On energetics of Black Holes

Energy extraction: Penrose Process vs Superradiance

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Cosmology and Quantum Space Time, Aug. 2, 2023, 임채호 교수님 추모 학회

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Motivation

When I heard the wave can be amplified after being scattered off from rotating black hole, I wondered if the dark matter can be detected through this procedure. Furthermore, the evidence of extra dimension in string theory can be discovered by this way. Although the neglect of backreaction in this analysis may oversimplify the problem, analytical study can be done in many cases.

Kerr black hole at a glance

$$ds^2 = \frac{\Delta}{\rho^2}\,[dt \ - \mathrm{a}\ \sin^2\theta\ \ d\phi]^2$$
$$- \frac{\sin^2\theta}{\rho^2}\,[(r^2 + a^2)d\phi - a\,dt]^2$$
$$- \frac{\rho^2}{\Delta}\,(dr)^2 \ - \rho^2(d\theta)^2$$

$$\rho^2 = \mathrm{r}^2 + \mathrm{a}^2\cos^2\theta, \Delta = r^2 + a^2 - 2\,M\,r$$

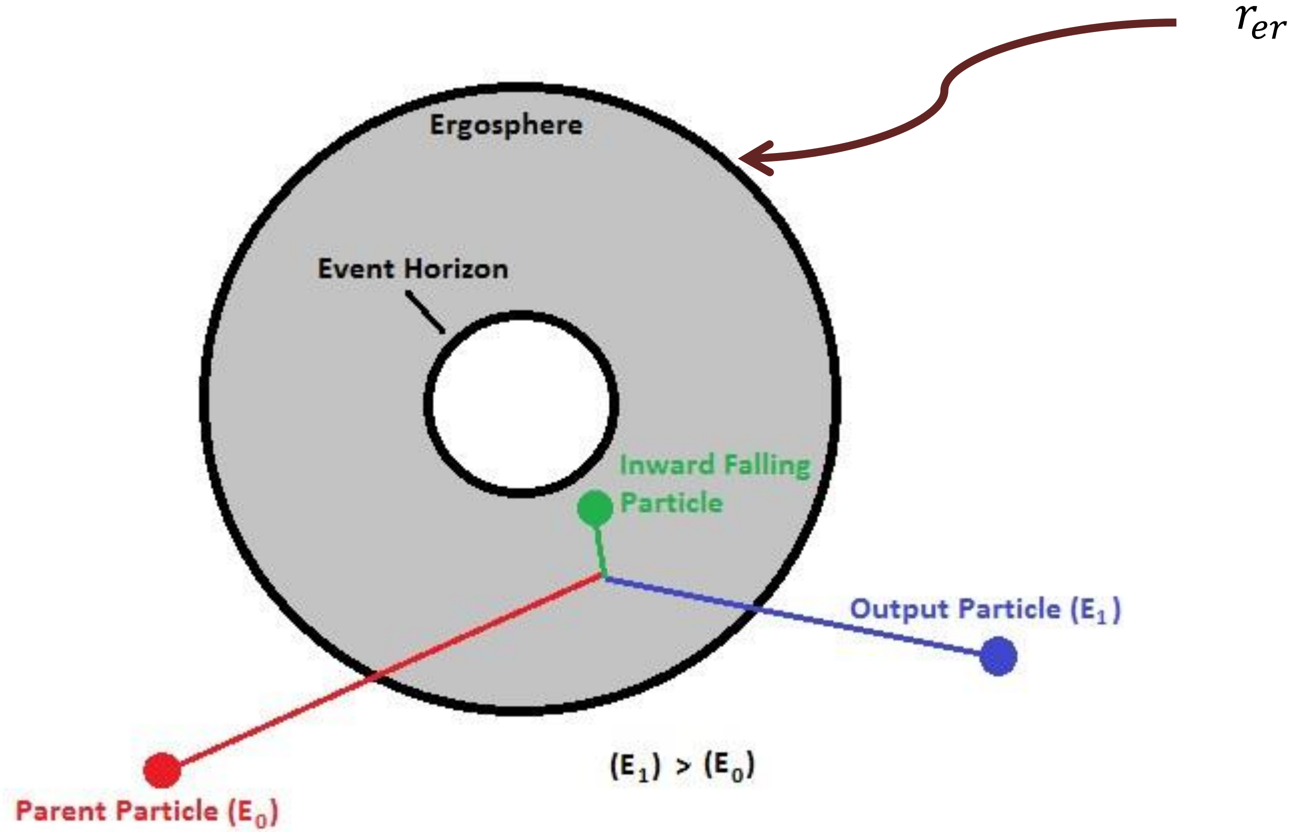
Horizon

$\Delta = 0$ gives $r = r_{\pm} = M \pm \sqrt{M^2 - a^2}$
Extreme Kerr: $M^2 = a^2$

Ergosphere

$k^2 = g_{tt} = -\frac{(\Delta - a^2 \sin^2\theta)}{\rho^2} = 0$ gives
$r_{er} = M + \sqrt{M^2 - a^2 \cos^2\theta}$
After passing over ergosphere ($r < r_{er}$) the Killing vector can possibly be positive meaning the Killing vector becomes spacelike.

Penrose Process



From infinity one particle with energy E_0 approaches to the rotating black hole. Assume that the particle breaks up into two inside the ergosphere and one of two falls into the hole while the other one escapes to infinity.

