.Y. Pan. Phys.Lett.B 660 (2008) 13-18.*



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# Quasinormal modes and second order thermodynamic phase transition for Reissner–Nordström black hole

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[b] Key Laboratory of Low Dimensional Quantum Structures and Quantum Control (Hunan Normal University), Ministry of Education, PR China

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Editor: L. Alvarez-Gaumé

**Abstract**

The relation between the quasinormal modes (QNMs) and the second order thermodynamic phase transition (SOTPT) for the Reissner–Nordström (RN) black hole is studied. It is shown that the quasinormal frequencies of the RN black hole start to get a spiral-like shape in the complex $\omega$ plane and both the real and imaginary parts become the oscillatory functions of the charge if the real part of the quasinormal frequencies arrives at its maximum at the second order phase transition point of Davies for given overtone number and angular quantum number. That is to say, we can find out the SOTPT point from the QNMs of the RN black hole. The fact shows that the quasinormal frequencies carry the thermodynamical information of the RN black hole.



*PACS:* 04.70.-s; 04.70.Bw; 04.70.Dy; 97.60.Lf

The QNMs of a black hole are defined as proper solutions of the perturbation equations belonging to certain complex characteristic frequencies which satisfy the boundary conditions appropriate for purely ingoing waves at the event horizon and purely outgoing waves at infinity [1]. They are entirely fixed by the structure of the background spacetime and irrelevant of the initial perturbations [1,2]. Thus, it is generally believed that QNMs carry the footprint to directly identify the existence of a black hole. Meanwhile, the study of QNMs may lead to a deeper understanding of the thermodynamical properties of black holes in loop quantum gravity [3,4], as well as the QNMs of anti-de Sitter black holes have a direct interpretation in terms of the dual conformal field theory [5–7].

On the other hand, one of important characteristics of a black hole is its thermodynamical properties. It is well known that the heat capacity of the Schwarzschild black hole is always negative and so the black hole is thermodynamically unstable. But for the RN black hole, the heat capacity is negative in some parameter region and positive in other region. Davies pointed out that the phase transition appears in black hole thermodynamics and the SOTPT takes place at the point where the heat capacity diverges [8–10].

Because the QNMs of a black hole are entirely fixed by the structure of the background spacetime and the SOTPT is only related to the parameters of the black hole, an interesting question is whether there are some relations between them. The aim of this Letter is to study this question for the RN black hole, and we find that the quasinormal frequencies do carry the thermodynamical information of the black hole.

Assuming that the azimuthal and time dependence of the fields will be the form $e^{-i(\omega t-m\varphi)}$ and using the Newman–Penrose formalism [11], we can obtain the separated equations for massless scalar, Dirac, and Rarita–Schwinger (RS) perturbations around a RN black hole [12–14]

$$[\Delta\mathcal{D}_{1-s}\mathcal{D}_0^{\dagger}+2(2s-1)i\omega r-(\lambda+2s)]\Delta^s R_s=0,$$
$$[\mathcal{L}_{1-s}^{\dagger}\mathcal{L}_s+(\lambda+2s)]S_s=0 \quad (s=0,1/2,3/2) \tag{1}$$

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Fig. 1: The relationship between thermodynamic critical points and QNMs.

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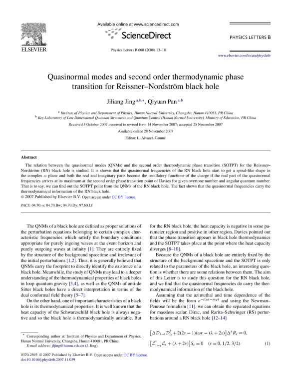
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Fig. 1: The relationship between thermodynamic critical points and QNMs.

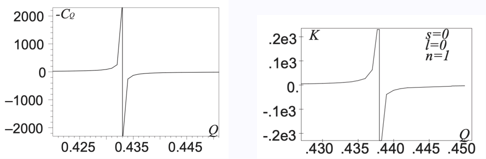


Fig. 2: From Phys.Lett.B 660 (2008) 13-18. $C_Q$ is the heat capacity at fixed charge $Q$. $K$ is a slope parameter of QNMs $\omega$ $K = d\omega_i/d\omega_r$.

