

Supersymmetry and complexified spectrum on Euclidean AdS₂

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Outline

- Motivation
- Problem with SUSY and standard basis
- Construction of supersymmetric Hilbert space
- 1-loop in SUSY Hilbert space
- Conclusion

Motivation: SUSY on EAdS₂

- All extremal black holes universally have an **AdS₂** factor in their near horizon geometry [Sen '07].
- **Euclidean path integral** approach provides thermodynamic properties of the black holes [Gibbons, Hawking '97]
- **Supersymmetry** provides us with a powerful tool for quantum study of black hole entropy.
e.g. SUSY localization method [Nekrasov '02, Pestun '07]

Motivation: SUSY on EAdS₂

- [Sen '08] Quantum formula of macroscopic entropy for extremal black holes is defined as a partition function on Euclidean AdS₂

$$S_{\text{BH}} = k_B \log Z_{\text{AdS}_2} = \frac{k_B A_H(q, p)}{4G_N \hbar / c^3} + \dots$$

Motivation: SUSY on EAdS₂

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$$S_{\text{BH}} = k_B \log Z_{\text{AdS}_2} = \frac{k_B A_H(q, p)}{4G_N \hbar / c^3} + c \log A_H(q, p) + \dots$$

- For various BPS black holes, perturbative 1-loop matches with corresponding microscopic result.

[Sen '08; Bhattacharyya, Panda, Sen '12; Keeler, Larsen, Lisão '14'15; Banerjee, Banerjee, Gupta, Mandal, Thakur, ...]

Motivation: SUSY on EAdS₂

- Application of supersymmetric localization:
One successful example is for entropy of 1/8 BPS black hole in type II supergravity, reproducing microstate degeneracy as an integer !

$$Z_{AdS_2}^{\text{sugra}} \equiv \langle e^{-iq \oint A} \rangle = \text{integer}$$

[Dabholkar, Gomes, Murthy '10,'11; Gupta, Murthy '12; Gupta, Ito, **IJ** '15; Murthy, Reys, de Wit Murthy, Reys '18; **IJ**, Murthy '18; Iliesiu, Murthy, Turiaci, '22]

- Despite those extensive results...

The problem

- Supersymmetric boundary condition and normalizable condition are not always compatible. [David, Gave, Gupta, Narain '18, '19]
- Standard normalizable eigenbasis [Camporesi, Higuchi '94] for bosonic and fermionic fluctuation are not mapped to each other by supersymmetry [Sen '23].
- Supersymmetry demands 'non-normalizable' modes?
 - path integral ill-defined?!
 - well-defined theories on AdS cannot have SUSY?!

The problem

Sen : “Then, we cannot use the localization for SUGRA ”, giving up SUSY.

- If there is no SUSY, what does it mean by ‘super’ gravity on AdS and its dual ‘supersymmetric’ field theory?
- How can the 1-loop test using the standard non-supersymmetric basis agree with results from supersymmetric microscopic theory?
- How is the localization method valid and capable of giving the correct exact result?
- We resolve this problem by showing that EAdS2 requires **complexified spectrum** and constructing the supersymmetric Hilbert space for scalar and spinor fields.

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Problem of SUSY and standard basis

- SUSY relation between boson and fermion is generically given by

$$Q\Phi = \varepsilon\Psi .$$

- On the AdS₂ geometry,

$$ds^2 = L^2(d\eta^2 + \sinh^2 \eta d\theta^2) \quad 0 \leq \eta < \infty, \quad 0 \leq \theta < 2\pi$$

the Killing spinor equation is given by

$$D_\mu \varepsilon^s = s \frac{1}{2L} \gamma_\mu \varepsilon^s, \quad s = \pm 1,$$

whose solutions are

$$\varepsilon_+^s = \sqrt{L} e^{\frac{i\theta}{2}} \begin{pmatrix} \cosh \frac{\eta}{2} \\ s \sinh \frac{\eta}{2} \end{pmatrix}, \quad \varepsilon_-^s = \sqrt{L} e^{-\frac{i\theta}{2}} \begin{pmatrix} s \sinh \frac{\eta}{2} \\ \cosh \frac{\eta}{2} \end{pmatrix}$$

- They have exponential asymptotic growth $\exp(\eta/2)$ for large η .

