Back reaction of cosmological perturbations during inflation

Inyong Cho (SeoulTech)

Workshop on Cosmology and Quantum Space Time
(CQUeST 2023)
Best Western Plus Hotel, Jeonju, 8/2/2023

PRD 106, 4, 084027 (2022)
co-work with Jinn-Ouk Gong (Ewha Womans U.) & Seung Hun Oh (Tech U. of Korea)

PRD 102, 043531 (2020): in Friedmann Universe

Outline

- Motivation
- Introduction to cosmological perturbations
- 2^{nd} order perturbations and back reaction
  :- 2^{nd} order effective Energy-Momentum Tensor (2EMT)
- Gauge Problem of 2EMT
- Several Results (3 gauges: Longitudinal, Spatially-flat, Comoving)

1. Scalar perturbations
2. Tensor perturbation
3. Scalar-Tensor cross terms

- Conclusions

Motivation

- Long wavelength mode of SOMETHING may play a role in the Universe

- It interacts barely with others
- It may be a source of dark energy

- SOMETHING could be cosmological perturbations

- Has been already investigated as a back reaction (in inflation)
- Abramo, Brandenberger, Mukhanov
- Has a GAUGE issue – Unruh, Ishibash & Wald

- We decided to investigate in FRW (perfect fluid)

- Gauge dependent
- Cannot explain dark energy

- We decided to investigate in Inflation (scalar field) for precision cosmology

- Gauge dependent

- But, still viable as precession cosmology

- investigate in the future in the TETRAD basis for local observations

Introduction to cosmological perturbations

Classification of perturbations

* FRW-metric w/ perturbations

$$ds^2 = \left[{}^{(0)}g_{\alpha\beta} + \delta g_{\alpha\beta}(x^\gamma)\right] dx^\alpha dx^\beta \quad , \quad |\delta g_{\alpha\beta}| \ll |{}^{(0)}g_{\alpha\beta}|$$

$$\Downarrow$$

$${}^{(0)}g_{\alpha\beta}\, dx^\alpha dx^\beta = a^2(\eta)\left[d\eta^2 - \delta_{ij}\, dx^i dx^j\right]$$: use conformally flat metric.

* $\delta g_{\alpha\beta}$: ① scalar ② vector ③ tensor perturbation.

: based on the sym. properties of hom. & iso background

invariant under rotations and translations.

① $\delta g_{00} = 2a^2 \phi(x^j)$: scalar under 3-rotations.

② $\delta g_{0i} = a^2 (B_{,i} + S_i)$: vector

$S^i_{\ ,i} = 0$: zero divergence

③ $\delta g_{ij} = a^2\left(2\psi\delta_{ij} + 2E_{,ij} + F_{i,j} + F_{j,i} + h_{ij}\right)$: tensor

3-scalar

$F^i_{\ ,i} = 0$

3 tensor : $\boxed{h^i_{\ i} = 0 \ , \ h^i_{\ j,i} = 0 \quad \text{traceless \& transverse}}$

4-cond's

1. Scalar Perturbation

$$ds^2 = a^2\left[(1+2\phi)\,d\eta^2 + B_{,i}\,dx^i d\eta - \left[(1-2\psi)\delta_{ij} - 2E_{,ij}\right]dx^i dx^j\right]$$

2. Vector Perturbation

$$ds^2 = a^2\left[d\eta^2 + 2S_i\,dx^i d\eta - (\delta_{ij} - F_{i,j} - F_{j,i})\,dx^i dx^j\right]$$

3. Tensor Perturbation

$$ds^2 = a^2\left[d\eta^2 - (\delta_{ij} - h_{ij})\,dx^i dx^j\right]$$

$$\Rightarrow ds^2 = a^2\left[(1+2\phi)\,d\eta^2 + (B_{,i}+2S_i)\,dx^i d\eta - \left[(1-2\psi)\delta_{ij} - 2E_{,ij} - F_{i,j} - F_{j,i} - h_{ij}\right]dx^i dx^j\right]$$

Number of independent functions of $\delta g_{\alpha\beta}$

$\boxed{S_i,\ F_i \text{ with 2 divergence-free conditions}}$

(4 for scalar) + (4 for vector) + (2 for tensor)

$\boxed{\phi,\ B,\ \psi,\ E}$

$\boxed{h_{ij} \text{ with 4 gauge conditions}}$

= 10 functions = number of independent components of $\delta g_{\alpha\beta}$

Scalar mode :- induced by energy density inhomogeneity (matter dof)
:- exhibit gravitational instability → structure formation
:- most important

Vector mode :- rotational motion of fluid
:- decays very quickly as in Newtonian gravity
:- not very interesting cosmologically

Tensor mode :- gravity waves → dof. of gravitational field itself
:- do not induce any perturbations in perfect fluid

Gauge Transformations

$\boxed{x^\alpha \rightarrow \tilde{x}^\alpha = x^\alpha + \xi^\alpha}$ **: infinitesimal coord. transformation**

$$\boxed{\xi^\alpha = (\xi^0, \xi^i) = (\xi^0, \xi_\perp^i + \zeta^{,i})} \qquad \xi_{\perp,i}^i = 0, \qquad \xi_{\perp i} = \xi_\perp^i$$

At a given point, the metric tensor in $\tilde{x}$ coord. system: $\tilde{g}_{\alpha\beta}(\tilde{x}^\rho)$

$$g_{\alpha\beta}(x^\rho) = \boxed{g_{\alpha\beta}^{(0)}(x^\rho) + \delta g_{\alpha\beta}(x^\rho)} \qquad \textbf{: perturbation in } x$$

$$\Rightarrow \quad \tilde{g}_{\alpha\beta}(\tilde{x}^\rho) = \frac{\partial x^\gamma}{\partial \tilde{x}^\alpha}\frac{\partial x^\delta}{\partial \tilde{x}^\beta} g_{\gamma\delta}(x^\rho) \qquad \textbf{: tensor transformation}$$

$$\approx \boxed{\cancel{g_{\alpha\beta}^{(0)}(x^\rho)} + \delta g_{\alpha\beta}(x^\rho)} - g_{\alpha\delta}^{(0)}(x^\rho)\xi^\delta_{,\beta} - g_{\gamma\beta}^{(0)}(x^\rho)\xi^\gamma_{,\alpha}$$

$$= g_{\alpha\beta}^{(0)}(\tilde{x}^\rho) + \delta\tilde{g}_{\alpha\beta}(\tilde{x}^\rho) \qquad \textbf{: perturbation in } \tilde{x}$$

Taylor expansion: $\boxed{g_{\alpha\beta}^{(0)}(\tilde{x}^\rho)} = g_{\alpha\beta}^{(0)}(x^\rho + \xi^\rho) \approx \cancel{g_{\alpha\beta}^{(0)}(x^\rho)} + g_{\alpha\beta,\gamma}^{(0)}(x^\rho)\xi^\gamma$

Finally we have

$$\boxed{\delta\tilde{g}_{\alpha\beta}(\tilde{x}^\rho) = \delta g_{\alpha\beta}(x^\rho) - g_{\alpha\beta,\gamma}^{(0)}(x^\rho)\xi^\gamma - g_{\alpha\delta}^{(0)}(x^\rho)\xi^\delta_{,\beta} - g_{\gamma\beta}^{(0)}(x^\rho)\xi^\gamma_{,\alpha}}$$

eg) scalar perturbation

$$ds^2 = a^2\left[(1+2\phi)d\eta^2 + B_{,i}dx^i d\eta - [(1-2\psi)\delta_{ij} - 2E_{,ij})]\,dx^i dx^j\right] \qquad \textbf{in } x$$

$$\boxed{\delta\tilde{g}_{\alpha\beta}(\tilde{x}^\rho) = \delta g_{\alpha\beta}(x^\rho) - g^{(0)}_{\alpha\beta,\gamma}(x^\rho)\xi^\gamma - g^{(0)}_{\alpha\delta}(x^\rho)\xi^\delta_{,\beta} - g^{(0)}_{\gamma\beta}(x^\rho)\xi^\gamma_{,\alpha}}$$

$$ds^2 = a^2\left[(1+2\tilde{\phi})d\tilde{\eta}^2 + \tilde{B}_{,i}d\tilde{x}^i d\tilde{\eta} - \left[(1-2\tilde{\psi})\delta_{ij} - 2\tilde{E}_{,ij})\right] d\tilde{x}^i d\tilde{x}^j\right] \qquad \textbf{in } \tilde{x}$$

Gauge Invariant Variables

1. Scalar Perturbation

$$\boxed{\begin{cases} \phi \to \tilde{\phi} = \phi - \frac{1}{a}(a\xi^0)' \\ B \to \tilde{B} = B + \xi' - \xi^0 \\ \psi \to \tilde{\psi} = \psi + \frac{a'}{a}\xi^0 \\ E \to \tilde{E} = E + \xi \end{cases}}$$

- only ξ^0 and ξ contribute
$\Rightarrow$ can make two of ϕ, B, ψ, E vanish!

* Gauge invariant variables (perturbations):

$$\boxed{\begin{cases} \Phi \equiv \phi - \frac{1}{a}\left[a(B-E')\right]' \\ \Psi \equiv \psi + \frac{a'}{a}(B-E') \end{cases}}$$

span 2D space of physical perturbations.

Bardeen Variables (1980)

① do not change under coord. transf.

② If $\Phi = 0, \Psi = 0$ in one coord syst $\Rightarrow$ $\Phi = 0, \Psi = 0$ in all coord. syst's.
①

③ If $\Phi = \Psi = 0$, No physical perturbations : fictitious metric pert's can be removed by coord. transf.

④ Can construct ∞ # of gauge-invariant variables:
e.g.) $\alpha\Phi + \beta\Psi \Rightarrow$ also gauge-inv.

2. Vector Perturbation

$$\begin{cases} S_i \longrightarrow \tilde{S}_i = S_i + \xi'_{\perp i} \\ F_i \longrightarrow \tilde{F}_i = F_i + \xi_{\perp i} \end{cases} \longrightarrow \text{only } \xi_{\perp i} \text{ is involved.}$$

* $\boxed{\overline{V}_i = S_i - F'_i}$: gauge-inv. vector perturbation

① Only 2 fn's characterize physical perturbations ($\overline{V}_i{}^{,i} = 0$)

② spans 2D space of physical perturbations.

③ describes rotational motions

④ Corresponding rotational vel $\delta u_{\perp i}$ ($\delta u_{\perp i}{}^{,i} = 0$) : also gauge invariant.

3. Tensor Perturbation

$\boxed{h_{ij}}$: <u>no change under coord. transf.</u>

: describes already the G.W. in a gauge invariant manner.