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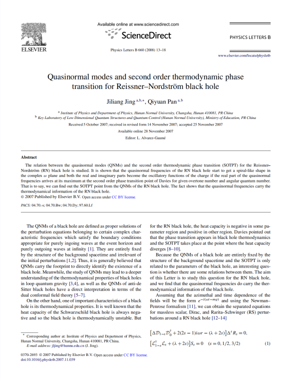
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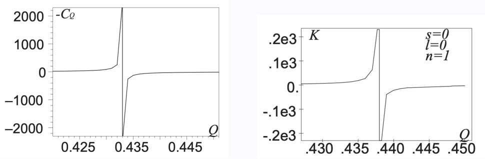


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Black hole **Thermodynamics & Dynamics**

## BH Thermodynamics and Dynamics

- **QNMs and BH phase transition:**
  Y. Liu, D.-C. Zou, and B. Wang. JHEP, 09:179, 2014, 1405.2644.
  C. Lan, Y.-G. Miao, and H. Yang. Nucl. Phys. B, 971:115539, 2021.
  ...
- **Lyapunov exponent of particle motion and BH phase transition:**
  R.H. Ali, Xiao-Mei Kuang. Eur.Phys.J.C 85 (2025) 10, 1131.
  Deyou Chen, Chuang Yang, Yongtao Liu. Phys.Lett.B 865 (2025) 139463.
  Shao-Wen Wei, Yu-Xiao Liu. Chin.Phys.C 44 (2020) 11, 115103.
  ...
- More ...

## Motivation

Black hole Thermodynamics & Dynamics(QNMs)

# Motivation

Black hole **Thermodynamics & Dynamics**(QNMs)

Black hole thermodynamics phase structure

$\updownarrow$

QNMs

1 Background and Motivation

2 Black Hole Model and Thermodynamics

3 Quasi-Normal Modes (QNMs)

4 BH Thermodynamics and QNMs

5 Summary and Discussion

# Euler-Heisenberg $F(R)$ Black Hole Model

*Y. Sekhmani, S.K. Maurya et al. Phys.Dark Univ. 46 (2024) 101701.*

$$\mathcal{I}_{h(r)} = \frac{1}{16\pi}\int_{\mathcal{M}} \mathrm{d}^4x\sqrt{-g}\Big(F(R) - \mathcal{L}(X,Y)\Big),$$
$$F(R) = R + \tilde{f}(R).$$

$$\mathcal{L}(X,Y) = -X + \frac{\lambda}{2}X^2 + \frac{7\lambda}{8}Y^2,$$

$$X = \frac{1}{4}F_{\mu\nu}F^{\mu\nu} = \frac{1}{2}\left(\mathbf{B}^2 - \mathbf{E}^2\right),$$
$$Y = \frac{1}{4}F_{\mu\nu}\,{}^*F^{\mu\nu} = \mathbf{E}\cdot\mathbf{B},$$

$\lambda$: The coupling parameter of nonlinear Electromagnetic.

$\mathbf{E}$: the electric field strength.

$\mathbf{B}$: the magnetic field strength.

$${}^*F^{\mu\nu} = \tfrac{1}{2}\tfrac{1}{\sqrt{-g}}\epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}.$$

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$$\mathbf{B} = 0$$

# Euler-Heisenberg $F(R)$ Black Hole Model

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$$ds^2 = -h(r)\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{h(r)} + r^2(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2),$$

$$h(r) = 1 - \frac{m_0}{r} - \frac{R_0 r^2}{12} + \frac{1}{1+f_{R_0}}\left(\frac{q^2}{r^2} - \frac{\lambda q^4}{20 r^6}\right),$$

$m_0$: The mass parameter of black hole.

$q$: the charge of black hole.

$$R_0 = 4\Lambda$$

Here, we set $R_0 > 0$.