Problem of SUSY and standard basis

- Eigenbasis of $-\nabla^2$ for scalar:

$$\phi_{\lambda,k}(\eta, \theta) \sim e^{ik\theta} \sinh^{|k|} \eta F\left(\alpha_s, \beta_s; |k|+1; -\sinh^2 \frac{\eta}{2}\right) \quad \alpha_s = \frac{1}{2} + |k| + i\lambda, \beta_s = \frac{1}{2} + |k| - i\lambda$$

with $k \in \mathbb{Z}$, $\lambda \in \mathbb{R}_{>0}$, where $F(\alpha, \beta; \gamma; z)$ is the hypergeometric function, which have eigenvalue, $L^{-2}(\lambda^2 + 1/4)$.

- Eigenbasis of $i\gamma^\mu D_\mu$ spinor field:

$$\psi_{\lambda,k}^+ \sim e^{i(k+\frac{1}{2})\theta} \begin{pmatrix} \cosh^{k+1} \frac{\eta}{2} \sinh^k \frac{\eta}{2} F(\alpha_f, \beta_f; k+1; -\sinh^2 \frac{\eta}{2}) \\ -i \frac{\lambda}{k+1} \cosh^k \frac{\eta}{2} \sinh^{k+1} \frac{\eta}{2} F(\alpha_f, \beta_f; k+2; -\sinh^2 \frac{\eta}{2}) \end{pmatrix}$$

$$\psi_{\lambda,k}^- \sim e^{-i(k+\frac{1}{2})\theta} \begin{pmatrix} i \frac{\lambda}{k+1} \cosh^k \frac{\eta}{2} \sinh^{k+1} \frac{\eta}{2} F(\alpha_f, \beta_f; k+2; -\sinh^2 \frac{\eta}{2}) \\ -\cosh^{k+1} \frac{\eta}{2} \sinh^k \frac{\eta}{2} F(\alpha_f, \beta_f; k+1; -\sinh^2 \frac{\eta}{2}) \end{pmatrix}$$

$$\alpha_f = k+1+i\lambda, \beta_f = k+1-i\lambda$$

with $\lambda \in \mathbb{R}$, $k \in \mathbb{Z}_{\geq 0}$, which have the eigenvalue, $L^{-1}\lambda$.

Problem of SUSY and standard basis

- Eigenbasis of scalar and spinor fields grow as

$$\phi_{\lambda,k}(\eta, \theta) \sim e^{-\frac{1}{2}\eta + ik\theta} (\alpha_{\lambda,k} e^{i\lambda\eta} + \alpha_{-\lambda,k} e^{-i\lambda\eta})$$

$$\psi_{\lambda,k}^{\pm} \sim e^{-\frac{\eta}{2} \pm i(k + \frac{1}{2})\theta} (e^{i\lambda\eta} \beta_{\lambda,k} v_{(-)} \pm e^{-i\lambda\eta} \beta_{-\lambda,k} v_{(+)})$$

having degree of growth **-1/2.**

- Since the bispinor

$$\varepsilon_{\pm}^s \psi_{\lambda,k}^{\pm} \sim e^{si\lambda\eta \pm i(k+1)\theta}, \quad \varepsilon_{\mp}^s \psi_{\lambda,k}^{\pm} \sim e^{si\lambda\eta \pm ik\theta}.$$

has the degree of growth **0,**

- the left and right hand side of the supersymmetry relation $Q\Phi = \varepsilon\Psi$ do not match when expressed in terms of standard basis.

Problem with SUSY on AdS₂

- Supersymmetry in terms of mode expansion coefficient

$$\Phi = \sum a_m \phi_m, \Psi = \sum b_n \psi_n,$$

$$Qa_m = \langle \phi_m | \varepsilon \Psi \rangle = \sum_n b_n \langle \phi_m | \varepsilon \psi_n \rangle$$

- The inner product is ill-defined as the integration diverges

$$\langle \phi_{\lambda', k'} | \varepsilon \psi_{\lambda, k} \rangle \sim \int_0^\infty d\eta \sqrt{g} e^{-\frac{1}{2}\eta} \rightarrow \infty$$

- We can obtain the inner product using analytic continuation: Introduce $\epsilon > 1/2$ such that the integration converges and take $\epsilon \rightarrow 0$ at the end.

$$\langle \phi_{\lambda', k} | \varepsilon^s \psi_{\lambda, k} \rangle \propto \delta(\lambda' + (\lambda - s\frac{i}{2})) + \delta(\lambda' - (\lambda - s\frac{i}{2}))$$

- Since the parameter λ and λ' are real, the inner product is zero. \rightarrow No SUSY.

