Superradiance in Kerr-Taub-NUT spacetime

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Outlines

- Superradiance
- Kerr-Taub-NUT spacetime
- Newman-Penrose formalism
- Teukolsky master equations
- Summary

Superradiant scattering

of waves/particles from a rotating spacetime

Teukolsky 1971 Bardeen-Press-Teukolsky 1972 Zel'dovich 1972

- Due to rotating of event horizon with frequency ω_H

$$R^s \overset{r\to r_+}{\sim} X^s_{\rm out} e^{i(\omega-\omega_H)r_\star} + X^s_{\rm in} \Delta^{-s} e^{-i(\omega-\omega_H)r_\star}$$

- When $\omega < \omega_H$, energy flows out of the horizon

$$v_g = \frac{d\omega}{dk} = -1 \quad v_p = -(1-\omega_H/\omega)$$

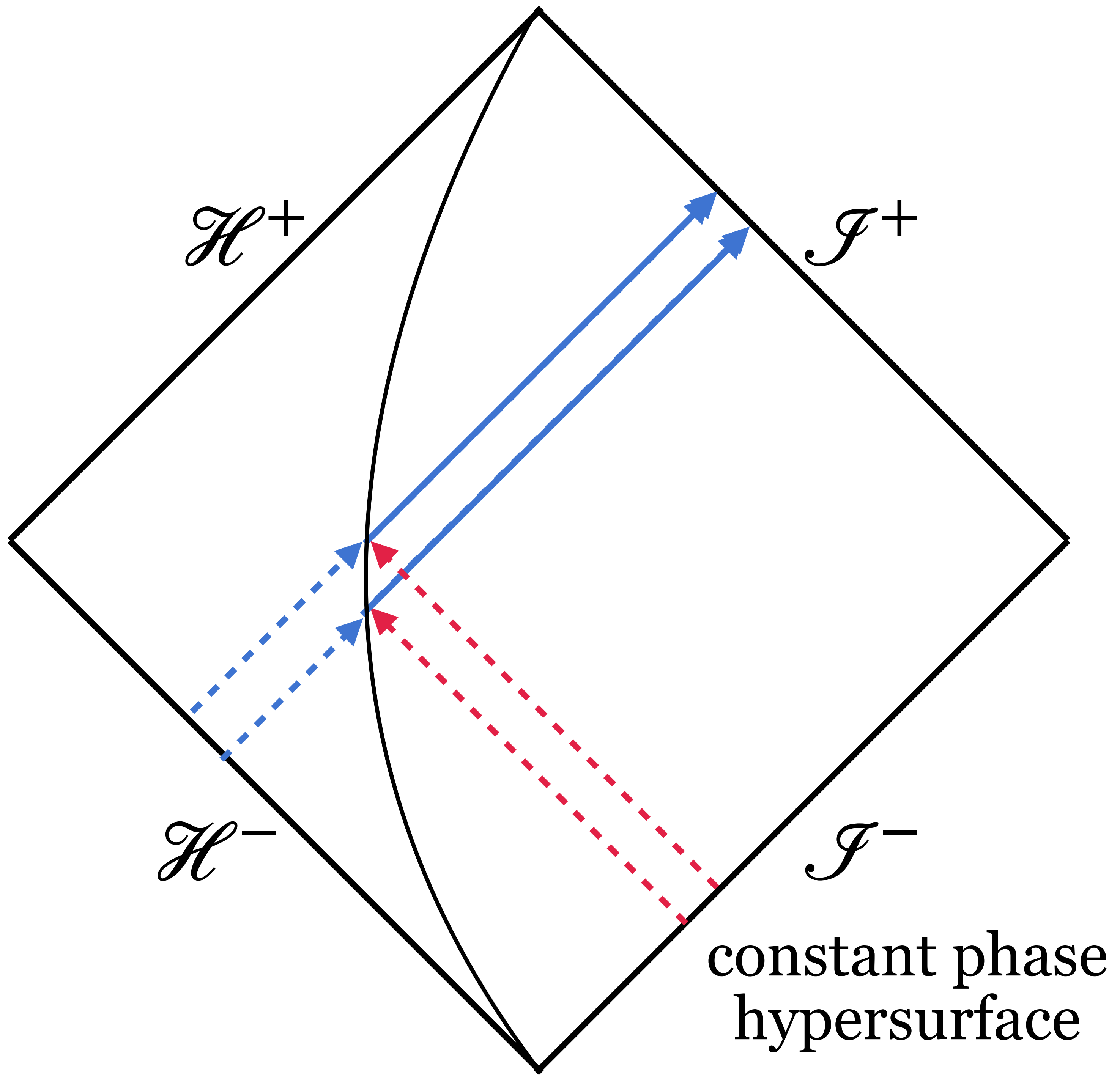
- Superradiance: at same wave frequency, the scattering amplitude of waves/particles are increasing during reflection from a rotating spacetime, i.e., it is amplified

$$R^s \overset{r\to\infty}{\sim} Y^s_{\rm out} \frac{e^{ik_\infty r_\star}}{r^{1+2s}} + Y^s_{\rm in} \frac{e^{-ik_\infty r_\star}}{r}$$

