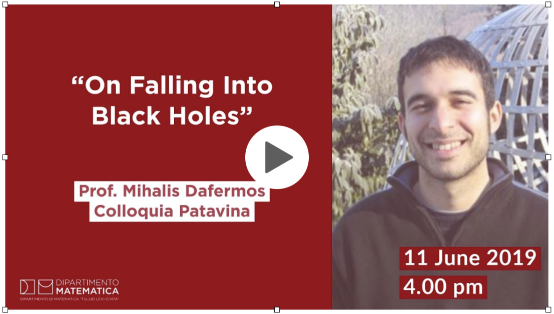
No Scalar-Haired Cauchy Horizon Theorem in Charged Gauss-Bonnet Black Holes

arXiv:2101.10116, 2307.10532

Deniz O. Devecioglu
CQUEST

August 3 2023

Outline



Solutions inside Cauchy horizon is not determined from initial data, i.e. determinism fails.

Strong cosmic censorship conjecture:

Conjecture (Penrose 1972):
The Kerr Cauchy horizon is unstable. For ''generic'' initial data, the spacetime uniquely predicted by the initial data cannot be continuously extended to a larger spacetime. In particular, observers who fall into black holes are necessarily destroyed by tidal forces.

Generically, the future is uniquely determined by the present.

Caption

No Cauchy horizon and scaling charge

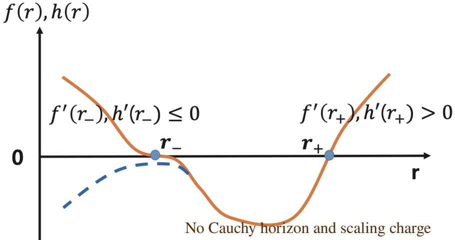
- First, let us start by a quick review of yesterday's talk by Mu-In.

- First, let us start by a quick review of yesterday's talk by Mu-In.
- Problem was to find that whether the Cauchy horizon (inner horizon) is allowed for black holes with charged scalar hairs in different horizon topologies $k = 0, \pm 1$.

- First, let us start by a quick review of yesterday's talk by Mu-In.
- Problem was to find that whether the Cauchy horizon (inner horizon) is allowed for black holes with charged scalar hairs in different horizon topologies $k = 0, \pm 1$.
- Yesterday he discussed the problem for Einstein-Maxwell-scalar (EMS) theories arXiv:2009.05520 and Einstein-Maxwell-Horndeski arXiv:2101.10116. (here by Horndeski I mean $G^{\mu\nu} D_\mu \varphi D_\nu \varphi$). Today I will try to cover what we have done in Einstein-Gauss-Bonnet arXiv:2307.10532 and may be some extras.

- First, let us start by a quick review of yesterday's talk by Mu-In.
- Problem was to find that whether the Cauchy horizon (inner horizon) is allowed for black holes with charged scalar hairs in different horizon topologies $k = 0, \pm 1$.
- Yesterday he discussed the problem for Einstein-Maxwell-scalar (EMS) theories arXiv:2009.05520 and Einstein-Maxwell-Horndeski arXiv:2101.10116. (here by Horndeski I mean $G^{\mu\nu} D_\mu \varphi D_\nu \varphi$). Today I will try to cover what we have done in Einstein-Gauss-Bonnet arXiv:2307.10532 and may be some extras.
- Basic idea is the following: Consider the metric and the possible horizon structure

$$ds^2 = -h(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d^2\Omega_k$$



Then assuming you have the following radially conserved charge $Q(r)$, i.e. $Q(r)' = 0$ on shell.

$$Q(r) = \text{LOCAL PART} + k \int^{r} \text{INTEGRAND}\; d\tilde{r}.$$

Then assuming you have the following radially conserved charge $Q(r)$, i.e. $Q(r)' = 0$ on shell.

$$Q(r) = \text{LOCAL PART} + k \int^{r} \text{INTEGRAND } d\tilde{r}.$$

as this is radially conserved: $Q(r_+) = Q(r_-)$ meaning

$$\text{LOCAL PART}(r_+) - \text{LOCAL PART}(r_-) = k \int_{r_-}^{r_+} \text{INTEGRAND } d\tilde{r}.$$

finding whether this equality can hold or not usually depends on the behaviour of the integrand.

Then assuming you have the following radially conserved charge $Q(r)$, i.e. $Q(r)' = 0$ on shell.

$$Q(r) = \text{LOCAL PART} + k \int^{r} \text{INTEGRAND } d\tilde{r}.$$

as this is radially conserved: $Q(r_+) = Q(r_-)$ meaning

$$\text{LOCAL PART}(r_+) - \text{LOCAL PART}(r_-) = k \int_{r_-}^{r_+} \text{INTEGRAND } d\tilde{r}.$$

finding whether this equality can hold or not usually depends on the behaviour of the integrand.

As a previous example discussed yesterday

$$\begin{aligned} &\frac{(D-2)}{\kappa}\left[r_+^{D-2}\frac{\sqrt{f}}{N}\,N^{2\prime}\Big|_{r_+} - r_-^{D-2}\frac{\sqrt{f}}{N}\,N^{2\prime}\Big|_{r_-}\right] \\ = \; & k(D-2)(D-3)\int_{r_-}^{r_+} dx\; x^{D-4}\left[-\frac{\gamma q^2 A_t^2|\varphi|^2}{\sqrt{N^2 f}} + \frac{N}{\sqrt{f}}\left(-\frac{2}{\kappa} + \gamma f|\varphi'|^2\right)\right]. \end{aligned}$$

In the following we will consider the D dimensional EMGB Lagrangian

$$\mathcal{L} = \kappa \left(R - \frac{Z(|\varphi|^2)}{4} F^{\mu\nu} F_{\mu\nu} \right) + \beta(|\varphi|^2) GB - \alpha (D_\mu \varphi)(D^\mu \varphi)^* - V(|\varphi|^2)$$

where $GB = \left(R_{\mu\nu\alpha\beta} R^{\mu\nu\alpha\beta} - 4 R_{\mu\nu} R^{\mu\nu} + R^2 \right)$ and $D_\mu = \nabla_\mu - iqA_\mu$ with $A_\mu = A_t(r)dt$.