$$E_0 = E_{fall} + E_1$$

At ∞,

$$E_0 = -k \cdot p > 0$$

Inside the ergo region $E_{fall} = -p_{fall} \cdot k < 0$ can possibly happen because the Killing vector k can be possibly positive leading to

$E_1 - E_0 > -E_{fall} > 0.$

At the horizon, Killing vector is

$$\xi = \frac{\partial}{\partial t} + \Omega_H \frac{\partial}{\partial \phi}$$

And so $E_{fall} = -p \cdot \xi = E - \Omega_H L \geq 0.$

$$\Omega_H = \frac{a}{r^2 + a^2}$$

$$\frac{E}{\Omega_H} \geq L$$

If E is negative L is also negative so hole's angular momentum is reduced.

Detailed Penrose Process

Begin with Lagrangian

$L = \frac{1}{2}\, g_{\mu\nu} \frac{dx^{\mu}}{d\tau}\frac{dx^{\nu}}{d\tau}$ with Kerr metric with

$\theta = \frac{\pi}{2}$ $and \frac{d\theta}{d\tau} = 0$ which is rewritten as

$$2L = \left(1 - \frac{2M}{r}\right)\left(\frac{dt}{d\tau}\right)^2 + \frac{4aM}{r}\frac{dt}{d\tau}\frac{d\phi}{d\tau} - \frac{r^2}{\Delta}\left(\frac{dr}{d\tau}\right)^2 - \left[(r^2 + a^2) + \frac{2a^2M}{r}\right]\left(\frac{d\phi}{d\tau}\right)^2$$

$t\ and\ \phi$ are cyclic coordinates and exists constant of motion $p_t\ and\ p_\phi$.

$$p_t = E = \left(1 - \frac{2M}{r}\right)\frac{dt}{d\tau} + \frac{2aM}{r}\frac{d\phi}{d\tau}$$

$$-p_E = L$$

$$= -\frac{2aM}{r}\frac{dt}{d\tau} + \left[(r^2 + a^2) + 2\frac{a^{2M}}{r}\right]\frac{d\phi}{d\tau}$$

Hamiltonian Hindependent on t

$$2H = E\frac{dt}{d\tau} - L\frac{d\phi}{d\tau} - \frac{r^2}{\Delta}\left(\frac{dr}{d\tau}\right)^2 = 1$$

Hamilton-Jacobi method is a easy way to analyze the geodesic equation.