Radiation at infinity

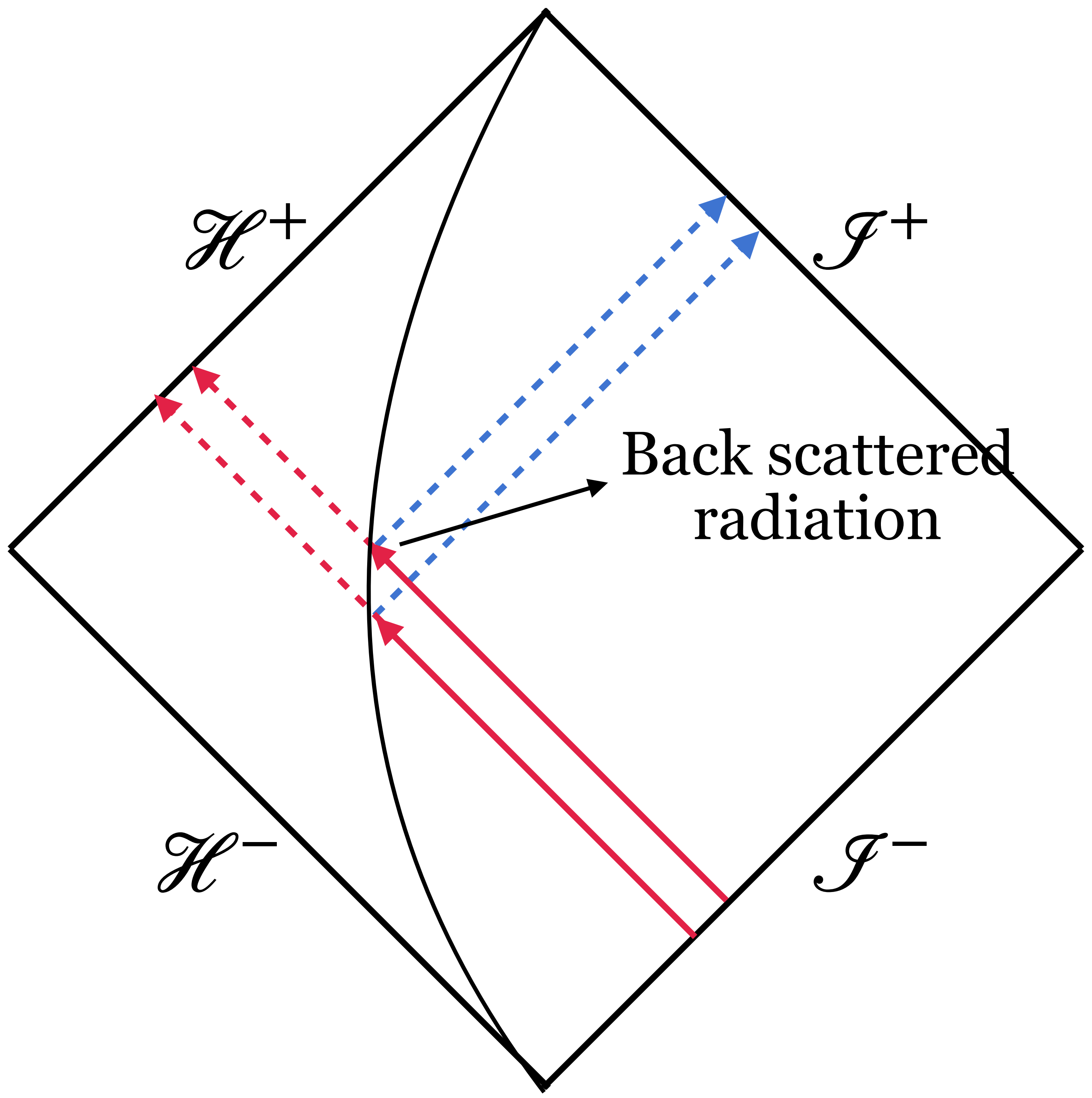
Out and in mode

- $|\text{out}\rangle$: no radiation ingoing to $\mathscr{H}^+$



$$R^s_{\text{out}} \overset{r\to\infty}{\sim} Y^s_{\text{out}} e^{i\omega r_\star} r^{-1-2s}$$

$|\text{in}\rangle$: no radiation outgoing from $\mathscr{H}^-$



$$R^s_{\text{in}} \overset{r\to\infty}{\sim} Y^s_{\text{in}} e^{-i\omega r_\star} r^{-1}$$

Taub-NUT spacetime

NUT (Newman-Unit-Tamburino)

- Taub-NUT metric is a stationary axisymmetric vacuum solution to Einstein's equation !

$$ds^2 = -f(r)[dt + 2n(\cos\theta + C)d\phi]^2 + \frac{dr^2}{f(r)} + (r^2 + n^2)d\Omega_2^2$$

- A generalization of Schwarzschild spacetime sourced by mass together with NUT parameter