In the following we will consider the D dimensional EMGB Lagrangian

$$\mathcal{L} = \kappa \left(R - \frac{Z(|\varphi|^2)}{4} F^{\mu\nu} F_{\mu\nu} \right) + \beta(|\varphi|^2) GB - \alpha (D_\mu \varphi)(D^\mu \varphi)^* - V(|\varphi|^2)$$

where $GB = \left(R_{\mu\nu\alpha\beta} R^{\mu\nu\alpha\beta} - 4 R_{\mu\nu} R^{\mu\nu} + R^2 \right)$ and $D_\mu = \nabla_\mu - iqA_\mu$ with $A_\mu = A_t(r) dt$.

For the sake of the discussion of the scaling charge this form of metric is more suitable

The alternate metric with $k = 0$

$$ds^2 = -A(\rho)^2 dt^2 + d\rho^2 + B(\rho)^2 d\mathbf{x}_{D-2}^2,$$

where the coordinate transformations to the usual f, h metric are

$$A = \sqrt{h}, \quad B = r, \quad \rho = \int \frac{dr}{\sqrt{f}}, \quad \frac{\partial \rho}{\partial r} = \frac{1}{\sqrt{f}}, \quad \frac{\partial}{\partial \rho} = \sqrt{f} \frac{\partial}{\partial r}.$$

With this choice of the planar metric ($k = 0$) the reduced action is invariant under the following finite scaling transformations

The scaling transformations

$$A(\rho) \to \lambda^{2-D} A(\rho), \quad B(\rho) \to \lambda B(\rho), \quad A_t(\rho) \to \lambda^{2-D} A_t(\rho)$$

With this choice of the planar metric ($k=0$) the reduced action is invariant under the following finite scaling transformations

The scaling transformations

$$A(\rho) \to \lambda^{2-D} A(\rho), \quad B(\rho) \to \lambda B(\rho), \quad A_t(\rho) \to \lambda^{2-D} A_t(\rho)$$

In order to find the Noether charge we use a field theory trick where the global symmetry parameter λ is *localized* as $\lambda(\rho)$ and then the Noether charge is the coefficient of $\lambda'(\rho)$ in the variation of the action. (arXiv:1605.07128)

$$\begin{aligned} Q &= 2\kappa(D-2)\left(A'B^{D-2} - AB^{D-3}B'\right) - \frac{\kappa(D-2)B^{D-2}A_t A_t' Z\left(|\varphi|^2\right)}{A} \\ &\quad +4\beta(|\varphi|^2)(D-4)(D-3)(D-2)B^{D-5}\left(AB'^3 - A'BB'^2\right) \\ &\quad +16\dot{\beta}(|\varphi|^2)|\varphi\varphi'|B^{D-4}B'(AB' - A'B). \end{aligned} \tag{1}$$

It is easy to show that the derivative of the charge is a combination of the field equations

$$\mathcal{Q}' = (D-2)\left(AE_A - \frac{BE_B}{(D-2)} + A_t E_{A_t}\right), \tag{2}$$

It is easy to show that the derivative of the charge is a combination of the field equations

$$\mathcal{Q}' \quad = \quad (D-2)\left(AE_A - \frac{BE_B}{(D-2)} + A_t E_{A_t}\right), \tag{2}$$

Remember this was for $k = 0$, but when we consider $k = \pm 1$ metrics

$$ds^2 = -A(\rho)^2 dt^2 + d\rho^2 + B(\rho)^2 d\Omega^2_{D-2},$$

the EMGB action in general will not be invariant under the scalings we have discussed. That is where we get the k dependent integral (non-local) pieces.

It is easy to show that the derivative of the charge is a combination of the field equations

$$\mathcal{Q}' \quad = \quad (D-2)\left(AE_A - \frac{BE_B}{(D-2)} + A_t E_{A_t}\right), \tag{2}$$

Remember this was for $k = 0$, but when we consider $k = \pm 1$ metrics

$$ds^2 = -A(\rho)^2 dt^2 + d\rho^2 + B(\rho)^2 d\Omega^2_{D-2},$$

the EMGB action in general will not be invariant under the scalings we have discussed. That is where we get the k dependent integral (non-local) pieces.

However, one can still *construct* a conserved charge by simply adding the *space integral* of all the non-conservation terms in (2) with a minus sign so that all the non-conservation terms are canceled. But in this case, the conserved charge has *non-local* (*i.e.* integral) terms as well as the usual local terms, and so it seems that there is no relation to the action invariance and it is beyond the Noether's theorem.