With the traceless energy-momentum tensor condition

$$R_0\left(1 + f_{R_0}\right) - 2\left(R_0 + \tilde{f}(R_0)\right) = 0,$$

$R_0$ is a constant scalar curvature $R = R_0 = \text{Constant}$ and $f_{R_0} = \left.\frac{\mathrm{d}\tilde{f}(R)}{\mathrm{d}R}\right|_{R=R_0}$. The $F(R)$ term of the theory can be characterized by the two parameters $R_0$ and $f_{R_0}$.

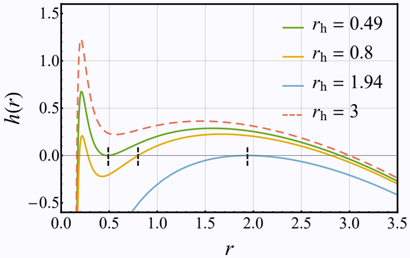
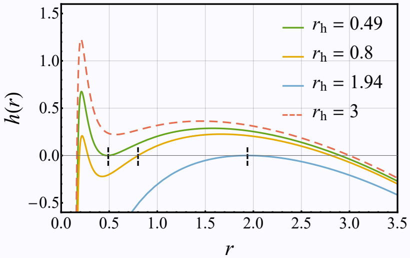



Fig. 3: The blackening factor $h(r)$ of $F(R)$-Euler-Heisenberg black holes with $R_0 = 1$, $q = 0.5$, $\lambda = 0.03$ and $f_{R_0} = 0.1$ for different $r_h$. The red dashed line does not correspond to an actual black hole solution, while the black dashed line marks the event horizon radius $r_h$ of the black hole solution.

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Defing a rescaled radius $\tilde{r}_h$ of the black hole event horizon

$$\tilde{r}_h = \frac{r_h - r_{extreme}}{r_{Nariai} - r_{extreme}} \in [0, 1].$$

# Euler-Heisenberg $F(R)$ Black Hole Thermodynamics

*Y. Sekhmani, S.K. Maurya et al. Phys.Dark Univ. 46 (2024) 101701.*

$$M = -\frac{1}{24}\left(1+f_{R_0}\right) r_{\mathsf{h}}\left(R_0 r_{\mathsf{h}}^2 - 12\right) - \frac{\lambda q^4 - 20 q^2 r_{\mathsf{h}}^4}{40 r_{\mathsf{h}}^5}$$

$$S = \pi(1+f_{R_0}) r_{\mathsf{h}}^2, \qquad Q = q$$

$$\Phi = \frac{q}{r_{\mathsf{h}}}\left(1 - \frac{\lambda q^2}{10 r_{\mathsf{h}}^4}\right)$$

$$T = \frac{1}{4\pi r_{\mathsf{h}}} - \frac{R_0 r_{\mathsf{h}}}{16\pi} + \frac{q^2}{16\pi\left(1+f_{R_0}\right) r_{\mathsf{h}}^3}\left(\frac{\lambda q^2}{r_{\mathsf{h}}^4} - 4\right)$$

**Local thermodynamics of the Black Hole Event Horizon.**

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**Local thermodynamics of the Black Hole Event Horizon.**

**Heat Capacity at Constant Charge:**

$$C_Q = T\left(\frac{\partial S}{\partial T}\right)_Q = \frac{2S^3\left[4\pi^2(1+f_{R_0})Q^2 - 4\pi(1+f_{R_0})S + R_0 S^2\right] - 2\pi^4\lambda(1+f_{R_0})^3 Q^4 S}{7\pi^4\lambda(1+f_{R_0})^3 Q^4 + S^2\left[-12\pi^2(1+f_{R_0})Q^2 + 4\pi(1+f_{R_0})S + R_0 S^2\right]}$$

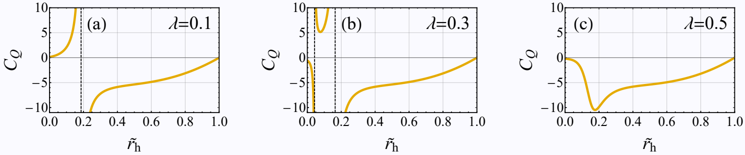


Fig. 4: Three distinct profiles of the heat capacity $C_Q$ as a function of the rescaled horizon radius $\tilde{r}_\mathrm{h}$ under variation of the Euler-Heisenberg parameter $\lambda$, in which (a) $\lambda = 0.1$, (b) $\lambda = 0.3$ and (c) $\lambda = 0.5$, while the other parameters are fixed as $R_0 = 1$, $Q = 0.5$, $f_{R_0} = 0.1$.