$$f(r) = 1 - \frac{2(Mr + n^2)}{r^2 + n^2} \overset{n=0}{=} 1 - \frac{2M}{r}$$

- Newtonian limit

$$f(r) \overset{M=0}{=} 1 - \frac{2n^2}{r^2 + n^2}$$

- The absence of closed timelike and null geodesics requires $|C| \leq 1$

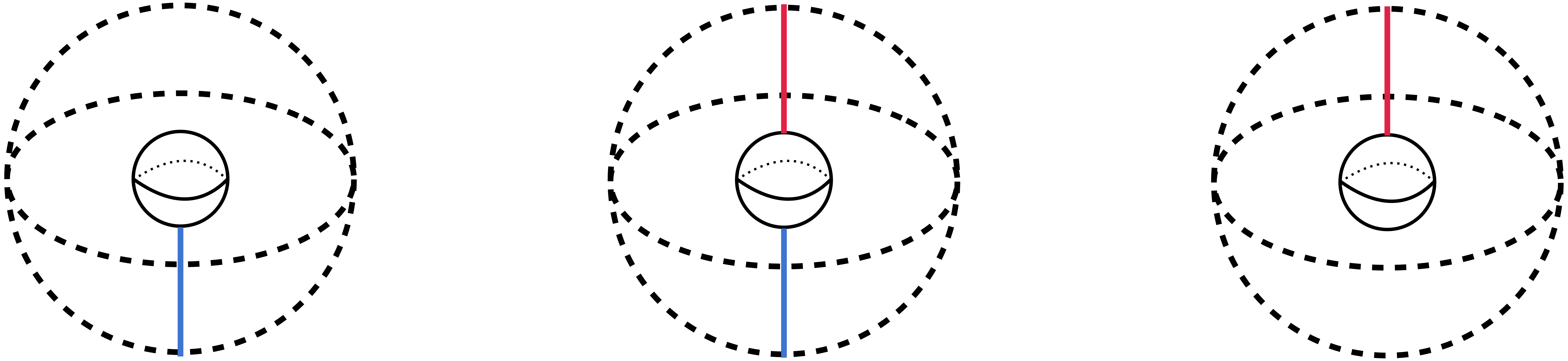
Hennigar, Kubiznak, Mann, 2019

Misner string/wire singularity

Topological line defect (metric singularity of spacetime) along symmetry polar axis

- There is an asymmetry (around the symmetric axis) : the co-rotating and counter rotating path (circular orbits) around the symmetric axis behave differently.

$C = -1$ $\quad$ $C = 0$ $\quad$ $C = 1$



$$\Omega_{\theta=\pi} = \frac{1}{4n} \qquad \Omega_{\theta=0,\pi} = \mp \frac{1}{2n} \qquad \Omega_{\theta=0} = -\frac{1}{4n}$$

- The orbits corotating in the direction of the source, is difference from the orbits rotating counter to the source. The difference is dragging of inertial frame.

Charged Kerr-Taub-NUT

- Brill spacetime: a generalization of RN spacetime without central singularity

$$f(r) = \frac{r^2 - 2Mr + Q^2 - n^2}{r^2 + n^2} \overset{n=0}{=} 1 - \frac{2M}{r} + \frac{Q^2}{r^2}$$

- We obtain the metric of charged Kerr-Taub-NUT via Newman-Janis algorithm

$$ds^2 = -\frac{\Delta}{\Sigma}[dt - (a\sin^2\theta - 2n(\cos\theta + C))d\phi]^2 + \frac{\Sigma}{\Delta}dr^2 + \Sigma d\theta^2 + \frac{\sin^2\theta}{\Sigma}[adt - (r^2 + a^2 + n^2 - 2anC)d\phi]^2$$

$$\Delta = r^2 - 2Mr + a^2 + Q^2 - n^2, \quad \Sigma = r^2 + (a\cos\theta + n)^2$$

- It's a solution to Einstein-Maxwell equations

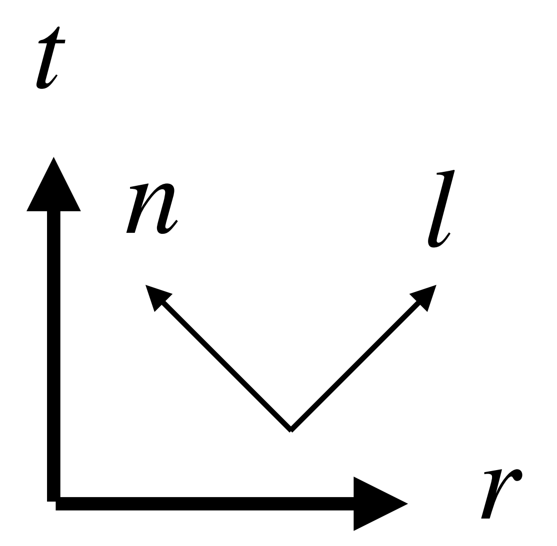
$$G_{\mu\nu} = \kappa_N T_{\mu\nu} \qquad A = -\frac{Qr}{\Sigma}[dt - [a\sin^2\theta - 2n(\cos\theta + C)]d\phi]$$

Newman-Penrose formalism

Local rest frame (Lagrangian observer) $e_t^a = u^a$

- Orthonormal Null vectors $(l, n, m, \bar{m})$ Newman-Penrose, 1962

$$n_a l^a = -1 \quad m_a \bar{m}^a = 1 \qquad g_{ab} = -2n_{(a}l_{b)} + 2m_{(a}\bar{m}_{b)}$$



- Generalized Kinnersley tetrad (in going null tetrad regular at $\mathscr{H}^-$)

$$l^a = \frac{1}{\Delta}(\Sigma + a^2\sin^2\theta - 2an(\cos\theta + C), \Delta, 0, a),$$

$$n^a = \frac{1}{2\Sigma}(\Sigma + a^2\sin^2\theta - 2an(\cos\theta + C), -\Delta, 0, a),$$

$$m^a = \frac{1}{\sqrt{2}\sqrt{r + i(a\cos\theta + n)}}(ia\cos\theta - 2n(\cos\theta + C)i\csc\theta, 0, 1, i\csc\theta),$$

Spin coefficients

In Newman-Penrose formalism

- NP spin coefficients $\kappa = \nu = -\kappa' = \sigma = \lambda = -\sigma' = \epsilon = -\gamma' = \pi = -\tau' = 0$