Transforming back to the f, h coordinates after finding the charge in A, B coordinates we have

The scaling charge for all k

$$
\begin{aligned}
\mathcal{Q} \;=\; & (D-2) r^{D-3} \sqrt{\frac{f}{h}} \left[\kappa \left(r h' - 2h - r Z A_t A_t{}' \right) - \frac{2(D-3)}{r^2} f \left(r h' - 2h \right) \left((D-4)\beta + 4 r |\varphi \varphi'| \dot{\beta} \right) \right] \\
& + k \int^r dr\, r^{D-4} \left\{ 2(D-2)(D-3) \sqrt{\frac{h}{f}} \left[\kappa - 4\dot{\beta} \left(|\varphi(f'\varphi' + 2f\varphi'')| + 2f|\varphi'|^2 \right) - 16\ddot{\beta} f |\varphi\varphi'|^2 \right] \right. \\
& + \frac{(D-4)(D-3)(D-2)}{r\sqrt{f} h^{3/2}} \left[\beta \left(r h (f'h' + 2 f h'') - 6 f' h^2 - r f h'^2 \right) + 4 \dot{\beta} f h (r h' - 6h) |\varphi\varphi'| \right] \\
& \left. + \frac{\sum_{n=5}^{D} (n-5)(n-4)(n-3)}{r^2 \sqrt{fh}} \, 8\beta \left[r f h' + 2(k - 2f) h \right] \right\}, \qquad (3)
\end{aligned}
$$

Transforming back to the f, h coordinates after finding the charge in A, B coordinates we have

The scaling charge for all k

$$\begin{aligned}
\mathcal{Q} \;=\;& (D-2)r^{D-3}\sqrt{\frac{f}{h}}\left[\kappa\left(rh' - 2h - rZA_tA_t{}'\right) - \frac{2(D-3)}{r^2}f\left(rh' - 2h\right)\left((D-4)\beta + 4r|\varphi\varphi'|\dot{\beta}\right)\right] \\
&+k\int^r dr\, r^{D-4}\left\{2(D-2)(D-3)\sqrt{\frac{h}{f}}\left[\kappa - 4\dot{\beta}\left(|\varphi(f'\varphi' + 2f\varphi'')| + 2f|\varphi'|^2\right) - 16\ddot{\beta}f|\varphi\varphi'|^2\right]\right. \\
&+\frac{(D-4)(D-3)(D-2)}{r\sqrt{f}h^{3/2}}\left[\beta\left(rh(f'h' + 2fh'') - 6f'h^2 - rfh'^2\right) + 4\dot{\beta}fh(rh' - 6h)|\varphi\varphi'|\right] \\
&\left.+\frac{\sum_{n=5}^{D}(n-5)(n-4)(n-3)}{r^2\sqrt{fh}}\,8\beta\left[rfh' + 2(k-2f)h\right]\right\}, \qquad (3)
\end{aligned}$$

Transforming back to the f, h coordinates after finding the charge in A, B coordinates we have

The scaling charge for all k

$$
\begin{aligned}
\mathcal{Q} \; = \; & (D-2) r^{D-3} \sqrt{\frac{f}{h}} \left[\kappa \left(r h' - 2h - r Z A_t A_t{}' \right) - \frac{2(D-3)}{r^2} f (r h' - 2h) \left((D-4)\beta + 4 r |\varphi \varphi'| \dot{\beta} \right) \right] \\
& + k \int^r dr \, r^{D-4} \Bigg\{ 2(D-2)(D-3) \sqrt{\frac{h}{f}} \left[\kappa - 4\dot{\beta} \left(|\varphi (f' \varphi' + 2 f \varphi'')| + 2 f |\varphi'|^2 \right) - 16 \ddot{\beta} f |\varphi \varphi'|^2 \right] \\
& + \frac{(D-4)(D-3)(D-2)}{r \sqrt{f} h^{3/2}} \left[\beta \left(r h (f' h' + 2 f h'') - 6 f' h^2 - r f h'^2 \right) + 4 \dot{\beta} f h (r h' - 6h) |\varphi \varphi'| \right] \\
& + \frac{\sum_{n=5}^{D} (n-5)(n-4)(n-3)}{r^2 \sqrt{fh}} \, 8\beta \left[r f h' + 2(k - 2f) h \right] \Bigg\}, \qquad (4)
\end{aligned}
$$

Now, due to the r-independence of the scaling charge $\mathcal{Q}$, one can consider the charges at the horizons, in particular, the outer event horizon r_+ and the inner Cauchy horizon r_-, if exists, so that we have

$$\begin{aligned}
&\kappa(D-2)\left[r_+^{D-2}\sqrt{\frac{f}{h}}\left(h'-ZA_tA_t'\right)\Big|_{r_+}-r_-^{D-2}\sqrt{\frac{f}{h}}\left(h'-ZA_tA_t'\right)\Big|_{r_-}\right]\\
=\ &-k\int_{r_-}^{r_+}dr\, r^{D-4}\left\{2(D-2)(D-3)\sqrt{\frac{h}{f}}\left[\kappa-4\dot{\beta}\left(|\varphi(f'\varphi'+2f\varphi'')|+2f|\varphi'|^2\right)-16\ddot{\beta}f|\varphi\varphi'|^2\right]\right.\\
&+\frac{(D-4)(D-3)(D-2)}{r\sqrt{f}h^{3/2}}\left[\beta\left(rh(f'h'+2fh'')-6f'h^2-rfh'^2\right)+4\dot{\beta}fh(rh'-6h)|\varphi\varphi'|\right]\\
&\left.+\frac{\sum_{n=5}^{D}(n-5)(n-4)(n-3)}{r^2\sqrt{fh}}\,8\beta\left[rfh'+2(k-2f)h\right]\right\}. \qquad (5)
\end{aligned}$$