**The transition for phase structure 1 to phase structure 2.**

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# Klein-Gordon Equation and Boundary Conditions

## Klein-Gordon Equation

$$\Box\Psi = \frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}\,g^{\mu\nu}\,\partial_\nu\Psi\right) = 0$$

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**Step 1: Ingoing EF Coordinates & Radial Decomposition**

Using $ds^2 = h(r)d\bar{t}^2 - 2drd\bar{t} + \dots$ and substituting $u = 1/r$:

$$\Big[\ell(1+\ell)u + 2\mathrm{i}\omega\Big]\psi + \Big[-2\mathrm{i}\omega u - u^3 h'\Big]\psi' - u^3 h\,\psi'' = 0$$

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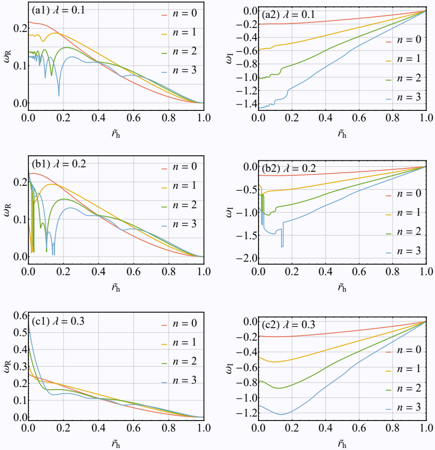
$$\Big[\ell(1+\ell)u + 2\mathrm{i}\omega\Big]\psi + \Big[-2\mathrm{i}\omega u - u^3 h'\Big]\psi' - u^3 h\,\psi'' = 0$$

**Step 2: Outgoing BCs at Cosmological Horizon ($u_c$)**

Applying the Ansatz $\psi = (u-u_c)^{\left(-\frac{2\mathrm{i}\omega}{u_c^2 h'(u_c)}-1\right)}\varphi$ and scaling $v = \frac{u-u_c}{u_b-u_c}$:

$$A_0(v)\frac{\mathrm{d}^2\varphi}{\mathrm{d}v^2} + \Big(\mathrm{i}\omega B_1(v) + B_0(v)\Big)\frac{\mathrm{d}\varphi}{\mathrm{d}v} + \Big(\omega^2 C_2(v) + \mathrm{i}\omega C_1(v) + C_0(v)\Big)\varphi = 0.$$

# QNMs Results




- Solving the equation by Chebyshev spectral method.
- Fixed the parameters:$\ell = 0$, $R_0 = 1$, $Q = 0.5$, $f_{R_0} = 0.1$.

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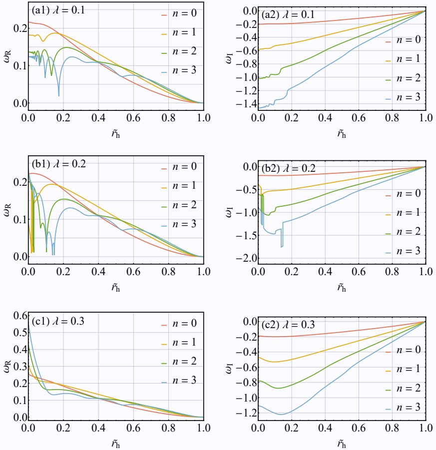


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Stability.

$\lambda$ affect the behavior $\omega_r$.