$$2L = \left(1 - \frac{2M}{\rho^2}\right)\left(\frac{dt}{d\tau}\right)^2 + \frac{4aMrsin^2\theta}{\rho^2}\frac{dt}{d\tau}\frac{d\phi}{d\tau}$$
$$- \frac{\rho^2}{\Delta}\left(\frac{dr}{d\tau}\right)^2 - \rho^2\left(\frac{d\theta}{d\tau}\right)^2$$
$$- \left[(r^2 + a^2)\right.$$
$$\left.+ \frac{2a^2Msin^2\theta}{\rho^2}\right]\sin^2\theta\left(\frac{d\phi}{d\tau}\right)^2$$

Hamilton-Jacobi equation

$$2\frac{\partial S}{\partial \tau} = g^{ij}\frac{\partial S}{\partial x^i}\frac{\partial S}{\partial x^j}$$

with $-m^2 = H$.

Inserting inverse metric into the Hamilton-Jacobi equation turns out to be

$$2\frac{\partial S}{\partial \tau} = \frac{1}{\rho^2\Delta}\left[(r^2+a^2)\frac{\partial S}{\partial t} + a\frac{\partial S}{\partial \phi}\right]^2 - \frac{1}{\rho^2\sin^2\theta}\left[a\sin^2\theta\frac{\partial S}{\partial t} + \frac{\partial S}{\partial \phi}\right]^2 - \frac{\Delta}{\rho^2}\left(\frac{\partial S}{\partial r}\right)^2 - \frac{1}{\rho^2}\left(\frac{\partial S}{\partial \theta}\right)^2$$

$$S = \frac{1}{2}\,\tau - E\,t + L_z\,\phi + S_r(r) + S_\theta(\theta)$$

Inserting this can make it separated into r and θ eqautions and let the separation constant Q.

We have

$$\rho^4\,\left(\frac{dr}{d\tau}\right)^2 = [(r^2 + a^2)E - a\,L_z]^2 - [Q + (L_z - a\,E)^2 + r^2]$$

From $\frac{dr}{d\tau} = 0$ (turning point) we read

$$E = \frac{1}{[r(r^2 + a^2) + 2\, a^2 M]} \Big[2\, a\, M\, L_z \pm \sqrt{\Delta}\sqrt{r^2 L_z^2 + [r(r^2 + a^2) + a\, a^2 M](\delta_1 r + Q r^{-1})}$$

Energy and angular momentum conservation can be expressed as

$$E_0 = 1 = E_1 + E_2,$$

$$L_{0z} = \alpha_0 = L_{1z} + L_{2z} = \alpha_1 E_1 + \alpha_2 E_2$$

$$E_1 = -\frac{1}{2}\sqrt{\frac{2M}{r}} + \frac{1}{2}, E_2 = \frac{1}{2}\sqrt{\frac{2M}{r}} + \frac{1}{2}$$

Therefore,

$$\Delta E = E_2 - E_0 = \frac{1}{2}\left(\sqrt{\frac{2M}{r}} - 1\right) = -E_1$$

Untouched:

Limitation of extracting energy.

The Penrose process does not violate the thermodynamic laws. The area of the black hole increases or stay constant under this process!

Superradiance

Wave analogue of Penrose process.

Spin 0:

Consider complex scalar field Φ of which equation of motion reads

$$\left(\nabla_\mu + iqA_\mu\right)(\nabla^\mu + iqA^\mu)\Phi - \mu^2\Phi = 0.$$

Consider Reissner-Nordstrom black hole with

$$ds^2 = -f\ dt^2 + f^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta\ d\phi^2),$$

where

$$f = 1 - \frac{2M}{r} + \frac{Q^2}{r^2}.$$

Choosing $\Phi = e^{-i\omega t + im\phi}Y(\theta)R(r)$, Separation of variable

$$\frac{1}{sin\theta}\frac{\partial}{\partial\theta}\left(sin\theta\frac{\partial}{\partial\theta}Y(\theta)\right)+$$

$$\left[l(l+1)-\frac{m^2}{\sin^2\theta}\right]Y(\theta)=0.$$

Introducing tortoise coordinate

$$r_* = \int\frac{dr}{f}$$

and new variables as

$$\psi = rR(r), \qquad \Delta = r^2 f$$

r-direction equation reduces to

$$\frac{d^2\psi}{dr_*^2} + [\omega^2 - V(r)] = 0,$$

where

$$V(r) = \omega^2 - (\omega + qA_t)^2 + f\left[\mu^2 + \frac{l(l+1)}{r^2} + \frac{1}{r}\frac{\partial f}{\partial r}\right]$$