Now, due to the r-independence of the scaling charge $\mathcal{Q}$, one can consider the charges at the horizons, in particular, the outer event horizon r_+ and the inner Cauchy horizon r_-, if exists, so that we have

$$\begin{aligned}
&\kappa(D-2)\left[r_+^{D-2}\sqrt{\frac{f}{h}}\left(h' - ZA_tA_t'\right)\bigg|_{r_+} - r_-^{D-2}\sqrt{\frac{f}{h}}\left(h' - ZA_tA_t'\right)\bigg|_{r_-}\right] \\
= \; & -k\int_{r_-}^{r_+} dr\, r^{D-4}\Bigg\{ 2(D-2)(D-3)\sqrt{\frac{h}{f}}\left[\kappa - 4\dot{\beta}\left(|\varphi(f'\varphi' + 2f\varphi'')| + 2f|\varphi'|^2\right) - 16\ddot{\beta}f|\varphi\varphi'|^2\right] \\
& + \frac{(D-4)(D-3)(D-2)}{r\sqrt{f}h^{3/2}}\left[\beta\left(rh(f'h' + 2fh'') - 6f'h^2 - rfh'^2\right) + 4\dot{\beta}fh(rh' - 6h)|\varphi\varphi'|\right] \\
& + \frac{\sum_{n=5}^{D}(n-5)(n-4)(n-3)}{r^2\sqrt{fh}}\, 8\beta\left[rfh' + 2(k-2f)h\right]\Bigg\}\,. \qquad (5)
\end{aligned}$$

Moreover, one can prove that A_t needs to be zero at the horizons $A_t(r_+) = A_t(r_-) = 0$, from the regularity at the horizons with "charged" (complex) scalar fields. This regularity condition is another key ingredient for the proof of the theorem.

Even for $D = 4$ it seems with $k \neq 0$, the right-hand side does not have a definite sign generally with the GB terms ($\beta \neq 0$), and so there is no simple condition for the (non) existence of the inner (or outer) horizon associated with non-planar topologies.

$$\begin{aligned} & 2\kappa \left[r_+^2 \sqrt{\frac{f}{h}} \left(h' - ZA_tA_t'\right)\bigg|_{r_+} - r_-^2 \sqrt{\frac{f}{h}} \left(h' - ZA_tA_t'\right)\bigg|_{r_-} \right] \\ = \quad & -k \int_{r_-}^{r_+} dr \, \left\{ 4\sqrt{\frac{h}{f}} \left[\kappa - 4\dot{\beta} \left(|\varphi(f'\varphi' + 2f\varphi'')| + 2f|\varphi'|^2 \right) - 16\ddot{\beta} f |\varphi\varphi'|^2 \right] \right\} . \end{aligned} \tag{6}$$

I will speculate on the integrand term and the form of it for some special solutions (If I have time).

Setup and field equations

In the following we will consider the D dimensional action

$$\mathcal{L} = \kappa\left(R - \frac{Z(|\varphi|^2)}{4}F^{\mu\nu}F_{\mu\nu}\right) + \beta(|\varphi|^2)GB - \alpha(D_\mu\varphi)(D^\mu\varphi)^* - V(|\varphi|^2)$$

where $GB = \left(R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta} - 4R_{\mu\nu}R^{\mu\nu} + R^2\right)$ and $D_\mu = \nabla_\mu - iqA_\mu$ with $A_\mu = A_t(r)dt$.

In the following we will consider the D dimensional action

$$\mathcal{L} = \kappa \left(R - \frac{Z(|\varphi|^2)}{4} F^{\mu\nu} F_{\mu\nu} \right) + \beta(|\varphi|^2) GB - \alpha (D_\mu \varphi)(D^\mu \varphi)^* - V(|\varphi|^2)$$

where $GB = \left(R_{\mu\nu\alpha\beta} R^{\mu\nu\alpha\beta} - 4 R_{\mu\nu} R^{\mu\nu} + R^2 \right)$ and $D_\mu = \nabla_\mu - iqA_\mu$ with $A_\mu = A_t(r) dt$.

The metric ansatz reads

$$ds^2 = -e^{A(r)} dt^2 + e^{B(r)} dr^2 + r^2 d_k \Omega^2.$$

In the following we will consider the D dimensional action

$$\mathcal{L} = \kappa\left(R - \frac{Z(|\varphi|^2)}{4}F^{\mu\nu}F_{\mu\nu}\right) + \beta(|\varphi|^2)GB - \alpha(D_\mu\varphi)(D^\mu\varphi)^* - V(|\varphi|^2)$$

where $GB = \left(R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta} - 4R_{\mu\nu}R^{\mu\nu} + R^2\right)$ and $D_\mu = \nabla_\mu - iqA_\mu$ with $A_\mu = A_t(r)dt$.