Gauge Fixing

- Gauge freedom: most important in scalar perturbation
- Free to choose ξ^0 and ζ → 2 conditions
- Imposing gauge conditions → Fixing coord. system

e.g.) Longitudinal Gauge

$$\boxed{B = E = 0 \quad \Rightarrow \xi^0 = \zeta = 0}$$: gauge conditions

$$\Rightarrow \boxed{\tilde{\phi} = \phi = \Phi, \quad \tilde{\psi} = \psi = \Psi}$$: the other 2 scalars → gauge invariant

$$\Rightarrow \boxed{ds^2 = a^2(\eta)[(1+2\Phi)\,d\eta^2 - (1-2\Psi)\gamma_{ij}\,dx^i\,dx^j]}$$

: so called, "Conformal Newtonian" gauge

1st order Einstein's equation

Brandenberger & Mukhanov (1992, Phys. Rep.)

$$\boxed{G^\mu{}_\nu = 8\pi G T^\mu{}_\nu}$$: Einstein's equation

$$G^\mu{}_\nu = {}^{(0)}G^\mu{}_\nu + \delta G^\mu{}_\nu + \cdots ,$$

$$\boxed{\delta G^\mu{}_\nu = 8\pi G\, \delta T^\mu{}_\nu}$$: 1st order Einstein's equation

NOT gauge-invariant, individually

Define Gauge-Invariant Quantities:

$$\delta G^{(\mathrm{gi})0}_0 = \delta G^0_0 + \boxed{({}^{(0)}G^0_0)'(B - E')}, \qquad \delta G^{(\mathrm{gi})0}_i = \delta G^0_i + ({}^{(0)}G^0_0 - \tfrac{1}{3}{}^{(0)}G^k_k)(B - E')_{|i} ,$$

$$\delta G^{(\mathrm{gi})i}_j = \delta G^i_j + ({}^{(0)}G^i_j)'(B - E') ,$$

$$\delta T^{(\mathrm{gi})0}_0 = \delta T^0_0 + \boxed{({}^{(0)}T^0_0)'(B - E')}, \qquad \delta T^{(\mathrm{gi})0}_i = \delta T^0_i + ({}^{(0)}T^0_0 - \tfrac{1}{3}{}^{(0)}T^k_k)(B - E')_{|i} ,$$

$$\delta T^{(\mathrm{gi})i}_j = \delta T^i_j + ({}^{(0)}T^i_j)'(B - E') .$$

$$\boxed{\delta G^\mu{}_\nu = 8\pi G\, \delta T^\mu{}_\nu} \;\Rightarrow\; \boxed{\delta G^{(\mathrm{gi})\mu}_\nu = 8\pi G\, \delta T^{(\mathrm{gi})\mu}_\nu}$$: can be written in gauge-invariant form

From now, consider Scalar Perturbations

$$\delta G^{(\mathrm{gi})\mu}_{\nu} = 8\pi G\, \delta T^{(\mathrm{gi})\mu}_{\nu}$$

: 1st order equation in gauge-invariant form

$$-3\mathcal{H}(\mathcal{H}\Phi + \Psi') + \nabla^2\Psi + 3\mathcal{K}\Psi = 4\pi G a^2\, \delta T^{(\mathrm{gi})0}_{0},$$

$$(\mathcal{H}\Phi + \Psi')_{,i} = 4\pi G a^2\, \delta T^{(\mathrm{gi})0}_{i},$$

$$[(2\mathcal{H}' + \mathcal{H}^2)\Phi + \mathcal{H}\Phi' + \Psi'' + 2\mathcal{H}\Psi' - \mathcal{K}\Psi + \tfrac{1}{2}\nabla^2 D]\delta^i_{\ j} - \tfrac{1}{2}\gamma^{ik}D_{|kj} = -4\pi G a^2\, \delta T^{(\mathrm{gi})i}_{j}$$

Einstein Equation

:- derived with NO gauge fixing
:- all written in gauge-invariant (Bardeen) variables
:- valid in arbitrary coord. system

$$\delta\varphi^{(\mathrm{gi})} = \delta\varphi + \varphi_0'(B - E')$$

$$\delta T^{(\mathrm{gi})0}_{0} = a^{-2}[-\varphi_0'^2\Phi + \varphi_0'\,\delta\varphi^{(\mathrm{gi})\prime} + V_{,\varphi}a^2\,\delta\varphi^{(\mathrm{gi})}]$$

$$\delta T^{(\mathrm{gi})0}_{i} = a^{-2}\varphi_0'\,\delta\varphi^{(\mathrm{gi})}_{,i}$$

$$\delta T^{(\mathrm{gi})i}_{j} = a^{-2}[+\varphi_0'^2\Phi - \varphi_0'\,\delta\varphi^{(\mathrm{gi})\prime} + V_{,\varphi}a^2\,\delta\varphi^{(\mathrm{gi})}]\delta^i_{\ j}$$

$$\delta\varphi^{(\mathrm{gi})\prime\prime} + 2\mathcal{H}\,\delta\varphi^{(\mathrm{gi})\prime} - \nabla^2\,\delta\varphi^{(\mathrm{gi})} + V_{,\varphi\varphi}a^2\,\delta\varphi^{(\mathrm{gi})} - 4\varphi_0'\Phi' + 2V_{,\varphi}a^2\Phi = 0$$

Scalar Field Equation

2^{nd} order Perturbations and Back Reaction

Picture

0^{th} order: $G^{(0)}_{\mu\nu} = T^{(0)}_{\mu\nu}$ $(8\pi G = 1)$

1^{st} order: $G^{(1)}_{\mu\nu} = T^{(1)}_{\mu\nu}$: solutions

plug in

2^{nd} order: $G^{(2)}_{\mu\nu} = T^{(2)}_{\mu\nu}$

$$\boxed{G^{(1)}_{\mu\nu}[g^{(2)}] + G^{(2)}_{\mu\nu}[g^{(1)}]}$$

Linear in $g^{(2)}$

Quadratic in $g^{(1)}$

quadratic terms only

$$\boxed{G^{(1)}_{\mu\nu}[g^{(2)}] = T^{(1)}_{\mu\nu}[\psi^{(2)}] + T^{(2)}_{\mu\nu}[g^{(1)}, \psi^{(1)}] - G^{(2)}_{\mu\nu}[g^{(1)}] \equiv T^{(2,\text{eff})}_{\mu\nu}}$$

Back-reaction source

$$\boxed{\textbf{cf)}\ \langle \delta\phi^{(2)} \rangle = 0,\ \langle \delta\rho^{(2)} \rangle = 0}$$

stochastic

2EMT: GAUGE DEPENDENT very possibly

$$G_{\mu\nu} = T_{\mu\nu}$$
$$\Rightarrow \quad G^{(0)}_{\mu\nu} + G^{(1)}_{\mu\nu} + G^{(2)}_{\mu\nu} + \cdots = T^{(0)}_{\mu\nu} + T^{(1)}_{\mu\nu} + T^{(2)}_{\mu\nu} + \cdots .$$

The equality is satisfied order by order, $G^{(n)}_{\mu\nu} = T^{(n)}_{\mu\nu}$.

1st order

$$G^{(1)}_{\mu\nu} = T^{(1)}_{\mu\nu} \longrightarrow \overline{G}^{(1)}_{\mu\nu} = \overline{T}^{(1)}_{\mu\nu}$$

$$\overline{G}^{(1)}_{\mu\nu} = G^{(1)}_{\mu\nu} + \delta G^{(1)}_{\mu\nu}, \qquad \overline{T}^{(1)}_{\mu\nu} = T^{(1)}_{\mu\nu} + \delta T^{(1)}_{\mu\nu}.$$

: terms making the quantity gauge invariant

2nd order

$$G^{(2)}_{\mu\nu} = T^{(2)}_{\mu\nu} \longrightarrow \overline{G}^{(2)}_{\mu\nu} = \overline{T}^{(2)}_{\mu\nu}$$

$$\overline{G}^{(2)}_{\mu\nu} = G^{(2)}_{\mu\nu} + \delta G^{(2)}_{\mu\nu} = G^{(1)}_{\mu\nu}[g^{(2)}] + \underline{G^{(2)}_{\mu\nu}[g^{(1)}]} + \delta G^{(1)}_{\mu\nu}[g^{(2)}] + \delta G^{(2)}_{\mu\nu}[g^{(1)}]$$

$$\overline{T}^{(2)}_{\mu\nu} = T^{(2)}_{\mu\nu} + \delta T^{(2)}_{\mu\nu} = T^{(1)}_{\mu\nu}[g^{(2)}, \psi^{(2)}] + \underline{T^{(2)}_{\mu\nu}[g^{(1)}, \psi^{(1)}]} + \delta T^{(1)}_{\mu\nu}[g^{(2)}, \psi^{(2)}] + \delta T^{(2)}_{\mu\nu}[g^{(1)}, \psi^{(1)}],$$

$$T^{(2,\text{eff})}_{\mu\nu} \equiv T^{(2)}_{\mu\nu}[g^{(1)}, \psi^{(1)}] - G^{(2)}_{\mu\nu}[g^{(1)}].$$

: 2EMT responsible for Back-Reaction

→ HARD to be gauge invariant

2EMT $$\tau_{\mu\nu} \equiv \langle T^{(2,\text{eff})}_{\mu\nu} \rangle$$

: spatial integration (over several wave lengths)

Interpretation

$$T^{\mu\ (2,\mathrm{eff})}_{\nu} = \begin{pmatrix} -\rho_{\mathrm{eff}} & 0 & 0 & 0 \\ 0 & p_{\mathrm{eff}} & 0 & 0 \\ 0 & 0 & p_{\mathrm{eff}} & 0 \\ 0 & 0 & 0 & p_{\mathrm{eff}} \end{pmatrix} \Longrightarrow \boxed{g^{(2)}_{\mu\nu}}$$

$$\boxed{T^{(2,\mathrm{eff})}_{\mu\nu} \equiv T^{(2)}_{\mu\nu}[g^{(1)}, \psi^{(1)}] - G^{(2)}_{\mu\nu}[g^{(1)}].}$$

Scalar Perturbations

$$ds^2 \boxed{= a^2(\eta)[-(1+2A)d\eta^2 - 2B_i d\eta dx^i + (\delta_{ij} + 2C_{ij})dx^i dx^j]}$$

$$\boxed{A = \alpha, \qquad B_i = \beta_{,i} + B_i^{(v)}, \qquad C_{ij} = -\psi\delta_{ij} + E_{,ij} + C_{(i,j)}^{(v)} + C_{ij}^{(t)}.}$$ **: general**

$$\boxed{A = \alpha\ , \quad B_i = \beta_{,i}\ , \quad C_{ij} = -\psi\delta_{ij} + E_{,ij}}$$ **: scalar only**

$$\boxed{= a^2(\eta)[-(1+2\alpha)d\eta^2 - 2\beta_{,i}d\eta dx^i + ((1-2\psi)\delta_{ij} + 2E_{,ij})dx^i dx^j]}$$