Note $-\infty < r_* < \infty$ whereas $r_+ < r < \infty$.

Reduced to reflection/transmission problem.

As $r_* \to \infty$,

$$\Psi \sim e^{-i\sqrt{(\omega^2-\mu^2)}\, r_*} + \mathcal{R}\, e^{i\sqrt{\omega^2-\mu^2}\, r_*}$$

and as $r_* \to -\infty$ $(f \to 0)$

$$\Psi \sim \mathcal{T} e^{-i\left(\omega-\frac{qQ}{r}\right)r_*}$$

From these

we have

$$|\mathcal{R}|^2 = 1 - \frac{\omega - \frac{qQ}{r}}{\sqrt{\omega^2 - \mu^2}} |\mathcal{T}|^2$$

If $\omega < \frac{qQ}{r}$,

$$|\mathcal{R}| > 1$$

Superradiance!

Same story goes for Kerr geometry as well.

Spin $\frac{1}{2}$.

Dirac equation

$$[\gamma^{\nu}(\partial_{\nu} - \Gamma_{\nu}) - \mu]\Psi = 0$$

Here

$$\Gamma_{\mu} = -\frac{1}{4}\,\omega_{\alpha\,bc}\,\hat{\gamma}^{b}\,\hat{\gamma}^{c}$$

$$\left\{\gamma^{\alpha}, \gamma^{\beta}\right\} = 2g^{\alpha\beta} I_4, \qquad \{\hat{\gamma}^{a}, \hat{\gamma}^{b}\} = 2\eta^{ab} I_4$$

$$\gamma^{\alpha} = e^{\alpha}_{a}\,\hat{\gamma}^{a}$$

$$\hat{\gamma}_{a} = \eta_{ab}\,\hat{\gamma}^{b}, \qquad \gamma_{\alpha} = g_{\alpha\beta}\gamma^{\beta}$$

$$\omega_{\mu ab} = \frac{1}{2}\,e^{c}_{\mu}\,(\lambda_{abc} + \lambda_{cab} - \lambda_{bca})$$

$$\lambda_{abc} = e_a^{\mu}\left(\partial_\nu e_{b\,\mu} - \partial_\mu e_{b\,\nu}\right) e_c^{\nu}$$

Writing $\Psi = \Delta^{-\frac{1}{4}} \begin{pmatrix} \varrho^{-\frac{1}{2}} \eta_- \\ \varrho^{*-\frac{1}{2}} \eta_+ \end{pmatrix}$ and

$$\eta_\pm(t, r, \theta, \phi) = e^{i(m\phi - \omega t)} \eta_\pm(r, \theta)$$

with ansatz

$$\eta_+ = \begin{pmatrix} R_1(r) S_1(\theta) \\ R_2(r) S_2(\theta) \end{pmatrix}, \eta_- = -\begin{pmatrix} R_2(r) S_1(\theta) \\ R_1(r) S_2(\theta) \end{pmatrix}$$

The Dirac equation can be separated. As a reflection/transmission problem in r-direction, the result shows no superradiance unlike spin 0 case.

Conslusion

Energy extraction from black holes is possible for charged and rotating holes in terms of Penrose process: particle point of view.

Wave point of view (superradiance) shows discrepancy between spin 0 and spin $\frac{1}{2}$.

Need more study to understand spin $\frac{1}{2}$ or higher spin or any? Rarita-Schwinger?

Quantum mechanics can be a solution? But how?