The metric ansatz reads

$$ds^2 = -e^{A(r)}dt^2 + e^{B(r)}dr^2 + r^2 d_k\Omega^2.$$

In this form the metric assuming there is an event horizon at $r = r_H$ we should have $e^A|_{\to r_H^+} \to 0^+$ outside the horizon and $e^A|_{\to r_H^-} \to 0^-$ where $r \to r_H^+$ means approaching the horizon from outside or $r \to r_H^-$ approaching from inside. Then from the near horizon expansion as $r \to r_H^\pm$ then $A'(r_H) \to \pm\infty$.

$$\begin{aligned} e^{A(r)} =& a_1(r - r_H) + a_2(r - r_H)^2 + \cdots \\ =& e^{A(r_H)}A'(r_H)(r - r_H) + \cdots \end{aligned}$$

Now we need to solve $A(r), B(r), A_t(r), \varphi(r)$. First focus on the equation for $B(r)$ which is algebraic

$$e^{2B}B_1(A, A_t, \varphi) + e^{B}B_2(A, A_t, \varphi) + B_3(A, A_t, \varphi) = 0.$$

Now we need to solve $A(r), B(r), A_t(r), \varphi(r)$. First focus on the equation for $B(r)$ which is algebraic

$$e^{2B}B_1(A, A_t, \varphi) + e^B B_2(A, A_t, \varphi) + B_3(A, A_t, \varphi) = 0.$$

The solution is

$$e^B = \frac{-\mu(r) \pm \sqrt{\mu(r)^2 - 4\nu(r)}}{2},$$

where

$$\mu(r) = \frac{2\left(4k|\varphi\varphi'|\dot{\beta} + r/\kappa\right)A' + \left(r^2 e^{-A} Z A_t'^2 + 4\right)/2\kappa - \alpha r^2 |\varphi'|^2}{-\alpha q^2 r^2 e^{-A}|\varphi|^2 A_t^2 - 2k/\kappa + r^2 V},$$

$$\nu(r) = \frac{-24|\varphi\varphi'|\dot{\beta}A'}{-\alpha q^2 r^2 e^{-A}|\varphi|^2 A_t^2 - 2k/\kappa + r^2 V}$$

From this solution it is useful to write down the derivative of $B(r)'$ as follows

$$B'(r) = -\frac{e^B \mu' + \nu'}{e^B \mu + 2e^{2B}}.$$

From this solution it is useful to write down the derivative of $B(r)'$ as follows

$$B'(r) = -\frac{e^B\mu' + \nu'}{e^B\mu + 2e^{2B}}.$$

Then employing the rest of the field equations we can find the rest of the equations satisfied by A'', φ'', A_t''

$$A'' = \frac{P}{S}, \quad \varphi'' = \frac{Q}{S}, \quad A_t'' = \frac{R}{Y}$$

with P, Q, R, S, Y are functions involving at most the first derivatives of the fields and second derivative of the coupling i.e. $\ddot{\beta}(|\varphi|^2)$.

As there are no known exact solutions we will resort to the numerical approach, which needs the information about initial values of the fields at the horizon.

Horizon expansion

No gauge field $A_t = 0$, q=0

No gauge field $A_t = 0$, q=0

- Einstein limit $\beta = 0$
  The solution for $e^{B(r)}$ is a single root.

No gauge field $A_t = 0$, q=0

- Einstein limit $\beta = 0$
  The solution for $e^{B(r)}$ is a single root.
- EGB limit $\beta \neq 0$ (some simplification) arXiv:1711.03390, arXiv:2305.19814...
  From a quick glance of the solution $e^{B(r)}$ one can see $e^{A(r)}$ terms are gone.

No gauge field $A_t = 0$, q=0

- Einstein limit $\beta = 0$
  The solution for $e^{B(r)}$ is a single root.
- EGB limit $\beta \neq 0$ (some simplification) arXiv:1711.03390, arXiv:2305.19814...
  From a quick glance of the solution $e^{B(r)}$ one can see $e^{A(r)}$ terms are gone.
- Now as we have discussed $e^B \longrightarrow \infty$ as $r \longrightarrow r_H$ meaning $A' \longrightarrow \infty$. In order to check the behaviour of $e^{B(r)}$ we take $A' = 1/c$ and expand around $c = 0$ (for the plus root)

No gauge field $A_t = 0$, q=0

- Einstein limit $\beta = 0$
  The solution for $e^{B(r)}$ is a single root.
- EGB limit $\beta \neq 0$ (some simplification) arXiv:1711.03390, arXiv:2305.19814...
  From a quick glance of the solution $e^{B(r)}$ one can see $e^{A(r)}$ terms are gone.
- Now as we have discussed $e^B \longrightarrow \infty$ as $r \longrightarrow r_H$ meaning $A' \longrightarrow \infty$. In order to check the behaviour of $e^{B(r)}$ we take $A' = 1/c$ and expand around $c = 0$ (for the plus root)

all evaluated ar $r = r_H$

$$e^B = \frac{8\varphi\varphi'\dot{\beta}(\varphi^2) + 2\kappa r}{\left(2\kappa - r^2 V(\varphi^2)\right)} A' + \frac{\kappa r\left(2\kappa - \alpha r^2\varphi'^2\right) - 4\varphi\varphi'\dot{\beta}(\varphi^2)\left(4\kappa + \alpha r^2\varphi'^2 - 3r^2 V(\varphi^2)\right)}{\left(2\kappa - r^2 V(\varphi^2)\right)\left(4\varphi\varphi'\dot{\beta}(\varphi^2) + \kappa r\right)} + \mathcal{O}(1/A')$$

- The condition $8\varphi\varphi'\dot{\beta}(\varphi^2) + 2\kappa r = 0$ actually forces $\beta(\varphi) \longrightarrow \infty$ at the horizon which will lead to divergent or trivial scalar field at the horizon.