Einstein Tensor

$$\begin{aligned}
G_{00} = {} & 3\mathcal{H}^2 + 2\mathcal{H}B_{k,k} + 2\mathcal{H}C'_{kk} - \Delta C_{kk} + C_{kl,kl} - (2\mathcal{H}' + \mathcal{H}^2)B_kB_k + \frac{1}{2}B_{k,k}B_{l,l} - \frac{1}{4}B_{k,l}B_{k,l} \\
& - \frac{1}{4}B_{k,l}B_{l,k} + B_kB_{l,lk} - B_k\Delta B_k - \frac{1}{2}C'_{kl}C'_{kl} + \frac{1}{2}C'_{kk}C'_{ll} - 4\mathcal{H}C_{kl}C'_{kl} + \frac{3}{2}C_{kl,m}C_{kl,m} \\
& - C_{kl,m}C_{km,l} - \frac{1}{2}(C_{kk,l} - 2C_{lk,k})(C_{mm,l} - 2C_{lm,m}) + 2C_{kl}(C_{mm,kl} + \Delta C_{kl} - 2C_{mk,ml}) \\
& - 4\mathcal{H}A_{,k}B_k + 2B_k(C'_{mm,k} - C'_{km,m}) + 2\mathcal{H}B_k(C_{mm,k} - 2C_{km,m}) - B_{k,l}(C'_{kl} + 4\mathcal{H}C_{kl}) \\
& + B_{k,k}C'_{ll} - 2A(\Delta C_{kk} - C_{kl,kl})\ , \qquad (26)
\end{aligned}$$

$$\begin{aligned}
G_{0i} = {} & 2\mathcal{H}A_{,i} + B_{[i,k]k} + (2\mathcal{H}' + \mathcal{H}^2)B_i + C'_{ik,k} - C'_{kk,i} - 4\mathcal{H}AA_{,i} - B_k(B'_{(i,k)} + 2\mathcal{H}B_{[i,k]}) \\
& + B_i(B'_{k,k} + 2\mathcal{H}B_{k,k}) + (C_{mm,k} - 2C_{km,m})C'_{ik} + 2C_{kl}(C'_{kl,i} - C'_{ik,l}) + C'_{kl}C_{kl,i} - 2\mathcal{H}A'B_i \\
& + A_{i,k}B_k + A_{,i}B_{k,k} - A_{,k}B_{(i,k)} - 2(2\mathcal{H}' + \mathcal{H}^2)AB_i - (\Delta A)B_i - 2\mathcal{H}B_kC'_{ik} + 2\mathcal{H}B_iC'_{kk} \\
& - 2B_{[i,k]l}C_{kl} + 2B_{[i,k]}(C_{ll,k} - 2C_{kl,l}) - 2B_{[k,l]}C_{i[k,l]} - B_i(\Delta C_{kk} - C_{kl,kl}) + A_{,i}C'_{kk} \\
& - A_{,k}C'_{ik}\ , \qquad (27)
\end{aligned}$$

$$
\begin{aligned}
G_{ij} = \delta_{ij}\{ & -(2\mathcal{H}'+\mathcal{H}^2) + 2\mathcal{H}A' + 2(2\mathcal{H}'+\mathcal{H}^2)A + \Delta A - B'_{k,k} - 2\mathcal{H}B_{k,k} - C''_{kk} - 2\mathcal{H}C'_{kk} \\
& + \Delta C_{kk} - C_{kl,kl} - 8\mathcal{H}AA' - (\nabla A)^2 - 2A\Delta A - 4(2\mathcal{H}'+\mathcal{H}^2)A^2 + 2\mathcal{H}B_kB'_k \\
& + (2\mathcal{H}'+\mathcal{H}^2)B_kB_k + \frac{3}{4}B_{k,l}B_{k,l} - \frac{1}{4}B_{k,l}B_{l,k} - \frac{1}{2}B_{k,k}B_{l,l} - B_lB_{k,kl} + B_k\Delta B_k \\
& + 2C_{kl}C''_{kl} + \frac{3}{2}C'_{kl}C'_{kl} - \frac{1}{2}C'_{kk}C'_{ll} + 4\mathcal{H}C_{kl}C'_{kl} + \frac{1}{2}(C_{kk,l} - 2C_{lk,k})(C_{mm,l} - 2C_{lm,m}) \\
& - \frac{3}{2}C_{lm,k}C_{lm,k} + C_{lm,k}C_{lk,m} - 2C_{kl}(C_{mm,kl} + \Delta C_{kl} - 2C_{mk,ml}) + A'B_{k,k} + 2AB'_{k,k} \\
& + 2\mathcal{H}A_{,k}B_k + 4\mathcal{H}AB_{k,k} + 2B'_{k,l}C_{kl} - B'_k(C_{ll,k} - 2C_{kl,l}) - 2B_k(C'_{ll,k} - C'_{kl,l}) - B_{k,k}C'_{ll} \\
& + B_{k,l}C'_{kl} + 4\mathcal{H}B_{k,l}C_{kl} - 2\mathcal{H}B_l(C_{kk,l} - 2C_{lk,k}) + 2AC''_{kk} + A'C'_{kk} + 4\mathcal{H}AC'_{kk} \\
& - 2A_{,kl}C_{kl} + A_{,k}(C_{ll,k} - 2C_{kl,l})\} \\
& - A_{,ij} + B'_{(i,j)} + 2\mathcal{H}B_{(i,j)} + C''_{ij} + 2\mathcal{H}C'_{ij} - \Delta C_{ij} - 2(2\mathcal{H}'+\mathcal{H}^2))C_{ij} + 2C_{k(i,j)k} - C_{kk,ij} \\
& + A_{,i}A_{,j} + 2AA_{,ij} + B_{k,k}B_{(i,j)} + B_kB_{(i,j)k} - \frac{1}{2}B_{i,k}B_{j,k} - \frac{1}{2}B_{k,i}B_{k,j} - B_kB_{k,ij} - 2C''_{kk}C_{ij} \\
& + C'_{kk}C'_{ij} - 2C'_{ik}C'_{jk} - 4\mathcal{H}C'_{kk}C_{ij} + 2(\Delta C_{kk} - C_{kl,kl})C_{ij} + 2C_{kl}(C_{ij,kl} + C_{kl,ij} - 2C_{k(i,j)l}) \\
& - (C_{kk,l} - 2C_{lk,k})(C_{ij,l} - 2C_{l(i,j)}) + C_{kl,i}C_{kl,j} + 2C_{ik,l}(C_{jk,l} - C_{jl,k}) - A'B_{(i,j)} \\
& - 2A(B'_{(i,j)} + 2\mathcal{H}B_{(i,j)}) - 2B_k(C'_{k(i,j)} - C'_{ij,k}) - (B'_k + 2\mathcal{H}B_k)(2C_{k(i,j)} - C_{ij,k}) \\
& + B_{k,k}(C'_{ij} - 4\mathcal{H}C_{ij}) - 2B'_{k,k}C_{ij} + B_{(i,j)}C'_{kk} - B_{i,k}C'_{jk} - B_{j,k}C'_{ik} - 2A(C''_{ij} + 2\mathcal{H}C'_{ij}) \\
& - A'(C'_{ij} - 4\mathcal{H}C_{ij}) + 4(2\mathcal{H}'+\mathcal{H}^2)AC_{ij} + A_{,k}(2C_{k(i,j)} - C_{ij,k}) + 2(\Delta A)C_{ij}\ . \qquad (28)
\end{aligned}
$$

Energy-Momentum Tensor (Inflaton)

$$
\begin{aligned}
T_{00} &= \frac{1}{2}({\phi_0}')^2 + a^2V(\phi_0) + \phi_0'\delta\phi' + 2AV(\phi_0) + a^2V_\phi(\phi_0)\delta\phi \\
&+ \frac{1}{2}(\delta\phi')^2 + \frac{1}{2}(\nabla\delta\phi)^2 + \frac{1}{2}a^2V_{\phi\phi}(\phi_0)\delta\phi^2 + 2a^2AV_\phi(\phi_0)\delta\phi + \frac{1}{2}B_kB_k(\phi_0')^2 - \phi_0'B_k\delta\phi_{,k}, \\
T_{0i} &= \phi_0'\delta\phi_{,i} - \frac{1}{2}B_i(\phi_0')^2 + a^2B_iV(\phi_0) \\
&+ \delta\phi'\delta\phi_{,i} - B_i\phi_0'\delta\phi' + AB_i(\phi_0')^2 + a^2B_iV_\phi(\phi_0)\delta\phi, \\
T_{ij} &= \delta_{ij}\left[\frac{1}{2}(\phi_0')^2 - a^2V(\phi_0)\right] + \delta_{ij}\left[\phi_0' - A(\phi_0')^2 - a^2V'(\phi_0)\delta\phi\right] + C_{ij}\left[(\phi_0')^2 - 2a^2V(\phi_0)\right] \\
&+ \delta\phi_{,i}\delta\phi_{,j} + 2C_{ij}\left[\phi_0'\delta\phi' - A(\phi_0')^2 - a^2V_\phi(\phi_0)\delta\phi\right] \\
&+ \delta_{ij}\left[\frac{1}{2}(\delta\phi')^2 - \frac{1}{2}(\nabla\delta\phi)^2 + \left(2A^2 - \frac{1}{2}B_kB_k\right)(\phi_0')^2 - 2A\phi_0'\delta\phi' + B_k\phi_0'\delta\phi_{,k} - \frac{1}{2}a^2V_{\phi\phi}(\phi_0)\delta\phi^2\right]
\end{aligned}
$$

0^{th} order (Background Friedmann Eqs.)

$$\mathcal{H}^2 = \frac{8\pi G}{3}\left[\frac{1}{2}(\phi_0')^2 + a^2 V\right],$$
$$2\mathcal{H}' + \mathcal{H}^2 = 8\pi G\left[-\frac{1}{2}(\phi_0')^2 + a^2 V\right],$$

$$\phi_0'' + 2\mathcal{H}\phi_0' + a^2 V_\phi = 0.$$

1st order → Obtain solutions

$$2(\mathcal{H}' + 2\mathcal{H}^2)A - 2\mathcal{H}B_{k,k} - 2\mathcal{H}C_{kk}' + \Delta C_{kk} - C_{kl,kl} = 8\pi G\left[(\phi_0'\delta\phi)' + 2\mathcal{H}\phi_0'\delta\phi - 2\phi_0'\delta\phi'\right],$$
$$2\mathcal{H}A_{,i} + B_{[i,k]k} + C_{ik,k}' - C_{kk,i}' = 8\pi G\phi_0'\delta\phi_{,i},$$
$$\delta_{ij}\left[2\mathcal{H}A' + 2(\mathcal{H}' + 2\mathcal{H}^2)A + \Delta A - B_{k,k}' - 2\mathcal{H}B_{k,k} - C_{kk}'' - 2\mathcal{H}C_{kk}' + \Delta C_{kk} - C_{kl,kl}\right] - A_{,ij} + B_{(i,j)}' + 2\mathcal{H}B_{(i,j)} + C_{ij}'' + 2\mathcal{H}C_{ij}' - \Delta C_{ij} + 2C_{k(i,j)k} - C_{kk,ij} = 8\pi G\delta_{ij}\left[(\phi_0'\delta\phi)' + 2\mathcal{H}\phi_0'\delta\phi\right].$$

$$\delta\phi'' + 2\mathcal{H}\delta\phi' - \Delta\delta\phi + a^2 V_{\phi\phi}\delta\phi - (A' - B_{k,k} - C_{kk}')\phi_0' + 2a^2 V_\phi A = 0.$$

1st order EQ in Gauge Invariant Variables

Matter perturbation: can be written in metric perturbations

$$\boxed{\Psi'' - \Delta\Psi + 2K\Psi' + 2L\Psi = 0.}$$

$$\boxed{\Psi = \Phi.}$$

T=diag

$$K = 3\mathcal{H} + a^2\frac{V_\phi}{\phi_0'},$$
$$L = \mathcal{H}' + 2\mathcal{H}^2 + a^2\mathcal{H}\frac{V_\phi}{\phi_0'}.$$

$$\overline{\delta\phi} = \delta\phi - \phi_0' Q,$$
$$\Phi = \alpha - Q' - \mathcal{H}Q,$$
$$\Psi = \psi + \mathcal{H}Q,$$

$$\boxed{Q = \beta + E'.}$$: gauge variable

$$\overline{\delta\phi} = \frac{\Psi' + \mathcal{H}\Psi}{4\pi G\phi_0'},$$
$$\overline{\delta\phi}' = \frac{\Delta\Psi - K\Psi' - L\Psi}{4\pi G\phi_0'},$$

Gauge Problem

INFLATON: scalar field

Abramo-Brandenberger-Mukhanov (1997)

2EMT: gauge invariant

Unruh (1998), Ishibashi-Wald (2006)