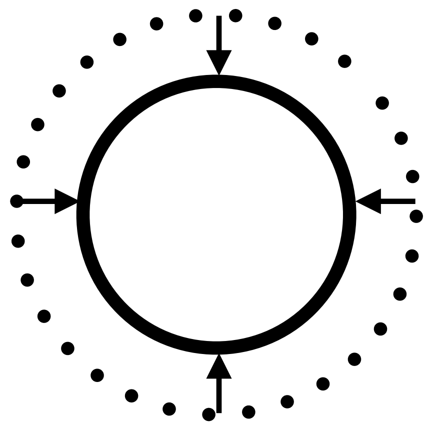
$$\rho = -\frac{1}{r - i(a\cos\theta + n)} \quad \beta = -\alpha' = -\frac{\cot\theta}{2\sqrt{2}}\bar{\rho}, \quad \alpha = -\beta' = \pi + \frac{\cot\theta}{2\sqrt{2}}\rho$$

$$\mu = -\rho' = \frac{\rho^2\bar{\rho}}{2}\Delta, \quad \tau = -\pi' = -\frac{ia\sin\theta}{\sqrt{2}}\rho\bar{\rho}, \quad \pi = -\tau' = \frac{ia\sin\theta}{\sqrt{2}}\rho^2, \quad \gamma = -\epsilon' = \mu + \frac{\rho\bar{\rho}}{4}\partial_r\Delta$$

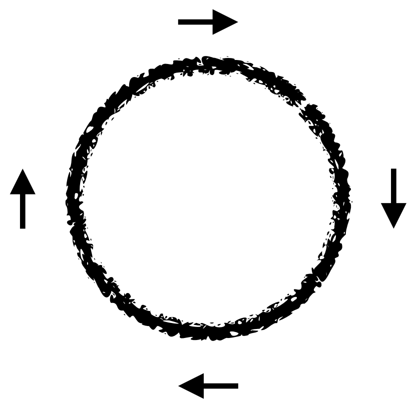
- Optical scalars of null congruence of geodesic field l^μ

$$\theta = -\frac{r}{r^2 + (a\cos\theta + n)^2} \quad \omega = -\frac{a\cos\theta + n}{r^2 + (a\cos\theta + n)^2} \quad \sigma = 0$$

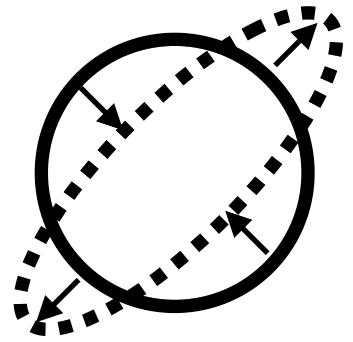
- NUT parameter result in compression, twist of null congruence at infinity



$\theta < 0$



$\omega < 0$



$\sigma \neq 0$

Generalized Teukolsky master equations

For spin-s massless particles

- The equation owns properties that its dependence on the angular and radial variables can be separated

$$\Psi^s = \int d\omega \sum_{l\bar{m}} e^{-i\omega t} e^{i\bar{m}\phi} S^{s,\bar{s}}_{l\bar{m}}(a\omega, \theta) R^{s,\bar{s}}_{l\bar{m}}(r)$$

$\theta \in [0,\pi]$

stationarity Axisymmetric

$\bar{s} = 2n\omega \quad \bar{m} = m + \bar{s}C$

- Radial Teukolsky equation

$$\Delta^{-s}\frac{\partial}{\partial r}\left(\Delta^{s+1}\frac{\partial}{\partial r}R^s\right) + \left(\frac{K^2 - 2isK(r-M)}{\Delta} + 4is\omega r - \lambda\right)R^s = 0$$

$$K = \omega(r^2 + a^2 + n^2 - 2anC) - am$$

Angular wave functions

Spheroidal harmonics

- Angular equation

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}S\right)+\left[-\left((s+\bar{s})\cot\theta+\frac{\bar{m}}{\sin\theta}-a\omega\sin\theta\right)^2+\lambda+s-2s(\bar{s}+2a\omega\cos\theta)\right]S=0$$

Misner condition: $\bar{s}=2n\omega\in\mathbb{Z}/2$

Dirac quantization condition: $2ge/\hbar\in\mathbb{Z}$

- Eigenvalues: angular and spin momentum

$$\lambda=\bar{\lambda}+a^2\omega^2-2am\omega \quad \bar{\lambda}=E-s-(s+\bar{s})^2-2a\omega\bar{s}C \qquad E=l(l+1)+O(a\omega)$$

- Eigenfunctions : (spin-weighted) spheroidal or spherical harmonics at $a\omega\ll 1$

$$S^{s+\bar{s}}_{l\bar{m}}(a\omega,\cos\theta,\phi)\overset{a\omega\ll1}{=}Y^{s}_{l\bar{m}}e^{-i\bar{m}\phi}+O(a\omega) \qquad \lambda=l(l+1)-s-(s+\bar{s})^2$$

Regge-Wheeler equations

- Regge-Wheeler equations in tortoise coordinates

$$\left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial r_\star^2} + V(r)\right)\Psi^s = 0 \qquad \Psi^s = \Delta^{s/2}\sqrt{r^2 + a^2 + n^2 - 2anC}R_{lm}^{s,\bar{s}} \qquad \frac{dr_\star}{dr} = \frac{r^2 + a^2 + n^2 - 2anC}{\Delta}$$