No gauge field $A_t = 0$, q=0

- Einstein limit $\beta = 0$
  The solution for $e^{B(r)}$ is a single root.
- EGB limit $\beta \neq 0$ (some simplification) arXiv:1711.03390, arXiv:2305.19814...
  From a quick glance of the solution $e^{B(r)}$ one can see $e^{A(r)}$ terms are gone.
- Now as we have discussed $e^B \longrightarrow \infty$ as $r \longrightarrow r_H$ meaning $A' \longrightarrow \infty$. In order to check the behaviour of $e^{B(r)}$ we take $A' = 1/c$ and expand around $c = 0$ (for the plus root)

all evaluated ar $r = r_H$

$$e^B = \frac{8\varphi\varphi'\dot{\beta}(\varphi^2) + 2\kappa r}{\left(2\kappa - r^2 V(\varphi^2)\right)} A' + \frac{\kappa r \left(2\kappa - \alpha r^2 \varphi'^2\right) - 4\varphi\varphi'\dot{\beta}(\varphi^2)\left(4\kappa + \alpha r^2 \varphi'^2 - 3r^2 V(\varphi^2)\right)}{\left(2\kappa - r^2 V(\varphi^2)\right)\left(4\varphi\varphi'\dot{\beta}(\varphi^2) + \kappa r\right)} + \mathcal{O}(1/A')$$

- The condition $8\varphi\varphi'\dot{\beta}(\varphi^2) + 2\kappa r = 0$ actually forces $\beta(\varphi) \longrightarrow \infty$ at the horizon which will lead to divergent or trivial scalar field at the horizon.

- On the other hand, the root with the minus sign has the following structure

- On the other hand, the root with the minus sign has the following structure

$$e^B = \frac{12\varphi\varphi'\dot{\beta}(\varphi^2)}{4\varphi\varphi'\dot{\beta}(\varphi^2) + \kappa r} + \mathcal{O}(1/A') + \cdots$$

- After making sure that you choose the correct root, apply the expansion to the remaning field equations

$$A'' = \frac{P}{S}, \quad \varphi'' = \frac{Q}{S}$$

- On the other hand, the root with the minus sign has the following structure

$$e^B = \frac{12\varphi\varphi'\dot{\beta}(\varphi^2)}{4\varphi\varphi'\dot{\beta}(\varphi^2) + \kappa r} + \mathcal{O}(1/A') + \cdots$$

- After making sure that you choose the correct root, apply the expansion to the remaning field equations

$$A'' = \frac{P}{S}, \quad \varphi'' = \frac{Q}{S}$$

all evaluated ar $r = r_H$

$$\varphi'' = \frac{Q}{S} = \frac{Q_1 A'^2 + Q_2 A' + Q_3 + \mathcal{O}(1/A')}{S_1 A' + S_2 + \mathcal{O}(1/A')} = f(T, \varphi, \varphi')\, A' + \mathcal{O}(1)$$

- In order to have a finite φ'' we need $f(T, \varphi, \varphi') = 0$

Solving this condition we have three roots for $\varphi'(r_H)$

$$\varphi'(r_H)_{1,2} = -\frac{\kappa r_H^2}{8 r_H \varphi \dot{\beta}} \left(1 \pm \sqrt{1 - \frac{192 \varphi^2 \dot{\beta}^2}{\alpha \kappa r_H^4}} \right), \quad \varphi'(r_H)_3 = -\frac{\kappa r_H}{4 \varphi \dot{\beta}} \tag{7}$$

among these solutions $\varphi(r_H)_{1,2}$ one of them gives the correct Einstein limit! (Remember yesterday Miok's talk.)

$A_t \neq 0$, q=0

$A_t \neq 0$, q=0

- This time the solution for $e^{B(r)}$ has $e^{A(r)}$ terms in it

$A_t \neq 0$, q=0

- This time the solution for $e^{B(r)}$ has $e^{A(r)}$ terms in it

$$e^B = \frac{-\mu(r) \pm \sqrt{\mu(r)^2 - 4\nu(r)}}{2},$$

where

$$\mu(r) = \frac{2\left(4k|\varphi\varphi'|\dot{\beta} + r/\kappa\right)A' + \left(r^2 e^{-A} Z A_t'^2 + 4\right)/2\kappa - \alpha r^2 |\varphi'|^2}{-2k/\kappa + r^2 V},$$

$$\nu(r) = \frac{-24|\varphi\varphi'|\dot{\beta}A'}{-2k/\kappa + r^2 V}$$

$A_t \neq 0$, q=0

- This time the solution for $e^{B(r)}$ has $e^{A(r)}$ terms in it

$$e^B = \frac{-\mu(r) \pm \sqrt{\mu(r)^2 - 4\nu(r)}}{2},$$

where

$$\mu(r) = \frac{2\left(4k|\varphi\varphi'|\dot{\beta} + r/\kappa\right) A' + \left(r^2 e^{-A} Z A_t'^2 + 4\right)/2\kappa - \alpha r^2 |\varphi'|^2}{-2k/\kappa + r^2 V},$$

$$\nu(r) = \frac{-24|\varphi\varphi'|\dot{\beta} A'}{-2k/\kappa + r^2 V}$$

Therefore we first need to compare e^{-A} and A' as $r \to r_H$. For this consider the following expansion of e^A near the horizon

$$e^A = a_1(r - r_H) + a_2(r - r_H) + \cdots, \tag{8}$$

then A' can be written as

$$A' = (e^A)' e^{-A} = a_1 + 2a_2(r - r_H) + \cdots \tag{9}$$

finally we can check the limit A'/e^{-A} as $r \longrightarrow r_H$

$$\lim_{r \to r_H} \frac{A'}{e^{-A}} = a_1 = T \tag{10}$$

so $e^{-A} \approx A'/T$. Employing this the $e^{B(r)}$ near horizon expansion will be finally we can check the limit A'/e^{-A} as $r \longrightarrow r_H$

$$\lim_{r \to r_H} \frac{A'}{e^{-A}} = a_1 = T \tag{10}$$

so $e^{-A} \approx A'/T$. Employing this the $e^{B(r)}$ near horizon expansion will be

all evaluated ar $r = r_H$

$$e^B = \frac{\kappa r^2 Z(\varphi) A_t'^2 + 16T\varphi\varphi'\dot{\gamma}(\varphi^2) + 4\kappa r T}{4\kappa T - 2r^2 T V(\varphi^2)} A' + \mathcal{O}(0) + \mathcal{O}(1/A')$$