2EMT: gauge dependent

2EMT cannot explain Dark Energy

2EMT for a scalar field

$\boxed{Q = \beta + E'\,.}$ $\boxed{E}$ **: GAUGE VARIABLES**

$$8\pi G\tau_{00} = \frac{1}{\mathcal{H}' - \mathcal{H}^2}\Big\{ -\langle(\nabla\Psi')^2\rangle - \langle(\Delta\Psi)^2\rangle - 2(K+\mathcal{H})\langle\nabla\Psi'\cdot\nabla\Psi\rangle - \left[3(\mathcal{H}'-\mathcal{H}^2) + K^2 + a^2V_{\phi\phi}\right]\langle(\Psi')^2\rangle$$
$$+ (9\mathcal{H}' - 10\mathcal{H}^2 - 2L)\langle(\nabla\Psi)^2\rangle + 2\left[6\mathcal{H}(\mathcal{H}'-\mathcal{H}^2) - KL + 2(\mathcal{H}'-\mathcal{H}^2)a^2\frac{V_\phi}{\phi_0'} - a^2\mathcal{H}V_{\phi\phi}\right]\langle\Psi\Psi'\rangle$$
$$- \left[L^2 - 4\mathcal{H}(\mathcal{H}'-\mathcal{H}^2)a^2\frac{V_\phi}{\phi_0'} + a^2\mathcal{H}^2V_{\phi\phi}\right]\langle\Psi^2\rangle\Big\}$$
$$+ 2\left\langle\left\{a^2\frac{V_\phi}{\phi_0'}Q' + \left[3\mathcal{H}' + (4\mathcal{H}+K)a^2\frac{V_\phi}{\phi_0'} + a^2V_{\phi\phi}\right]Q - \Delta Q + \Delta E' - 2\mathcal{H}\Delta E\right\}\Psi'\right\rangle$$
$$+ 2\Big\langle\Big\{ -\left(L + 6\mathcal{H}^2 - 2a^2\mathcal{H}\frac{V_\phi}{\phi_0'}\right)Q' + \left[2\mathcal{H}(L - 3\mathcal{H}') + (L - 2\mathcal{H}' + 4\mathcal{H}^2)a^2\frac{V_\phi}{\phi_0'} + \mathcal{H}a^2V_{\phi\phi}\right]Q$$
$$- \Delta Q' - (K-\mathcal{H})\Delta Q + 2\mathcal{H}\Delta E' + \Delta^2 E\Big\}\Psi\Big\rangle - (\mathcal{H}' + 2\mathcal{H}^2)\langle Q'^2\rangle - 2\left[\mathcal{H}(\mathcal{H}' - 4\mathcal{H}^2) + (\mathcal{H}'-\mathcal{H}^2)a^2\frac{V_\phi}{\phi_0'}\right]\langle Q'Q\rangle$$
$$- \left\{(\mathcal{H}'-\mathcal{H}^2)\left[4\mathcal{H}^2 + 8\mathcal{H}a^2\frac{V_\phi}{\phi_0'} + a^4\left(\frac{V_\phi}{\phi_0'}\right)^2 + a^2V_{\phi\phi}\right] + 3\mathcal{H}'(\mathcal{H}' - 4\mathcal{H}^2)\right\}\langle Q^2\rangle$$
$$- 2\mathcal{H}\langle\nabla Q'\cdot\nabla Q\rangle + \langle\left[4\mathcal{H}^2Q + 2\mathcal{H}\Delta E - (\mathcal{H}' + 2\mathcal{H}^2)E'\right]\Delta E'\rangle + 4\mathcal{H}\langle(\mathcal{H}Q' + \mathcal{H}'Q)\Delta E\rangle,$$

After spatial integration

$$
\begin{aligned}
8\pi G\tau_{0i} &= 0,\\
8\pi G\tau_{ij} &= \delta_{ij}\Bigg[\frac{1}{\mathcal{H}'-\mathcal{H}^2}\Bigg\{\frac{1}{3}\langle(\nabla\Psi')^2\rangle-\langle(\Delta\Psi)^2\rangle+2\left(\frac{\mathcal{H}}{3}-K\right)\langle\nabla\Psi'\cdot\nabla\Psi\rangle+\left(\mathcal{H}'-\mathcal{H}^2-K^2+a^2V_{\phi\phi}\right)\langle(\Psi')^2\rangle\\
&\quad+\left[\frac{1}{3}(11\mathcal{H}'-10\mathcal{H}^2)-2L\right]\langle(\nabla\Psi)^2\rangle+2\left[\mathcal{H}a^2V_{\phi\phi}-KL+2(\mathcal{H}'-\mathcal{H}^2)(K+2\mathcal{H})\right]\langle\Psi\Psi'\rangle\\
&\quad+\left[4(\mathcal{H}'-\mathcal{H}^2)(\mathcal{H}'+2\mathcal{H}^2+L)-L^2+\mathcal{H}^2a^2V_{\phi\phi}\right]\langle\Psi^2\rangle\Bigg\}\\
&\quad+2\left\langle\left[Q''+3(3\mathcal{H}-K)Q'+\left(\mathcal{H}'+12\mathcal{H}^2+K^2-7\mathcal{H}K-a^2V_{\phi\phi}\right)Q+\frac{1}{3}\Delta E'-\frac{4}{3}(K-\mathcal{H})\Delta E\right]\Psi'\right\rangle\\
&\quad+2\Bigg\langle\Bigg\{4\mathcal{H}Q''+(6\mathcal{H}'+10\mathcal{H}^2-3L)Q'+\Delta Q'-(K-3\mathcal{H})\Delta Q\\
&\qquad-\left[2\mathcal{H}''+a^2\mathcal{H}V_{\phi\phi}-\mathcal{H}K^2+(5\mathcal{H}'+2\mathcal{H}^2)K-11\mathcal{H}'\mathcal{H}-9\mathcal{H}^3\right]Q-\frac{1}{3}\Delta E''-\frac{2}{3}\mathcal{H}\Delta E'-\frac{4}{3}L\Delta E+\frac{2}{3}\Delta^2E\Bigg\}\Psi\Bigg\rangle\\
&\quad+\frac{2}{3}\langle\nabla Q''\cdot\nabla Q\rangle+\frac{2}{3}\langle(\nabla Q')^2\rangle+\frac{4\mathcal{H}}{3}\langle\nabla Q'\cdot\nabla Q\rangle+2\mathcal{H}\langle Q''Q'\rangle-2\mathcal{H}'\langle Q''Q\rangle-(\mathcal{H}'-2\mathcal{H}^2)\langle Q'^2\rangle\\
&\quad-2\left[2\mathcal{H}''+3\mathcal{H}'\mathcal{H}+(\mathcal{H}'-\mathcal{H}^2)(K+\mathcal{H})\right]\langle Q'Q\rangle\\
&\quad-\left\{4\mathcal{H}\mathcal{H}''+3\mathcal{H}'^2+4\mathcal{H}'\mathcal{H}^2+(\mathcal{H}'-\mathcal{H}^2)\left[4\mathcal{H}^2+a^2V_{\phi\phi}+a^4\left(\frac{V_\phi}{\phi_0'}\right)^2\right]\right\}\langle Q^2\rangle\\
&\quad-\frac{2}{3}\langle(2\mathcal{H}Q-3\mathcal{H}E'+\Delta E)\Delta E''\rangle-\frac{1}{3}\left\langle\left[8\mathcal{H}Q'+8(\mathcal{H}'+\mathcal{H}^2)Q-3(\mathcal{H}'+2\mathcal{H}^2)E'+2\Delta E'+4\mathcal{H}\Delta E\right]\Delta E'\right\rangle\\
&\quad-\frac{4}{3}\left\langle\left\{\mathcal{H}Q''+2(\mathcal{H}'+\mathcal{H}^2)Q'+2\left[3\mathcal{H}\mathcal{H}'-\mathcal{H}^3-(\mathcal{H}'-\mathcal{H}^2)K\right]Q\right\}\Delta E\right\rangle\Bigg].
\end{aligned}
$$

Therefore, 2EMT is GAUGE DEPENDENT

Several Results with Gauge Choices

Would like to examine if 2EMT converges??
in some limit of different gauge choices

Longitudinal Gauge

$$\boxed{\beta = 0\ , \qquad E = 0 \qquad \Longrightarrow \qquad Q = 0}$$

$$\tau_{00} = \frac{1}{8\pi G(\mathcal{H}' - \mathcal{H}^2)} \Big\{ -\langle(\nabla\Psi')^2\rangle - \langle(\Delta\Psi)^2\rangle - 2(K + \mathcal{H})\langle\nabla\Psi' \cdot \nabla\Psi\rangle - [3(\mathcal{H}' - \mathcal{H}^2) + K^2 + a^2 V_{\phi\phi}]\langle(\Psi')^2\rangle$$
$$+ (9\mathcal{H}' - 10\mathcal{H}^2 - 2L)\langle(\nabla\Psi)^2\rangle + 2\left[6\mathcal{H}(\mathcal{H}' - \mathcal{H}^2) - KL + 2(\mathcal{H}' - \mathcal{H}^2)a^2\frac{V_\phi}{\phi_0'} - a^2\mathcal{H}V_{\phi\phi}\right]\langle\Psi\Psi'\rangle$$
$$- \left[L^2 - 4\mathcal{H}(\mathcal{H}' - \mathcal{H}^2)a^2\frac{V_\phi}{\phi_0'} + a^2\mathcal{H}^2 V_{\phi\phi}\Big\}\langle\Psi^2\rangle\right],$$

$$\tau_{ij} = \frac{\delta_{ij}}{8\pi G(\mathcal{H}' - \mathcal{H}^2)} \Big\{ \frac{1}{3}\langle(\nabla\Psi')^2\rangle - \langle(\Delta\Psi)^2\rangle + 2\left(\frac{\mathcal{H}}{3} - K\right)\langle\nabla\Psi' \cdot \nabla\Psi\rangle + (\mathcal{H}' - \mathcal{H}^2 - K^2 + a^2 V_{\phi\phi})\langle(\Psi')^2\rangle$$
$$+ \left[\frac{1}{3}(11\mathcal{H}' - 10\mathcal{H}^2) - 2L\right]\langle(\nabla\Psi)^2\rangle + 2[\mathcal{H}a^2 V_{\phi\phi} - KL + 2(\mathcal{H}' - \mathcal{H}^2)(K + 2\mathcal{H})]\langle\Psi\Psi'\rangle$$
$$+ [4(\mathcal{H}' - \mathcal{H}^2)(\mathcal{H}' + 2\mathcal{H}^2 + L) - L^2 + \mathcal{H}^2 a^2 V_{\phi\phi}]\langle\Psi^2\rangle \Big\}.$$