- Effective potential $s = 0$

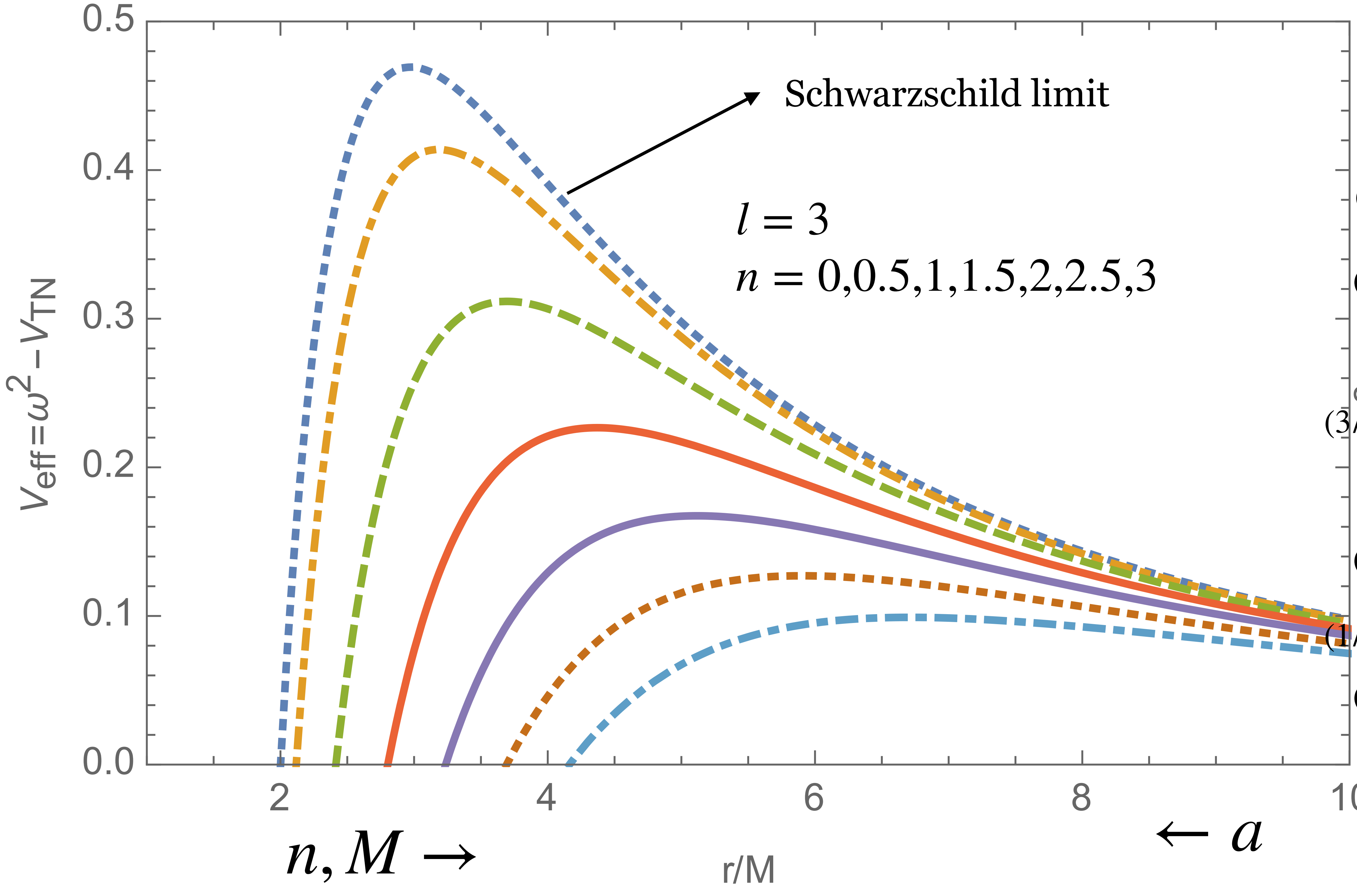
$$\left(\frac{dr^2}{dr_\star^2} + \omega^2 - V(r)\right)\Psi = 0$$