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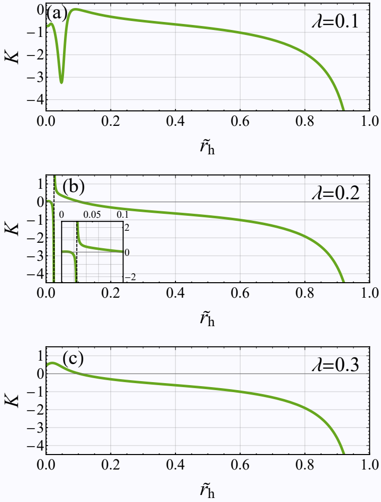
# Slope parameter of QNMs $K$




$$K = \frac{\mathrm{d}\omega_{\mathrm{I}}/\mathrm{d}\tilde{r}_{\mathrm{h}}}{\mathrm{d}\omega_{\mathrm{R}}/\mathrm{d}\tilde{r}_{\mathrm{h}}}.$$

The detail parameter settings are $R_0 = 1$, $Q = 0.5$, $f_{R_0} = 0.1$. With the quantum number of QNMs $\ell = 0$ and $n = 0$.

## Correspondence

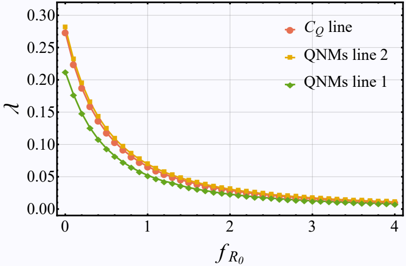



Fig. 5: Three kinds of structure transitions under the parameters setting $R_0 = 1$ and $Q = 1/2$...

**The black hole thermodynamic phase structure and QNM behavior have the same transition point.**

# Relative Error Parameter

$$\Delta_\lambda = \left(\lambda_{\mathrm{C.QNMs}} - \lambda_{\mathrm{C.}C_Q}\right)/\lambda_{\mathrm{C.}C_Q}.$$

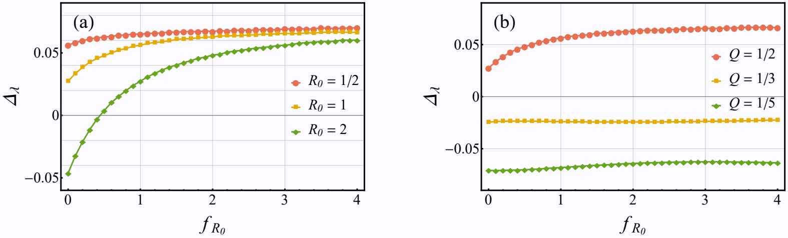



Fig. 6: The influences on the relative error $\Delta_\lambda$ as a function of $f_{R_0}$ of varying the parameter (a) $R_0$ with fixed $Q = \frac{1}{2}$ and (b) $Q$ with fixed $R_0 = 1$...

## Effect of Quantum Numbers $l$ and $n$ in QNMs

$N_{\text{div}} =$ **the number of the divergence points in the specific** $K - \tilde{r}_{\text{h}}$ curve.

Take the change of $N_{\text{div}}$ as the sign of transition in QNMs behavior.

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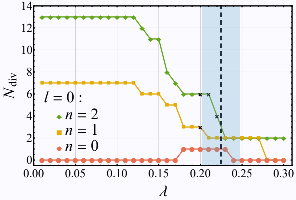

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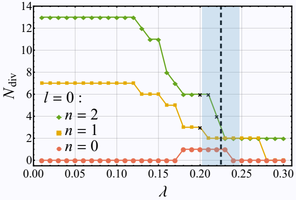


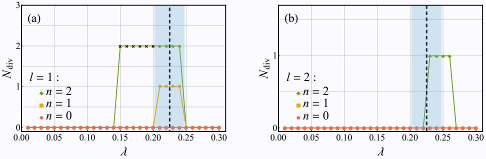


There is a competitive relationship between $n$ and $\ell$.

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## Summary and Discussion

# Summary:

- A relationship between black hole thermodynamic phase structures and QNM behaviors in $F(R)$-Euler-Heisenberg theory;
- The coincidence between heat capacity and QNM transitions remains valid across the parameter space;
- A competitive relationship exists, the correspondence is obscured by larger angular quantum numbers ($l$) but reemerges at higher overtone numbers ($n$).

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## Future Directions:

- Extend the study to rotating black holes;
- Connect to the Lyapunov exponent of particle motion and black hole phase transition;
- Connect theoretical transitions to gravitational wave ringdown signals for potential observational verification of thermodynamic properties.

# Thank you for your attention!