Gauge Invariant

Spatially-flat Gauge

$$\boxed{\psi = 0, \qquad E = 0} \qquad \boxed{\Psi = \alpha - \beta' - \mathcal{H}\beta = \mathcal{H}\beta = \mathcal{H}Q.}$$

$$\begin{aligned}
\tau_{00} = \frac{1}{8\pi G(\mathcal{H}' - \mathcal{H}^2)} \Bigg\{ & -\langle(\nabla\Psi')^2\rangle - \langle(\Delta\Psi)^2\rangle + 2(L - M)\langle\nabla\Psi'\cdot\nabla\Psi\rangle \\
& - \left[\left(K + \frac{3(\mathcal{H}' - \mathcal{H}^2)}{\mathcal{H}}\right)^2 - \frac{4}{\mathcal{H}^2}(\mathcal{H}' - \mathcal{H}^2)(2\mathcal{H}' + \mathcal{H}^2) + a^2 V_{\phi\phi}\right]\langle(\Psi')^2\rangle \\
& - \frac{1}{\mathcal{H}^2}[(\mathcal{H}' - 2\mathcal{H}^2)(\mathcal{H}' - 2\mathcal{H}^2 - 2L) + (\mathcal{H}' - 4\mathcal{H}^2)(\mathcal{H}' - \mathcal{H}^2)]\langle(\nabla\Psi)^2\rangle \\
& - \left[\frac{2L(\mathcal{H}' - \mathcal{H}^2)}{\mathcal{H}^2}\left(3K - \frac{\mathcal{H}' + 11\mathcal{H}^2}{\mathcal{H}}\right) - \frac{2(\mathcal{H}' - 2\mathcal{H}^2)}{\mathcal{H}} a^2 V_{\phi\phi}\right]\langle\Psi'\Psi\rangle \\
& - \frac{(\mathcal{H}' - 2\mathcal{H}^2)^2}{\mathcal{H}^2}\left[(L - \frac{2(\mathcal{H}' - \mathcal{H}^2)}{\mathcal{H}})^2 + 12(\mathcal{H}' - \mathcal{H}^2) + a^2 V_{\phi\phi}\right]\langle\Psi^2\rangle \Bigg\},
\end{aligned}$$

$$\begin{aligned}
\tau_{ij} = \frac{\delta_{ij}}{8\pi G\mathcal{H}^2(\mathcal{H}' - \mathcal{H}^2)} \Bigg\{ & \frac{1}{3}(2\mathcal{H}' - \mathcal{H}^2)\langle(\nabla\Psi')^2\rangle - \frac{1}{3}(\mathcal{H}' + 2\mathcal{H}^2)\langle(\Delta\Psi)^2\rangle \\
& - \frac{2}{3\mathcal{H}}[4\mathcal{H}'^2 + 3\mathcal{H}'\mathcal{H}^2 - 8\mathcal{H}^4 + (2\mathcal{H}' + \mathcal{H}^2)\mathcal{H}K]\langle\nabla\Psi'\cdot\nabla\Psi\rangle \\
& - [3(\mathcal{H}' - \mathcal{H}^2)(3\mathcal{H}' - 7\mathcal{H}^2) + 14\mathcal{H}(\mathcal{H}' - \mathcal{H}^2)K + \mathcal{H}^2K^2 - \mathcal{H}^2 a^2 V_{\phi\phi}]\langle(\Psi')^2\rangle \\
& + \frac{1}{3\mathcal{H}^2}(\mathcal{H}' - 2\mathcal{H}^2)[2\mathcal{H}'^2 + 10\mathcal{H}'\mathcal{H}^2 - 13\mathcal{H}^4 + 2(2\mathcal{H}' + \mathcal{H}^2)L]\langle(\nabla\Psi)^2\rangle \\
& + \left[\frac{2}{\mathcal{H}}(\mathcal{H}' - 2\mathcal{H}^2)(L^2 + 12(\mathcal{H}' - \mathcal{H}^2)L - 4(\mathcal{H}' + 2\mathcal{H}^2)(\mathcal{H}' - \mathcal{H}^2)) - 2\mathcal{H}(\mathcal{H}' - \mathcal{H}^2)a^2 V_{\phi\phi}\right]\langle\Psi'\Psi\rangle \\
& - \frac{(\mathcal{H}' - 2\mathcal{H}^2)}{\mathcal{H}^2}[L^2 + 12(\mathcal{H}' - \mathcal{H}^2)L - \mathcal{H}^2 a^2 V_{\phi\phi} - 4(\mathcal{H}' + 2\mathcal{H}^2)(\mathcal{H}' - \mathcal{H}^2)]\langle\Psi^2\rangle \Bigg\}.
\end{aligned}$$

Gauge Invariant

Comoving Gauge

$$\boxed{\delta\phi = E = 0,}$$

$$\boxed{\begin{aligned}\Psi &= \psi + \mathcal{H}\beta, \\ Q &= \frac{\Psi' + \mathcal{H}\Psi}{\mathcal{H}' - \mathcal{H}^2}.\end{aligned}}$$

$$\begin{aligned}\tau_{00} = \frac{1}{8\pi G(\mathcal{H}' - \mathcal{H}^2)^2}\Bigg[&-2\mathcal{H}\langle\Delta\Psi'\Delta\Psi\rangle - (4\mathcal{H}' - 3\mathcal{H}^2)\langle(\Delta\Psi)^2\rangle \\ &+ \frac{F_1}{\mathcal{H}' - \mathcal{H}^2}\langle(\nabla\Psi')^2\rangle + \frac{2F_2}{\mathcal{H}' - \mathcal{H}^2}\langle\nabla\Psi' \cdot \nabla\Psi\rangle + \frac{F_3}{\mathcal{H}' - \mathcal{H}^2}\langle(\nabla\Psi)^2\rangle \\ &- \frac{F_4}{(\mathcal{H}' - \mathcal{H}^2)^2}\langle(\Psi')^2\rangle - \frac{2F_5}{(\mathcal{H}' - \mathcal{H}^2)^2}\langle\Psi'\Psi\rangle - \frac{F_6}{(\mathcal{H}' - \mathcal{H}^2)^2}\langle\Psi^2\rangle\Bigg],\end{aligned}$$

$$\begin{aligned}\tau_{ij} = \frac{\delta_{ij}}{8\pi G(\mathcal{H}' - \mathcal{H}^2)^2}\Bigg[&-\frac{2}{3}\langle(\Delta\Psi')^2\rangle - \frac{2}{3}\langle\Delta^2\Psi\Delta\Psi\rangle + \frac{2P_1}{3(\mathcal{H}' - \mathcal{H}^2)}\langle\Delta\Psi'\Delta\Psi\rangle - \frac{P_2}{3(\mathcal{H}' - \mathcal{H}^2)}\langle(\Delta\Psi)^2\rangle \\ &+ \frac{P_3}{3(\mathcal{H}' - \mathcal{H}^2)^2}\langle(\nabla\Psi')^2\rangle - \frac{2P_4}{3(\mathcal{H}' - \mathcal{H}^2)^2}\langle\nabla\Psi' \cdot \nabla\Psi\rangle + \frac{P_5}{3(\mathcal{H}' - \mathcal{H}^2)^2}\langle(\nabla\Psi)^2\rangle \\ &- \frac{P_6}{3(\mathcal{H}' - \mathcal{H}^2)^3}\langle(\Psi')^2\rangle + \frac{2P_7}{(\mathcal{H}' - \mathcal{H}^2)^3}\langle\Psi'\Psi\rangle - \frac{P_8}{3(\mathcal{H}' - \mathcal{H}^2)^3}\langle\Psi^2\rangle\Bigg].\end{aligned}$$

Gauge Invariant

$$
\begin{aligned}
F_1 =& 2\mathcal{H}\mathcal{H}'' + (\mathcal{H}')^2 + 4\mathcal{H}(K - 2\mathcal{H})\mathcal{H}' - \mathcal{H}^3(4K - 3\mathcal{H}),\\
F_2 =& -(2\mathcal{H}' - \mathcal{H}^2)\mathcal{H}'' - (4K - 5\mathcal{H})(\mathcal{H}')^2 + 2\mathcal{H}(2K\mathcal{H} + L - 3\mathcal{H}^2)\mathcal{H}' - \mathcal{H}^3(2L - 3\mathcal{H}^2),\\
F_3 =& -4\mathcal{H}\mathcal{H}'\mathcal{H}'' + 13(\mathcal{H}')^3 - \left[8L - 2\left(K - \frac{a^2V_\phi}{\phi_0'}\right)\mathcal{H} + 12\mathcal{H}^2\right](\mathcal{H}')^2\\
& - \mathcal{H}^2\left(4\mathcal{H}K - 8L + \mathcal{H}^2 - 4\mathcal{H}\frac{a^2V_\phi}{\phi_0'}\right)\mathcal{H}' + 2\mathcal{H}^5\left(K + 4\mathcal{H} - \frac{a^2V_\phi}{\phi_0'}\right),\\
F_4 =& (\mathcal{H}' + 2\mathcal{H}^2)(\mathcal{H}'')^2 - 4\big[2\mathcal{H}(\mathcal{H}')^2 - K(\mathcal{H}')^2 - K\mathcal{H}^2\mathcal{H}' + \mathcal{H}^4(2K + \mathcal{H})\big]\mathcal{H}''\\
& + \left[5K^2 - 16K\mathcal{H} + 7\mathcal{H}^2 - 2Ka^2V'(\phi_0)/\phi_0' + \left(\frac{a^2V_\phi}{\phi_0'}\right)^2\right](\mathcal{H}')^3\\
& - \left[3K^2 - 16\mathcal{H}K + 5\mathcal{H}^2 - 6Ka^2V'(\phi_0)/\phi_0' + 3\left(\frac{a^2V_\phi}{\phi_0'}\right)^2\right]\mathcal{H}^2(\mathcal{H}')^2\\
& - \left[9K^2 + 8\mathcal{H}K - 17\mathcal{H}^2 + 6Ka^2V'(\phi_0)/\phi_0' - 3\left(\frac{a^2V_\phi}{\phi_0'}\right)^2\right]\mathcal{H}^4(\mathcal{H}')\\
& + \left[7K^2 + 8\mathcal{H}K - 7\mathcal{H}^2 + 2Ka^2V'(\phi_0)/\phi_0' - \left(\frac{a^2V_\phi}{\phi_0'}\right)^2\right]\mathcal{H}^6,
\end{aligned}
$$

$$
\begin{aligned}
P_1 =& 4\mathcal{H}'' + (6K - 11\mathcal{H})\mathcal{H}' - 3\mathcal{H}^2(2K - \mathcal{H}),\\
P_2 =& 4\mathcal{H}\mathcal{H}'' + 4(\mathcal{H}')^2 + (8\mathcal{H}K - 8L - 15\mathcal{H}^2)\mathcal{H}' - \mathcal{H}^2(8\mathcal{H}K - 8L - 3\mathcal{H}^2)\\
P_3 =& -2(\mathcal{H}' - \mathcal{H}^2)\mathcal{H}''' + 6(\mathcal{H}'')^2 + 2\left[(8K - 13\mathcal{H})\mathcal{H}' - \mathcal{H}^2(8K - \mathcal{H})\right]\mathcal{H}'' + 9(\mathcal{H}')^3\\
&+ \left[8K(2K - 5\mathcal{H}) - 4L + 27\mathcal{H}^2\right](\mathcal{H}')^2 - \left[16K(2K - 3\mathcal{H}) - 8L + 17\mathcal{H}^2\right]\mathcal{H}^2\mathcal{H}'\\
&+ \left[8K(2K - \mathcal{H}) - 4L + 5\mathcal{H}^2\right]\mathcal{H}^4,\\
P_4 =& -\mathcal{H}(\mathcal{H}' - \mathcal{H}^2)\mathcal{H}''' + 6\mathcal{H}(\mathcal{H}'')^2 - \left[3(\mathcal{H}')^2 - (22\mathcal{H}K - 8L - 19\mathcal{H}^2)\mathcal{H}' + 2\mathcal{H}^2(11\mathcal{H}K - 4L + \mathcal{H}^2)\right]\mathcal{H}''\\
&- (8K - 29\mathcal{H})(\mathcal{H}')^3 + \left[4K(5K\mathcal{H} - 3L - 7\mathcal{H}^2) + \mathcal{H}(16L - 69\mathcal{H}^2)\right](\mathcal{H}')^2\\
&- \mathcal{H}^2\left[4\mathcal{H}K(10K - 9\mathcal{H}) - 8L(3K - 2\mathcal{H}) - 107\mathcal{H}^3\right]\mathcal{H}' + \mathcal{H}^4\left[4K(5\mathcal{H}K - 3L) - 43\mathcal{H}^3\right],\\
P_5 =& 4\mathcal{H}^2(\mathcal{H}' - \mathcal{H}^2)\mathcal{H}''' - 18\mathcal{H}^2(\mathcal{H}'')^2 + \mathcal{H}\left[60(\mathcal{H}')^2 - 4(3\mathcal{H}K + 8L + 11\mathcal{H}^2)\mathcal{H}'\right.\\
&\left.+ 4\mathcal{H}^2(3\mathcal{H}K + 8L + 14\mathcal{H}^2)\right]\mathcal{H}'' - 23(\mathcal{H}')^4 + (60\mathcal{H}K + 4L - 131\mathcal{H}^2)(\mathcal{H}')^3\\
&- \left[4\mathcal{H}K(10L + 39\mathcal{H}^2) - 4L(2L + 17\mathcal{H}^2) - 333\mathcal{H}^4\right](\mathcal{H}')^2\\
&+ \mathcal{H}^2\left[4\mathcal{H}K(20L + 39\mathcal{H}^2) - 4L(4L + 21\mathcal{H}^2) - 357\mathcal{H}^4\right]\mathcal{H}'\\
&- 2\mathcal{H}^4\left[10\mathcal{H}K(2L + 3\mathcal{H}^2) - 2L(2L + 3\mathcal{H}^2) - 53\mathcal{H}^4\right],
\end{aligned}
$$

However, the results look ALL DIFFERENT in different gauge choices.
IN SOME LIMITS, can we have convergences in results?