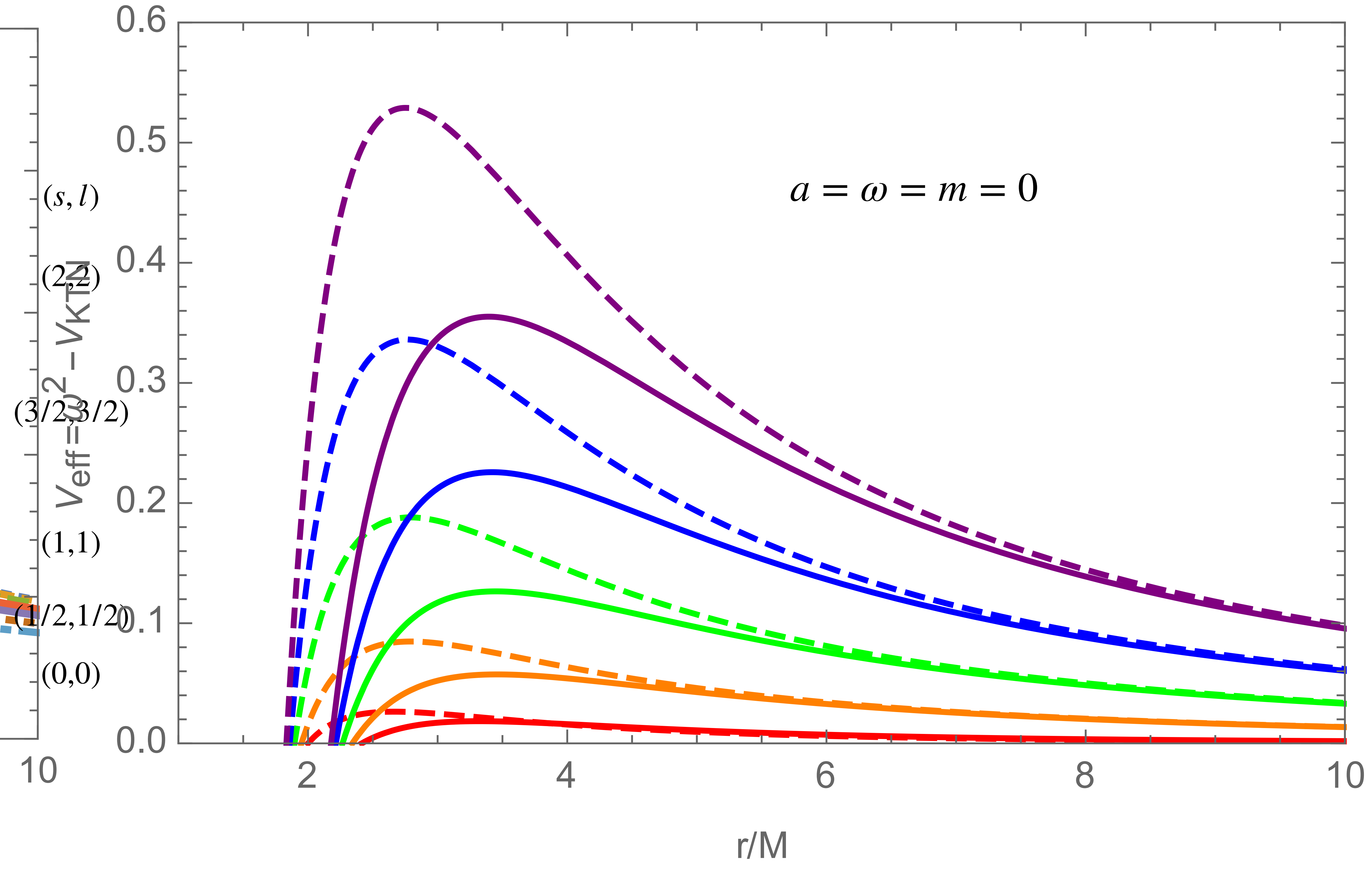
$$V_{KTN}(r) = \omega^2 - \frac{H^2 - \Delta(\mu^2 r^2 + \lambda) + M^2 + n^2 - a^2 - Q^2}{\Delta^2} \qquad H = \omega(r^2 + a^2 + n^2 - 2anC) - (am + qQr)$$

$$V_{TN} = \frac{r^2 - 2Mr - n^2}{r^2 + n^2}\left(\mu^2 + \frac{\lambda_{lm}}{r^2 + n^2} + \frac{2Mr + 5n^2}{(r^2 + n^2)^2} - \frac{3n^2(2Mr + 2n^2)}{(r^2 + n^2)^3}\right) + O(a)$$

Effective potential

Kerr-Taub-NUT





$$k_H^2 \equiv V_{\text{eff}}(r_\star \to -\infty) = (\omega - \omega_H - is\kappa_+)^2$$

$$k_\infty^2 \equiv V_{\text{eff}}(r_\star \to \infty) = \omega^2 - \mu^2 + 2is\omega/r$$

Black hole thermal dynamics

Kerr-Taub-NUT (Newman-Unti-Tamburino)

- Hawking temperature

$$T_H \equiv \frac{\hbar \kappa_+}{2\pi} = \frac{2\sqrt{M^2+n^2-a^2}}{A} \overset{a=M=0}{=} \frac{1}{4\pi n}$$

- Event horizon area

$$A = 8\pi[M^2+n^2-anC+M\sqrt{M^2-a^2+n^2}] \overset{a=M=0}{=} 8\pi n^2$$

- Horizon angular frequency and electric potential

$$\omega_H = \frac{4\pi(am+qQr_+)}{A} \equiv m\Omega_H + q\Phi_H$$

Absorption probability

In Kerr-Taub-NUT

- Amplification factor $\mathscr{A}$, reflection coefficient $\mathscr{R}$ and absorption coefficient Γ

$$\mathscr{A}^s_{lm} \equiv \frac{dE_{\mathrm{out}}}{dE_{\mathrm{in}}} - 1 = \left| \frac{Y^s_{\mathrm{out}}}{Y^s_{\mathrm{in}}} \frac{Z^s_{\mathrm{out}}}{Z^s_{\mathrm{in}}} \right| - 1 = \mathscr{R}^s_{lm} - 1 = -\Gamma^s_{lm}$$

- Absorption probability (grey body factor) at low frequency for boson and fermion

$$\Gamma^s_{lm} = \left(2^l l! \frac{(l-s)!(l+s)!}{(2l)!(2l+1)!} \right)^2 \prod_{n=1}^{l} \left[1 + \left(\frac{1}{n} \frac{\omega - m\Omega_H}{2\pi T_H} \right)^2 \right] \frac{\omega - m\Omega_H}{\pi T_H} (AT_H\omega)^{2l+1} \quad 2s \in \text{even}$$

$$= \left(2^l l! \frac{(l-s)!(l+s)!}{(2l)!(2l+1)!} \right)^2 \prod_{n=1}^{l+1/2} \left[1 + \left(\frac{1}{n-1/2} \frac{\omega - m\Omega_H}{2\pi T_H} \right)^2 \right] \frac{2}{\pi} (AT_H\omega)^{2l+1} \quad 2s \in \text{odd}$$

Emission rate and Power spectrum

In Kerr-Taub-NUT

- Energy and angular momentum flux due to the loss of rotation/static inertial

$$\frac{d}{dt}\begin{pmatrix} M \\ J \end{pmatrix} = -\sum_{s,l,m}\int \frac{dN^s_{lm}}{dt}\begin{pmatrix} \omega \\ m \end{pmatrix}$$