Again the other root is not suitable, i.e. finite at the horizon. This time we have three field equations

$$A'' = \frac{P}{S}, \quad \varphi'' = \frac{Q}{S}, \quad A_t'' = \frac{R}{Y} \tag{11}$$

Expanding φ''

$$\varphi'' = \frac{Q}{S} = \frac{Q_1 A'^2 + Q_2 A' + Q_3 + \mathcal{O}(1/A')}{S_1 A' + S_2 + \mathcal{O}(1/A')} = f(T, \varphi, \varphi', A_t') A' + \mathcal{O}(1) \tag{12}$$

$$
\begin{aligned}
\varphi' =& f_{1\pm}/f_2\\
f_{1\pm} =& 4\kappa^2 r^4 TZA_t'^2\dot{\gamma}\dot{V} - 2\alpha\kappa^3 r^4 TZA_t'^2 - 8\alpha\kappa^3 r^3 T^2 + 16\kappa^2 r^3 T^2\dot{\gamma}\dot{V} + 16 r^3 T^2 V^2\dot{\gamma}^2\\
&-4\kappa^3 r^2 TA_t'^2\dot{\gamma}\dot{Z} + \kappa r TV\left(\alpha\kappa r^4\left(rZA_t'^2+4T\right) + 2\kappa r^3 A_t'^2\dot{\gamma}\dot{Z} - 24\dot{\gamma}^2\left(rZA_t'^2+4T\right)\right) + 16\kappa^2 TZA_t'^2\dot{\gamma}^2\\
&\quad\pm\left\{T^2\left(r^2V-2\kappa\right)^2\left[\alpha^2\kappa^4 r^6\left(rZA_t'^2+4T\right)^2 - 4\alpha\kappa^4 r^5 A_t'^2\dot{\gamma}\dot{Z}\left(rZA_t'^2+4T\right)\right.\right.\\
&+4\kappa^2 r^2\dot{\gamma}^2\left(\kappa\left(\kappa r^2 A_t'^4\dot{Z}^2 + 4\alpha r^2 Z^2 A_t'^4 - 16\alpha r TZA_t'^2 - 192\alpha T^2\right) + 8\alpha r^2 TV\left(rZA_t'^2+8T\right)\right)\\
&-64\dot{\gamma}^4\left(-4r^2T^2V^2 + 12\kappa TV\left(rZA_t'^2+4T\right) - \kappa^2 Z^2 A_t'^4\right)\\
&\left.\left.+32\kappa^2 r^2\dot{\gamma}^3\left(4T\dot{V}\left(rZA_t'^2+6T\right) + A_t'^2\dot{Z}\left(2rTV-\kappa ZA_t'^2\right)\right)\right]\right\}^{1/2}\\
f_2 =& 16T^2\dot{\gamma}\left(V\left(8\dot{\gamma}^2-\alpha\kappa r^4\right) + 2\kappa r^2\left(\alpha\kappa-\dot{\gamma}\dot{V}\right)\right).
\end{aligned}
$$

$A_t \neq 0, q \neq 0$

$A_t \neq 0, q \neq 0$

- Doing the e^B expansion as usual we find

$$e^B = \frac{1}{4\alpha q^2 r^2 \varphi^2 A_t^2} (\cdots) + \mathcal{O}(1/A') \tag{13}$$

both of the roots are finite!

$A_t \neq 0, q \neq 0$

- Doing the e^B expansion as usual we find

$$e^B = \frac{1}{4\alpha q^2 r^2 \varphi^2 A_t^2}(\cdots) + \mathcal{O}(1/A') \tag{13}$$

  both of the roots are finite!
- We should set the gauge field zero at the horizon, i.e. $A_t(r_H) = 0$. Therefore, we should take two limits to understand near horizon behaviour: one for $A' \longrightarrow \infty$ and the other $A_t \longrightarrow 0$ both as $r \longrightarrow r_H$.

$A_t \neq 0, q \neq 0$

- Doing the e^B expansion as usual we find

$$e^B = \frac{1}{4\alpha q^2 r^2 \varphi^2 A_t^2}(\cdots) + \mathcal{O}(1/A') \tag{13}$$

  both of the roots are finite!
- We should set the gauge field zero at the horizon, i.e. $A_t(r_H) = 0$. Therefore, we should take two limits to understand near horizon behaviour: one for $A' \longrightarrow \infty$ and the other $A_t \longrightarrow 0$ both as $r \longrightarrow r_H$.
- We call $A' = 1/c$ and $A_t = d$ then expand around both parameters. The final result should not depend on the expansion, however if one does first c expansion then d expansion, the lowest order in c can not be obtained. Therefore we first expand in d then expand in c the result is