Field Equation

$$\boxed{\Psi'' - \Delta\Psi + 2K\Psi' + 2L\Psi = 0.}$$

$$K = 3\mathcal{H} + a^2\frac{V_\phi}{\phi_0'},$$
$$L = \mathcal{H}' + 2\mathcal{H}^2 + a^2\mathcal{H}\frac{V_\phi}{\phi_0'}.$$

Fourier Mode Expansion

$$\Psi(\eta, \boldsymbol{x}) = \sum_{\boldsymbol{k}} \Psi_{\boldsymbol{k}}(\eta) e^{i\boldsymbol{k}\cdot\boldsymbol{x}}$$

Solutions

$$\Psi_{\boldsymbol{k}}(\eta) = \begin{cases} A_1\epsilon & \text{(long-wavelength)} \\ 4\pi G\dot{\phi}_0[c_1\sin(k\eta) + c_2\cos(k\eta)] & \text{(short-wavelength)} \end{cases}$$

Slow-roll parameters

$$\epsilon \equiv -\frac{\dot{H}}{H^2} = 4\pi G\frac{\dot{\phi}_0^2}{H^2},$$
$$\delta \equiv -\frac{\ddot{\phi}_0}{H\dot{\phi}_0} = \epsilon - \frac{\dot{\epsilon}}{2H\epsilon}.$$

Quantities in slow-roll parameters

$$\mathcal{H}' = (1-\epsilon)\mathcal{H}^2,$$
$$V = \frac{3\mathcal{H}^2}{8\pi G a^2}, \quad \frac{dV}{d\phi} = 4\sqrt{\pi G\epsilon}V, \quad \frac{d^2V}{d\phi^2} = 8\pi G(\epsilon+\delta)V,$$
$$\phi' = -\frac{a^2}{3\mathcal{H}}\frac{dV}{d\phi}, \quad \ddot{\phi} = \frac{a^2}{9\mathcal{H}^2}\frac{d^2V}{d\phi^2}\frac{dV}{d\phi} - \frac{\epsilon}{3}\frac{dV}{d\phi}.$$

1. Long-wavelength limit ($k \ll \mathcal{H}$)

$\varepsilon \ni$ slow-roll parameters (ϵ, δ) and wavelength parameter $(\mathcal{H}/k)$

A. Longitudinal gauge

$$\tau_{00} \approx \frac{|A_1|^2 k^2}{8\pi G}\Big[\underbrace{2\epsilon}_{\mathcal{O}(\varepsilon)} + 3\underbrace{\frac{\mathcal{H}^2}{k^2}(-3\epsilon^2 + \epsilon\delta)}_{\mathcal{O}(\varepsilon^0)}\Big],$$

$$\tau_{ij} \approx \frac{|A_1|^2 k^2}{8\pi G}\left[-\frac{2}{3}\epsilon + 3\frac{\mathcal{H}^2}{k^2}(3\epsilon^2 - \epsilon\delta)\right],$$

1-1. Ultra LW limit ($k \lesssim \sqrt{\epsilon}\mathcal{H}$) $\boxed{w \equiv \frac{\mathfrak{p}}{\varrho} \approx -1,}$ $\boxed{\varrho \approx -\frac{3H^2|A_1|^2(3\epsilon^2 - \epsilon\delta)}{8\pi G}}$

1-2. Infra LW limit ($\sqrt{\epsilon}\mathcal{H} \lesssim k$) $\boxed{w \approx -\frac{1}{3},}$ $\boxed{\varrho \approx \frac{k^2|A_1|^2\epsilon}{4\pi G a^2} > 0}$

B. Spatially flat gauge

$$\tau_{00} \approx \frac{|A_1|^2 k^2}{8\pi G}\left[\epsilon + 3\frac{\mathcal{H}^2}{k^2}(-3\epsilon^2 + \epsilon\delta)\right],$$

$$\tau_{ij} \approx \frac{|A_1|^2 k^2}{8\pi G}\left[-\frac{1}{3}\epsilon + 3\frac{\mathcal{H}^2}{k^2}(3\epsilon^2 - \epsilon\delta)\right],$$

1-1. Ultra LW limit ($k \lesssim \sqrt{\epsilon}\mathcal{H}$) **: same with Longitudinal gauge**

1-2. Infra LW limit ($\sqrt{\epsilon}\mathcal{H} \lesssim k$) $\boxed{w = w_{LG}, \qquad \varrho = \varrho_{LG}/2}$

C. Comoving gauge

$$\tau_{00} \approx \underbrace{\frac{|A_1|^2 \mathcal{H}^2}{8\pi G}}_{\mathcal{O}(\varepsilon^{-2})}\left[9 + (49\epsilon - 36\delta) + 4(16\epsilon^2 - 25\epsilon\delta + 9\delta^2)\right],$$

$$\tau_{ij} \approx \frac{|A_1|^2 \mathcal{H}^2}{8\pi G}\left[-3 + (5\epsilon + 12\delta) + 4(13\epsilon^2 - 8\epsilon\delta - 3\delta^2)\right].$$

$\boxed{w = w_{LG},}$ $\boxed{\varrho \approx \frac{9\mathcal{H}^2|A_1|^2}{8\pi G a^2} = \frac{9H^2|A_1|^2}{8\pi G}}$

2. Short-wavelength limit ($k \gg \mathcal{H}$)

A. Longitudinal gauge

$\mathcal{O}(\varepsilon^0)$

$$\tau_{00} \approx (|c_1|^2 + |c_2|^2)\frac{k^4}{4a^2}\left[6 + \frac{\mathcal{H}^2}{k^2}(14\epsilon - \delta)\right],$$

$$\tau_{ij} \approx (|c_1|^2 + |c_2|^2)\frac{k^4}{4a^2}\left[\frac{10}{3} + \frac{\mathcal{H}^2}{k^2}\left(\frac{4\epsilon}{3} - \frac{5\delta}{3}\right)\right]$$

$$\boxed{w \approx \frac{5}{9},} \qquad \boxed{\varrho \approx \frac{3k^4(|c_1|^2 + |c_2|^2)}{2a^4}}$$

B. Spatially flat gauge

$$\tau_{00} \approx (|c_1|^2 + |c_2|^2)\frac{k^4}{4a^2}\left[2 + \frac{\mathcal{H}^2}{k^2}(-4\epsilon + \delta)\right],$$

$$\tau_{ij} \approx (|c_1|^2 + |c_2|^2)\frac{k^4}{4a^2}\left[\frac{2}{3} + \frac{\mathcal{H}^2}{k^2}\left(2\epsilon - \frac{7\delta}{3}\right)\right]$$

$$\boxed{w \approx \frac{1}{3},} \qquad \boxed{\varrho \approx \frac{k^4(|c_1|^2 + |c_2|^2)}{2a^4}}$$

$\mathcal{O}(\varepsilon^{-1})$

C. Comoving gauge

$$\tau_{00} \approx (|c_1|^2 + |c_2|^2)\frac{k^4}{4a^2}\left[-\frac{1}{\epsilon} + \left(3 - \frac{2\delta}{\epsilon}\right) + \frac{\mathcal{H}^2}{k^2}(5\epsilon + 2\delta)\right],$$

$$\tau_{ij} \approx -(|c_1|^2 + |c_2|^2)\frac{k^4}{4a^2}\left[\frac{11}{3\epsilon} + \left(1 + \frac{2\delta}{3\epsilon}\right) + \frac{2\delta^2}{\epsilon}\right].$$

$$\boxed{w \approx \frac{11}{3},} \qquad \boxed{\varrho \approx -\frac{k^4(|c_1|^2 + |c_2|^2)}{4a^4\epsilon}}$$

Tensor Perturbation

$$ds^2 \boxed{= a^2(\eta)[-(1+2\alpha)d\eta^2 - 2\beta_{,i}d\eta dx^i + ((1-2\psi)\delta_{ij} + 2E_{,ij})dx^i dx^j]}$$

$\boxed{+h_{ij} \text{ with TT gauge}}$

1st order eq.

$$h''_{ij} + 2\mathcal{H}h'_{ij} - h_{ij,nn} = 0$$

Solution

$$v''(\eta) + \left(k^2 - \frac{\mu^2 - 1/4}{\eta^2}\right) v(\eta) = 0$$

$$v = \frac{a*h}{\sqrt{16\pi G}}$$

$$\mu^2 = \frac{9}{4} + 3\epsilon + \cdots$$

$$v(\eta) = \sqrt{-\eta}\,[b_1 J_\mu(-k\eta) + b_2 Y_\mu(-k\eta)]$$

Expand in ϵ **and** $\sigma_L = -k\eta \ (\ll 1)$ **(for LW)**

$\sigma_S = -1/k\eta \ (\ll 1)$ **(for SW)**

2EMT for h-only

$$\tau_{00} = -\frac{3}{2}\left(h_{ij,n1}\right)^2 - 2h_{ij}h_{ij,n1n1} + 4\mathcal{H}h_{ij}h'_{ij} + \frac{1}{2}(h'_{ij})^2$$

$$\tau_{ij} = -2h_{in1,n2}h_{jn1,n2} + h_{n1n2,i}h_{n1n2,j} + 2h'_{in1}h'_{jn1} + \delta_{ij}\left(\frac{3}{2}\left(h_{n1n2,n3}\right)^2 - \frac{3}{2}(h'_{n1n2})^2\right)$$