- Emission rate of thermal radiation

Hawking 1975

$$\frac{dN^s_{lm}}{dt} = \frac{\Gamma_{slm}}{2\pi}\frac{d\omega}{e^{(\omega-\omega_H)/T_H} - (-1)^{2s}}$$

- Power spectrum (Angular dependent) of radiation at low frequencies

$$\frac{dE^s_{lm}}{dtd\omega d\cos\theta} \overset{a\omega \ll 1}{=} \frac{1}{2\pi}\left(2^l l!\frac{(l-s)!(l+s)!}{(2l)!(2l+1)!}\right)^2 (2\sqrt{M^2+n^2})^{2l+1}\omega^{2l+2}|S^s_{lm}(\theta,\phi)|^2 d\phi$$

Absorption cross sections

Kerr-Taub-NUT

- Absorption cross section at low frequency

$$\sigma^s = \sum_{l=s}^{\infty} \sigma_l^s, \quad \sigma_l^s = \frac{\pi}{\omega^2} \sum_{m=-l}^{l} \Gamma^s_{lm}$$

- Dominant $l = s$ spin mode
  - Scalar wave $\sigma_0^0 \overset{\omega \to 0}{=} A$
  - Neutrino wave $\sigma_{1/2}^{1/2} \overset{\omega \to 0}{=} 2\pi(M^2 + n^2)$
  - Electromagnetic wave $\sigma_1^1 \overset{\omega \to 0}{=} 4A[3(M^2 + n^2) - a^2]\omega^2/9$
  - Rarita-Schwinger wave $\sigma_{3/2}^{3/2} \overset{\omega \to 0}{=} \pi(M^2 + n^2)[9(M^2 + n^2) + 32a^2]\omega^2/18$
  - Gravitational waves $\sigma_2^2 \overset{\omega \to 0}{=} 8A[10(M^2 + n^2)^2 + 5(M^2 + n^2)a^2 + 2a^4]\omega^4/225$

Summary

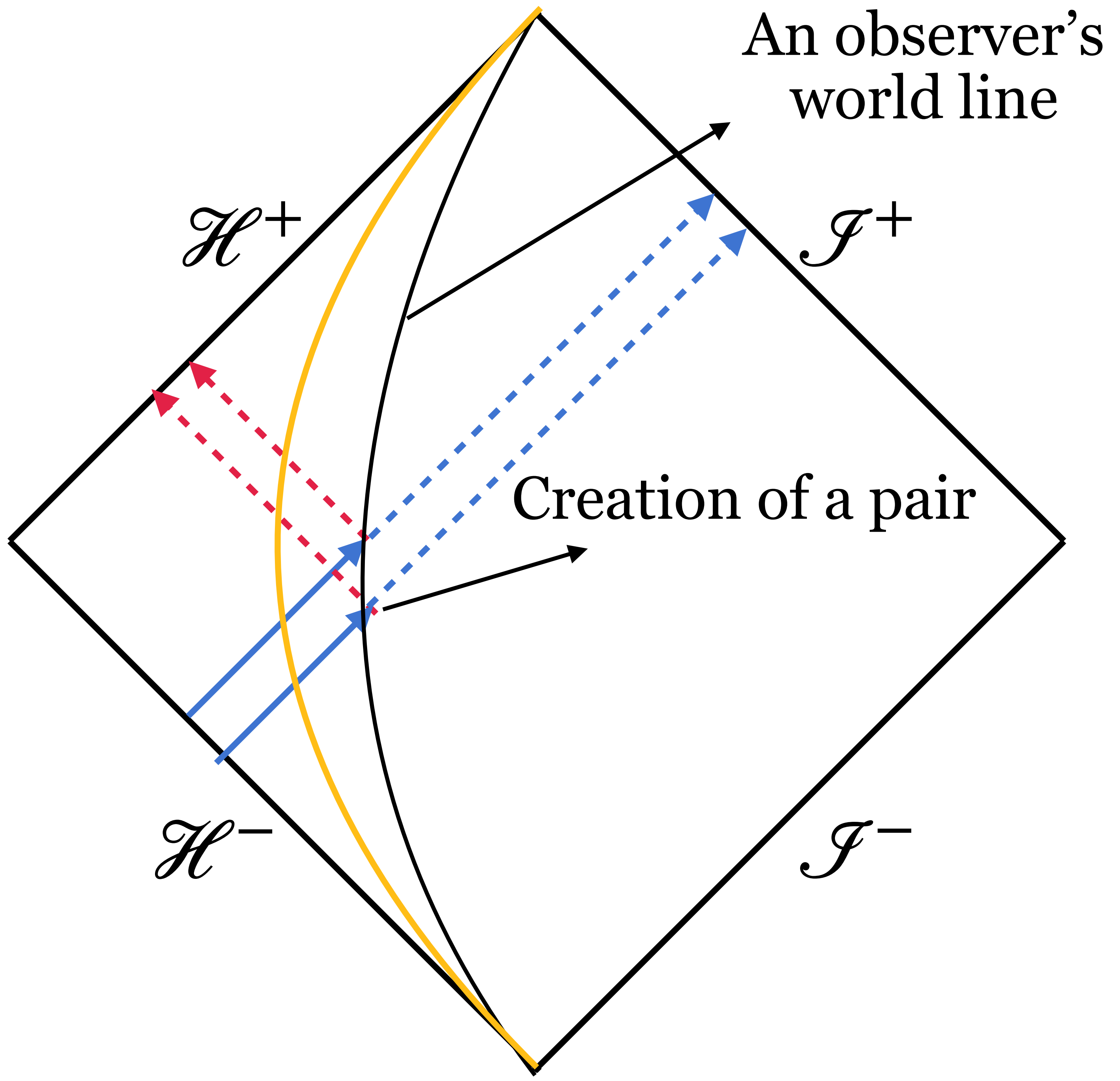
- We study superradiance in Kerr-Taub-NUT spacetime
  - The is a rotating spacetime with a NUT parameter : magnetic mass, or gravitomagnetic charge
  - The rotating spacetime can amplify both classical wave and quantum fluctuation
- We derived a generalized Teukolsky master equations via Newman-Penrose formalism
  - In the formalism, NUT parameter leads to the twist of null congruence at infinity
  - The radial and angular equations are separable: an effective spin $\bar{s} = 2n\omega \in \mathbb{Z}/2$ is quantized
- The superradiantly scattering of all spin-weighted massless particles
  - Absorption probability/cross section, Emission rate and power spectrum of Hawking radiation
  - Scalar, neutrino, photon/electromagnetic wave, Rarita-Schwinger field, and gravitational wave