$A_t \neq 0, q \neq 0$

- Doing the e^B expansion as usual we find

$$e^B = \frac{1}{4\alpha q^2 r^2 \varphi^2 A_t^2} (\cdots) + \mathcal{O}(1/A') \tag{13}$$

  both of the roots are finite!
- We should set the gauge field zero at the horizon, i.e. $A_t(r_H) = 0$. Therefore, we should take two limits to understand near horizon behaviour: one for $A' \longrightarrow \infty$ and the other $A_t \longrightarrow 0$ both as $r \longrightarrow r_H$.
- We call $A' = 1/c$ and $A_t = d$ then expand around both parameters. The final result should not depend on the expansion, however if one does first c expansion then d expansion, the lowest order in c can not be obtained. Therefore we first expand in d then expand in c the result is

$$e^{B(r)} = \frac{k\kappa r^2 Z\left(\varphi^2\right) A_t'^2 + 16T\varphi\varphi'\dot{\gamma}\left(\varphi^2\right) + 4k\kappa r T}{2T\left(2\kappa - kr^2 V\left(\varphi^2\right)\right)} \frac{1}{c} + \frac{\alpha q^2 r^2 \varphi^2 \left(\alpha r^2 \varphi'^2 - 2\kappa\right)}{T\left(r^2 V\left(\varphi^2\right) - 2k\kappa\right)^2} \frac{d^2}{c} + \cdots \tag{14}$$

- In order to be able to solve the φ'' expansion here we assume $d^2/c \to 0.$ Then the results of the previous section applies for the solution of $\varphi(r_H)'$ as there are no q^2 contributions.
- Armed with the knowledge of initial condition on the φ' we are now in a a position to solve the field equations numerically.

Numerical solutions

Here we first present numerical solution of the four-dimensional ($D = 4$) hyperbolic charged GB black hole in EMGBS gravity for the choice of the model

$$\beta(|\varphi|^2) = \lambda|\varphi|^2,\ V(|\varphi|^2) = -6 + m^2|\varphi|^2,\ Z(|\varphi|^2) = 1, \quad \lambda = 10^{-3},\ m^2 = -0.18,\ q = 2.5. \tag{15}$$

Here we first present numerical solution of the four-dimensional ($D = 4$) hyperbolic charged GB black hole in EMGBS gravity for the choice of the model

$$\beta(|\varphi|^2) = \lambda|\varphi|^2,\ V(|\varphi|^2) = -6 + m^2|\varphi|^2,\ Z(|\varphi|^2) = 1, \quad \lambda = 10^{-3},\ m^2 = -0.18,\ q = 2.5. \quad (15)$$

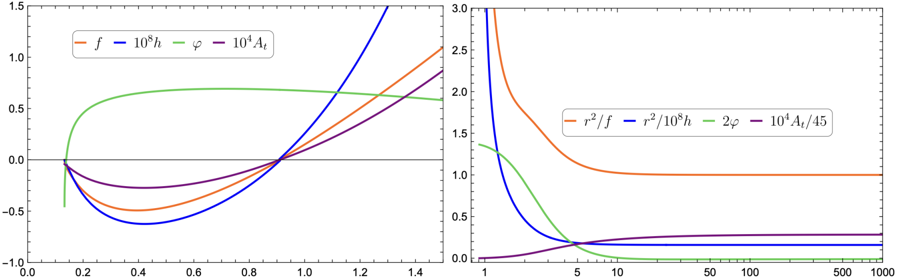


FIG. 6: Numerical solutions of a $D = 4$ hyperbolic charged GB black hole with the model (27) and initial condition (28). In the left panel, we plot f (orange), 10^8h (blue), φ (green), 10^4A_t (purple), and no solution is found for the deep interior region due to high numerical errors at the inner horizon $r_- \approx 0.132916515$. In the right panel, we plot $r^2/f, r^2/(10^8h), 2\varphi, 10^4A_t/45$, and they show the *AdS-like* behaviors in f, h, $\varphi \sim 0$ and finite A_t as in Fig. 2.

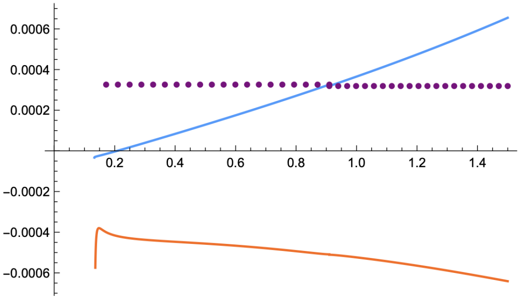


FIG. 7: Plots of the scaling charge $\mathcal{Q}$ (purple) and its local (blue), integrand (orange) parts for the numerical solution in Fig. 6.

Conclusions-Future Work

- In this work utilizing the scaling charge we tried to determine whether the black holes of EMGB admits Cauchy horizon.
- However, other than $k = 0$ case it was not possible to determine the topologies that admit Cauchy horizon because of the complicated structure of the non-local term.
- Employing numerical techniques we were able to the existence of the inner horizon for $k = -1$ hyperbolic black holes.

- In this work utilizing the scaling charge we tried to determine whether the black holes of EMGB admits Cauchy horizon.
- However, other than $k = 0$ case it was not possible to determine the topologies that admit Cauchy horizon because of the complicated structure of the non-local term.
- Employing numerical techniques we were able to the existence of the inner horizon for $k = -1$ hyperbolic black holes.
- Is there a way to discuss the positivity of the integral piece ? May be, for the outside of the black hole (in 4 dimensions) I can give an argument depending on the perturbation stability (gradient stability).
- Any other uses of this charge ? Actually the local version was already used to compute charge and entropy of the Horndeski type black holes for $k = 0$.

THANK YOU!