Long-wavelength $\boxed{|k\eta| \ll 1}$

$$\boxed{\text{PolyGamma}[n,z] \equiv \frac{d^n}{dz^n}\left[\frac{\Gamma'(z)}{\Gamma(z)}\right]}$$

$$h(\eta) = \frac{1}{9\eta a[\eta]}\sqrt{2}\sqrt{G}\left(12k^3\eta^3 b_1 + 18k^2\eta^2 b_2\left(1 - \epsilon\text{Log}[-k\eta] + \epsilon\left(-2 + \text{Log}[2] + \text{PolyGamma}\left[0,\tfrac{3}{2}\right]\right)\right)\right.$$
$$+b_2\left(36 - 6\epsilon^3\text{Log}[-k\eta]^3 + 36\epsilon\left(\text{Log}[2] + \text{PolyGamma}\left[0,\tfrac{3}{2}\right]\right)\right.$$
$$+3\epsilon^2\left(-24 + 3\pi^2 + 6\text{Log}[2]^2 - 4\text{PolyGamma}\left[0,\tfrac{3}{2}\right]\right.$$
$$\left.+6\text{PolyGamma}\left[0,\tfrac{3}{2}\right]^2 + 4\text{Log}[2]\left(-1 + 3\text{PolyGamma}\left[0,\tfrac{3}{2}\right]\right)\right)$$
$$+3\epsilon^2\text{Log}[-k\eta]^2\left(6 + \epsilon\left(-4 + \text{Log}[64] + 6\text{PolyGamma}\left[0,\tfrac{3}{2}\right]\right)\right)$$
$$-\epsilon\text{Log}[-k\eta]\left(36 + 12\epsilon\left(-1 + \text{Log}[8] + 3\text{PolyGamma}\left[0,\tfrac{3}{2}\right]\right)\right.$$
$$+\epsilon^2\left(9\pi^2 + 2\left(-32 + 9\text{Log}[2]^2 - 2\text{Log}[64] - 12\text{PolyGamma}\left[0,\tfrac{3}{2}\right]\right.\right.$$
$$\left.\left.\left.+\text{Log}[262144]\text{PolyGamma}\left[0,\tfrac{3}{2}\right] + 9\text{PolyGamma}\left[0,\tfrac{3}{2}\right]^2\right)\right)\right)$$
$$+\epsilon^3\left(\pi^2\left(-6 + \text{Log}[512] + 9\text{PolyGamma}\left[0,\tfrac{3}{2}\right]\right)\right.$$
$$+2\left(24 + 3\text{Log}[2]^3 - 32\text{PolyGamma}\left[0,\tfrac{3}{2}\right] - 6\text{PolyGamma}\left[0,\tfrac{3}{2}\right]^2 + 3\text{PolyGamma}\left[0,\tfrac{3}{2}\right]^3\right.$$
$$+\text{Log}[2]^2\left(-6 + 9\text{PolyGamma}\left[0,\tfrac{3}{2}\right]\right)$$
$$\left.\left.\left.\left.+\text{Log}[2]\left(-32 - 12\text{PolyGamma}\left[0,\tfrac{3}{2}\right] + 9\text{PolyGamma}\left[0,\tfrac{3}{2}\right]^2\right) + 3\text{PolyGamma}\left[2,\tfrac{3}{2}\right]\right)\right)\right)\right)\left(\frac{1}{k}\right)^{3/2}$$

$\mathcal{O}(\varepsilon^{-2})$ $\mathcal{O}(\varepsilon^{-2})$ $\mathcal{O}(\varepsilon^{-1})$

pure de Sitter

$|k\eta| \ll 1$

$$\tau_{00} = \boxed{-\frac{256 G\epsilon^2 b_2 b_2^*}{3a^2k^3\eta^4}}\ \boxed{-\frac{224 G b_2 b_2^*}{a^2 k\eta^2} - \frac{16G(72 b_2 b_1^* + 72 b_1 b_2^*)}{9a^2\eta}}$$

$$\boxed{+\frac{\epsilon}{k^2\eta^2}}\left(\frac{448 Gk b_2 b_2^* \mathrm{Log}[-k\eta]}{a^2} + \frac{16Gk\left(144 b_2 b_2^* - 252 b_2 b_2^* \mathrm{Log}[2] - 252 b_2 b_2^* \mathrm{PolyGamma}[0,\tfrac{3}{2}]\right)}{9a^2}\right)$$

$\mathcal{O}(\varepsilon^{-1})$

Lowest-order in slow-roll parameter

$$\boxed{+\frac{\epsilon^3}{k^4\eta^4}}\left(\frac{16Gk b_2 b_2^* \mathrm{Log}[-k\eta]\left(96 - 6\mathrm{Log}[64]^2 + \mathrm{Log}[2]\left(-780 - 684\mathrm{PolyGamma}[0,\tfrac{3}{2}] + 36\left(13 + 6\mathrm{PolyGamma}[0,\tfrac{3}{2}]\right) + 12\left(26 + 6\mathrm{Log}[8] + 39\mathrm{PolyGamma}[0,\tfrac{3}{2}]\right)\right)\right)}{9a^2}\right.$$

$$+\frac{1}{9a^2} 16Gk\left(-16 b_2 b_2^* - 96 b_2 b_2^* \mathrm{PolyGamma}\left[0, \tfrac{3}{2}\right] - 12 b_2 b_2^* \mathrm{Log}[2]^2\left(11 + 3\mathrm{PolyGamma}\left[0, \tfrac{3}{2}\right]\right)\right.$$

$$+\mathrm{Log}[2]\left(72 b_2 b_2^* + 312 b_2 b_2^* \mathrm{PolyGamma}\left[0, \tfrac{3}{2}\right] + 36 b_2 b_2^* \mathrm{PolyGamma}\left[0, \tfrac{3}{2}\right]^2\right.$$

$$+\mathrm{Log}[4]\left(66 b_2 b_2^* + 18 b_2 b_2^* \mathrm{PolyGamma}\left[0, \tfrac{3}{2}\right]\right)$$

$$\left.\left.\left.-12 b_2 b_2^*\left(14 + 26\mathrm{PolyGamma}\left[0, \tfrac{3}{2}\right] + 3\mathrm{PolyGamma}\left[0, \tfrac{3}{2}\right]^2\right)\right)\right)\right)$$

propagating along x^3 direction

$$\tau_{11} = \boxed{\frac{32 G b_2 b_2^*}{a^2 k\eta^2}} + \boxed{\frac{\epsilon}{k^2\eta^2}}\left(-\frac{64 Gk b_2 b_2^* \mathrm{Log}[-k\eta]}{a^2} + \frac{32 Gk b_2 b_2^*\left(\mathrm{Log}[4] + 2\mathrm{PolyGamma}[0,\tfrac{3}{2}]\right)}{a^2}\right)$$

$$\boxed{+\frac{\epsilon^3}{k^4\eta^4}}\left(-\frac{16Gk b_2 b_2^*\left(-\mathrm{Log}[64]^2 - 12\mathrm{Log}[2]^2\left(-1 + 3\mathrm{PolyGamma}[0,\tfrac{3}{2}]\right) + \mathrm{Log}[8]\mathrm{Log}[16]\left(2 + 3\mathrm{PolyGamma}[0,\tfrac{3}{2}]\right)\right)}{9a^2}\right.$$

$$\left.-\frac{16Gk b_2 b_2^* \mathrm{Log}[-k\eta]\left(2\mathrm{Log}[64]^2 + \mathrm{Log}[2]\left(180 + 180\mathrm{PolyGamma}[0,\tfrac{3}{2}] - 36\left(3 + 2\mathrm{PolyGamma}[0,\tfrac{3}{2}]\right) - 12\left(6 + \mathrm{Log}[64] + 9\mathrm{PolyGamma}[0,\tfrac{3}{2}]\right)\right)\right)}{9a^2}\right)$$

$$\tau_{33} = \boxed{\frac{160 G b_2 b_2^*}{a^2 k\eta^2}}$$

$$\boxed{+\frac{\epsilon}{k^2\eta^2}}\left(-\frac{320 Gk b_2 b_2^* \mathrm{Log}[-k\eta]}{a^2} + \frac{32 Gk b_2 b_2^*\left(\mathrm{Log}[1024] + 10\mathrm{PolyGamma}[0,\tfrac{3}{2}]\right)}{a^2}\right)$$

$$\boxed{+\frac{\epsilon^3}{k^4\eta^4}}\left(-\frac{16Gk b_2 b_2^*\left(-\mathrm{Log}[64]^2 - 12\mathrm{Log}[2]^2\left(-1 + 3\mathrm{PolyGamma}[0,\tfrac{3}{2}]\right) + \mathrm{Log}[8]\mathrm{Log}[16]\left(2 + 3\mathrm{PolyGamma}[0,\tfrac{3}{2}]\right)\right)}{3a^2}\right.$$

$$\left.-\frac{16Gk b_2 b_2^* \mathrm{Log}[-k\eta]\left(2\mathrm{Log}[64]^2 + \mathrm{Log}[2]\left(180 + 180\mathrm{PolyGamma}[0,\tfrac{3}{2}] - 36\left(3 + 2\mathrm{PolyGamma}[0,\tfrac{3}{2}]\right) - 12\left(6 + \mathrm{Log}[64] + 9\mathrm{PolyGamma}[0,\tfrac{3}{2}]\right)\right)\right)}{3a^2}\right)$$

Short-wavelength

$|k\eta| \gg 1$

$$h(\eta) = \frac{1}{432\sqrt{2}\eta^3 a[\eta]}\sqrt{G}\,(1728\epsilon\,(\mathrm{Sin}[k\eta]b_1 + \mathrm{Cos}[k\eta]b_2)$$
$$-1296k\epsilon\eta\,(\mathrm{Cos}[k\eta]\,(-((2+3\epsilon)b_1) + \pi\epsilon b_2) + \mathrm{Sin}[k\eta]\,(\pi\epsilon b_1 + (2+3\epsilon)b_2))$$
$$-24k^2\eta^2\,(\mathrm{Cos}[k\eta]\,(3\pi^3\epsilon^3 b_1 + 4\pi\epsilon\,(-18-21\epsilon+5\epsilon^2)\,b_1 - 72(2+3\epsilon)b_2 + 3\pi^2\epsilon^2(6+5\epsilon)b_2)$$
$$+\mathrm{Sin}[k\eta]\,(3\,(-48-72\epsilon+6\pi^2\epsilon^2+5\pi^2\epsilon^3)\,b_1 + \pi\epsilon\,(72+84\epsilon-(20+3\pi^2)\,\epsilon^2)\,b_2))$$
$$-k^3\eta^3\,(\mathrm{Sin}[k\eta]\,(-72\pi^3(-1+\epsilon)\epsilon^3 b_1 + 64\pi\epsilon\,(-27+9\epsilon-6\epsilon^2+5\epsilon^3)\,b_1$$
$$-3456b_2 - 9\pi^4\epsilon^4 b_2 + 48\pi^2\epsilon^2\,(9-6\epsilon+5\epsilon^2)\,b_2)$$
$$+\mathrm{Cos}[k\eta]\,(3\,(1152+3\pi^4\epsilon^4-16\pi^2\epsilon^2\,(9-6\epsilon+5\epsilon^2))\,b_1$$
$$-8\pi\epsilon\,(216-72\epsilon+(48-9\pi^2)\,\epsilon^2+(-40+9\pi^2)\,\epsilon^3)\,b_2)))\,\left(\frac{1}{k}\right)^{7/2}$$

$\mathcal{O}(\varepsilon^2)$ $\mathcal{O}(\varepsilon^3)$

$$\tau_{00} = \boxed{\frac{32Gk(b_1^*b_1+b_2^*b_2)}{a^2} - \frac{112G(b_1^*b_1+b_2^*b_2)}{a^2k\eta^2}} - \frac{224G\boxed{\epsilon}(b_1^*b_1+b_2^*b_2)}{a^2k\eta^2}$$

$$\tau_{11} = \boxed{\frac{16G(b_1^*b_1+b_2^*b_2)}{a^2k\eta^2}} + \frac{16G\boxed{\epsilon}(b_1^*b_1+b_2^*b_2)}{a^2k\eta^2}$$

$$\tau_{33} = \boxed{\frac{32Gk(b_1^*b_1+b_2^*b_2)}{a^2} + \frac{80G(b_1^*b_1+b_2^*b_2)}{a^2k\eta^2}} + \frac{96G\boxed{\epsilon}(b_1^*b_1+b_2^*b_2)}{a^2k\eta^2}$$