Radiation on horizons

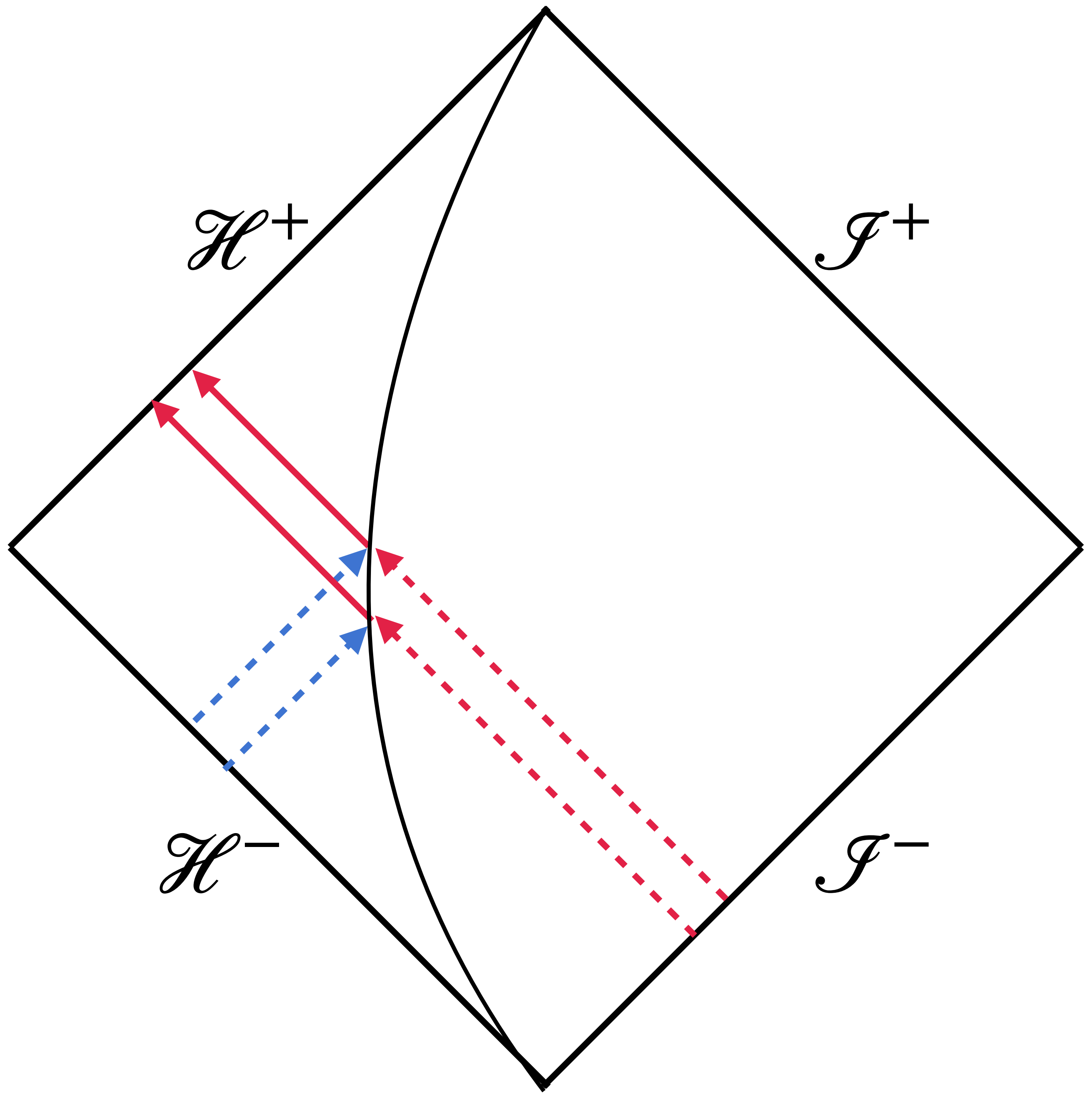
Up and down mode

- $|\,\mathrm{up}\rangle$: no radiation incoming from $\mathscr{F}^-$

$|\,\mathrm{down}\rangle$: no radiation outgoing to $\mathscr{F}^+$



$$R^s_{\mathrm{up}} \overset{r\to r_H}{\sim} Y^s_{\mathrm{up}} e^{i(\omega-\omega_H)r_\star}$$



$$R^s_{\mathrm{down}} \overset{r\to r_+}{\sim} Y^s_{\mathrm{down}} \Delta^{-s} e^{-i(\omega-\omega_H)r_\star}$$