Strong in Short Wave

Strong in Long Wave

$$
\tau^{\mathrm{Total}}_{\mu\nu} = \begin{pmatrix}
\tau^{\mathrm{S}}_{00} + \boxed{\tau^{\mathrm{T}}_{00}} & 0 & 0 & 0 \\
0 & \tau^{\mathrm{S}}_{11} + \tau^{\mathrm{T}}_{11} + \tau^{\mathrm{ST}} + \tau^{\mathrm{ST}} & +\tau^{\mathrm{ST}} & 0 \\
0 & & \tau^{\mathrm{S}}_{11} + \tau^{\mathrm{T}}_{11} - \tau^{\mathrm{ST}} & 0 \\
0 & 0 & 0 & \tau^{\mathrm{S}}_{11} + \tau^{\mathrm{T}}_{33}
\end{pmatrix}
$$

- **Pure de Sitter: only τ^{T} exists**
- **In slow-roll**
  - **LW: Tensor and Cross modes are dominant**
  - **SW: Scalar mode is dominant (stronger than pure de Sitter)**

Scalar-Tensor Coupled

Solutions Ψ & h : same, 2EMT: only (ij)-compts

$$\tau_{\rm ij} = h'_{\rm ij}\left(Q_{,\rm n1n1} - 2E'_{,\rm n1n1} + 2\mathcal{H}Q' + 2Q\mathcal{H}' + Q''\right) +$$
$$h_{\rm ij}\left(2\mathcal{H}Q_{,\rm n1n1} - 2\mathcal{H}E'_{,\rm n1n1} + 2Q'_{,\rm n1n1} - E''_{,\rm n1n1} - E_{,\rm n1n1n2n2} + Q'\left(4\mathcal{H}^2 + 4\mathcal{H}'\right) + \Psi\left(8\mathcal{H}^2 + 4\mathcal{H}' + \frac{4a^2\mathrm{V1}\mathcal{H}}{\phi0'}\right)\right.$$
$$\left.+Q\left(48a^2G\pi\mathrm{V0}\mathcal{H} + 12\mathcal{H}^3 - 22\mathcal{H}\mathcal{H}' + 16a^2G\pi\mathrm{V1}\phi0' - 64G\pi\mathcal{H}\phi0'^2\right) + 8\mathcal{H}\Psi' + \frac{4a^2\mathrm{V1}\Psi'}{\phi0'} + 2\mathcal{H}Q''\right)$$

GAUGE DEPENDENT

For $+$-polization: $\tau_{11} = -\tau_{22}$

For $\times$-polization: $\tau_{12} = \tau_{11}$

Long-wavelength

$$\tau^{\mathrm{LG}}{}_{11} = -\frac{8\sqrt{2}\sqrt{G}\mathcal{H}^2\epsilon(3\epsilon-2\delta)(b_2^*A_1+A_1^*b_2)}{a\eta}\left(\frac{1}{k}\right)^{3/2}$$

$$\tau^{\mathrm{SF}}{}_{11} = \frac{2\sqrt{2}\sqrt{G}\mathcal{H}\epsilon^2(-1+3\mathcal{H}\eta)(b_2^*A_1+A_1^*b_2)}{a\eta^2}\left(\frac{1}{k}\right)^{3/2}$$

$\mathcal{O}(\varepsilon^{-1})$

$$\tau^{\mathrm{CM}}{}_{11} = -\frac{8\sqrt{2}\sqrt{G}\mathcal{H}(b_2^*A_1+A_1^*b_2)}{a\eta^2}\left(\frac{1}{k}\right)^{3/2} \longrightarrow \mathcal{O}(\varepsilon^{-3})$$

Short-wavelength

$$\tau^{\mathrm{LG}}{}_{11} = -\frac{8G\sqrt{2\pi}\mathcal{H}^2\sqrt{\epsilon}(c_1^*b_1+c_2^*b_2+b_1^*c_1+b_2^*c_2)}{\sqrt{\frac{1}{k}}a^2}$$

$$\tau^{\mathrm{SF}}{}_{11} = \frac{4G\sqrt{2\pi}\mathcal{H}\sqrt{\epsilon}\delta(c_2^*b_1-c_1^*b_2-b_2^*c_1+b_1^*c_2)}{\left(\frac{1}{k}\right)^{3/2}a^2}$$

$\mathcal{O}(\varepsilon^{5/2})$

$$\tau^{\mathrm{CM}}{}_{11} = \frac{20G\sqrt{2\pi}\mathcal{H}(c_2^*b_1-c_1^*b_2-b_2^*c_1+b_1^*c_2)}{\left(\frac{1}{k}\right)^{3/2}a^2\sqrt{\epsilon}} \longrightarrow \mathcal{O}(\varepsilon^{1/2})$$

Conclusions

- Evaluated the 2nd order effective energy-momentum tensor (2EMT) for a scalar field

- 2EMT is gauge dependent

- Obtained 2EMT in 3 gauge conditions (Longitudinal, Spatially-flat, Comoving) in long- and short-wavelength limit in slow-roll regime of inflation for

1. Scalar
2. Tensor
3. Scalar-Tensor coupled

- In pure de Sitter, Tensor mode contribution only
- In slow-roll,
- Scalar mode correction is stronger for SHORT Wave
- The others are stronger for LONG Wave

I. DUST $(w=0)$

$$\boxed{\rho \propto \frac{1}{a^2}, \quad p \propto \frac{1}{a^2}}$$

$$\boxed{\begin{aligned}\rho &\equiv \rho^{(2)}_{\rm eff}\\ p &\equiv p^{(2)}_{\rm eff}\end{aligned}}$$

(i) Longitudinal Gauge

$$\tau_{00} = \frac{1}{8\pi G}\left[\frac{19k^2}{3}|\tilde{C}_1|^2 - \frac{48}{\eta^2}|\tilde{C}_1|^2 + \cdots\right.$$

$$\tau_{ij} = \frac{1}{8\pi G}\delta_{ij}\left[\frac{k^2}{9}|\tilde{C}_1|^2 - \frac{k^2}{\eta^5}(\tilde{C}_1, \tilde{C}_2) + \cdots\right.$$

$$\rightarrow p \approx 57\rho$$

(ii) Flat Gauge

$$\tau_{00} = \frac{1}{8\pi G}\left[\frac{4k^2}{3}|\tilde{C}_1|^2 - \frac{300}{\eta^2}|\tilde{C}_1|^2 \cdots\right.$$

$$\tau_{ij} = \frac{1}{8\pi G}\delta_{ij}\left[\left(4 + \frac{17k^2}{18}\right)|\tilde{C}_1|^2 - \frac{16}{\eta^2}|\tilde{C}_1|^2 \cdots\right.$$

$$\rightarrow p \approx \frac{4 + 17k^2/18}{4k^2/3}\rho$$

$= p \approx 4\rho$ ($k \ll$: **long wave**)

$= p \approx \frac{17}{24}\rho$ ($k \gg$: **short wave**)

(iii) Comoving Gauge

$$\tau_{00} = \frac{1}{8\pi G}\left[\frac{77k^2}{9}|\tilde{C}_1|^2 + \cdots\right.$$

$$\tau_{ij} = \frac{1}{8\pi G}\delta_{ij}\left[\left(\frac{16}{9} + \frac{13k^2}{27}\right)|\tilde{C}_1|^2 - \frac{64}{9\eta^2}|\tilde{C}_1|^2 + \cdots\right.$$

$$\rightarrow p \approx \frac{16 + 13k^2/3}{77k^2}\rho$$

$= p \approx 16\rho$ ($k \ll$: **long wave**)

$= p \approx \frac{13}{231}\rho$ ($k \gg$: **short wave**)

II. RADIATION ($w = \frac{1}{3}$)

(A) Long-wave limit ($\sqrt{w}k\eta \ll 1$)

(i) Longitudinal Gauge

$$\tau_{00} = \frac{1}{8\pi G}\Big[-\frac{39}{\eta^8}|\tilde{D}_2|^2 + \frac{6}{\eta^8}|\tilde{D}_2|^2(k\eta/\sqrt{3})^2 + \cdots$$

$$\tau_{ij} = \frac{1}{8\pi G}\delta_{ij}\Big[-\frac{19}{\eta^8}|\tilde{D}_2|^2 - \frac{14}{\eta^8}|\tilde{D}_2|^2(k\eta/\sqrt{3})^2 + \cdots$$

$$\boxed{\rho \propto \frac{1}{a^{10}}, \quad p \propto \frac{1}{a^{10}}}$$

(ii) Flat Gauge

$$\tau_{00} = \frac{1}{8\pi G}\Big[\frac{9}{\eta^8}|\tilde{D}_2|^2(k\eta/\sqrt{3})^2 \cdots$$

$$\tau_{ij} = \frac{1}{8\pi G}\delta_{ij}\Big[4\Big(\frac{1}{\eta^6} - \frac{1}{\eta^8}\Big)|\tilde{D}_2|^2 + \Big(\frac{4}{\eta^6} + \frac{11}{\eta^8}\Big)|\tilde{D}_2|^2(k\eta/\sqrt{3})^2 \cdots$$

$$\boxed{\rho \propto \frac{k^2}{a^8}, \ll p \propto \frac{1}{a^8}}$$

(iii) Comoving Gauge

$$\tau_{00} = \frac{1}{8\pi G}\Big[\frac{45}{2\eta^8}|\tilde{D}_2|^2(k\eta/\sqrt{3})^2 + \cdots$$

$$\tau_{ij} = \frac{1}{8\pi G}\delta_{ij}\Big[\Big(\frac{1}{\eta^6} - \frac{1}{\eta^8}\Big)|\tilde{D}_2|^2 + \Big(\frac{1}{\eta^6} + \frac{1}{2\eta^8}\Big)|\tilde{D}_2|^2(k\eta/\sqrt{3})^2 + \cdots$$

(B) Short-wave limit ($\sqrt{w}k\eta \gg 1$)

(i) Longitudinal Gauge

$$\tau_{00} = \frac{2}{8\pi G\eta^8}\left[3\left(|\tilde{D}_1|^2 + |\tilde{D}_2|^2\right)(k\eta/\sqrt{3})^6 + \cdots\right.$$

$$\tau_{ij} = \frac{2}{8\pi G\eta^8}\delta_{ij}\left[\left(|\tilde{D}_1|^2 + |\tilde{D}_2|^2\right)(k\eta/\sqrt{3})^6 + \cdots\right.$$

Dominant terms: $\tau_{00} = 3\tau_{ii}$

$$\boxed{\rho \propto \frac{1}{a^4}, \qquad p = \frac{1}{3}\rho}$$

: behaves as radiation !

(ii) Flat Gauge

$$\tau_{00} = \frac{2}{8\pi G\eta^8}\left[3\left(|\tilde{D}_1|^2 + |\tilde{D}_2|^2\right)(k\eta/\sqrt{3})^6 + \cdots\right.$$

$$\tau_{ij} = \frac{2}{8\pi G\eta^8}\delta_{ij}\left[\left(|\tilde{D}_1|^2 + |\tilde{D}_2|^2\right)(k\eta/\sqrt{3})^6 + \cdots\right.$$

Dominant terms: same with in Longitudinal Gauge

(iii) Comoving Gauge

$$\tau_{00} = \frac{2}{8\pi G\eta^8}\left[\frac{45}{2}\left(|\tilde{D}_1|^2 + |\tilde{D}_2|^2\right)(k\eta/\sqrt{3})^4 + \cdots\right.$$

$$\tau_{ij} = \frac{2}{8\pi G}\delta_{ij}\left[\frac{3}{2\eta^8}\left(|\tilde{D}_1|^2 + |\tilde{D}_2|^2\right)(k\eta/\sqrt{3})^4 + \frac{1}{\eta^6}\left(\left(|\tilde{D}_1|^2 + |\tilde{D}_2|^2\right)\right)(k\eta/\sqrt{3})^2 + \cdots\right.$$

$$\boxed{\rho \propto \frac{k^4}{a^6}, \qquad p = \frac{1}{15}\rho}$$

Behaves in a different way