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Supersymmetric Hilbert space

- The above result suggest to consider $\lambda \rightarrow \lambda + s i/2$.
- The bi-spinor with shifted λ by $i/2$ is exactly proportional to the eigenfunctions for scalar as

$$-\nabla^2(\varepsilon^s \psi_{\lambda,k}^\pm) = \frac{1}{L^2} \left((\lambda - s \frac{i}{2})^2 + \frac{1}{4} \right) (\varepsilon^s \psi_{\lambda,k}^\pm)$$

$$\varepsilon_\pm^s \psi_{\lambda+s \frac{i}{2},k}^\pm \propto \phi_{\lambda,\pm(|k|+1)}, \quad \varepsilon_\mp^s \psi_{\lambda+s \frac{i}{2},k}^\pm \propto \phi_{\lambda,\pm|k|}.$$

Therefore, the mapping between boson and fermion becomes manifest.

- We propose a SUSY Hilbert space

Scalar:	$\left\{ \phi_{\lambda,k}(\eta, \theta) \mid \lambda \in \mathbb{R}_{>0}, \quad k \in \mathbb{Z} \right\}$
Spinor:	$\left\{ \psi_{\lambda+s \frac{i}{2},k}^\pm(\eta, \theta) \mid \lambda \in \mathbb{R}, \quad k \in \mathbb{Z}_{\geq 0} \right\}.$

Supersymmetric Hilbert space

- Complex eigenvalue:

$$i\gamma^\mu D_\mu \psi_{\lambda+s\frac{i}{2},k}^\pm = L^{-1}(\lambda + s\frac{i}{2})\psi_{\lambda+s\frac{i}{2},k}^\pm$$

- It is natural: for a space with boundary, the Dirac operator is no longer hermitian.

c.f. non-hermiticity in a box [[Bonneau, Faraut, Valent '01](#)]

- Unlike the standard basis, there is no fermionic zero mode:
If mass is $\pm 1/2$, the kinetic operator $i(\gamma^\mu D_\mu - s/2)$ vanishes at $\lambda = 0$. But there is no spectrum at this point as spectral density of the spinor basis is zero.

$$\mu_{\psi^\pm}(\lambda + s\frac{i}{2}) = \frac{1}{4\pi} \left(\lambda + s\frac{i}{2} \right) \tanh \pi \lambda$$

Supersymmetric Hilbert space

- Asymptotic behaviour : not vanishing at the asymptotic boundary as

$$\psi_{\lambda+s\frac{i}{2},k}^{\pm} \sim e^{\pm i(k+\frac{1}{2})\theta - si\lambda\eta} v_{(s)}, \quad \frac{1}{2}(1 \pm \sigma_1)v_{(\pm)} = v_{(\pm)}$$

- Nevertheless, the SUSY basis form a delta-function normalizable orthonormal basis as

$$\langle \psi_{\lambda+s\frac{i}{2},k}^{\pm} | \psi_{\lambda'+s\frac{i}{2},k'}^{\pm} \rangle \equiv \pm i \int d\eta d\theta \sqrt{g} (\psi_{\lambda+s\frac{i}{2},k}^{\pm})^T C \psi_{\lambda'+s\frac{i}{2},k'}^{\pm} = \delta_{kk'} \delta(\lambda - \lambda')$$

- We define the appropriate inner product without using hermitian conjugate, “Euclidean inner product”.
- The projection condition of $v_{(\pm)}$ cancels the dominant term in there inner product, making the inner product well-defined.

Supersymmetric Hilbert space

- Compatible with asymptotic boundary condition.
Asymptotic fall-off behavior of the fluctuation of fields is dictated by variational principle.

$\delta S \equiv 0$ at on-shell saddle requires

$$\delta\phi = \delta\phi_{(0)} e^{-\Delta_\phi \eta} + \dots, \quad \Delta_\phi > \frac{1}{2},$$

$$\delta\psi = \delta\psi_{(0)} e^{-\Delta_\psi \eta} + \dots, \quad \Delta_\psi > 0.$$

- Note that the SUSY basis have $\phi_{\lambda,k} \sim e^{-\frac{1}{2}\eta}$, $\psi_{\lambda+s\frac{i}{2},k} \sim e^{-0\cdot\eta}$ saturate the bounds, being able to span all fluctuations above the bound.

cf. For the case $1/2 \geq \Delta_\psi > 0$, standard basis having $1/2$ growth cannot span the corresponding fluctuation.

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1-loop in SUSY Hilbert space

- Let us compare 1-loop partition function using SUSY and non-SUSY basis.

- Focus on contribution of spinors having kinetic term,

$$-i\bar{\psi}(\not{D} + M_{\psi})\psi$$

- A difference: SUSY Hilbert space does not suffer from zero modes

- If there are zero modes, one need to separate out their regularized contribution

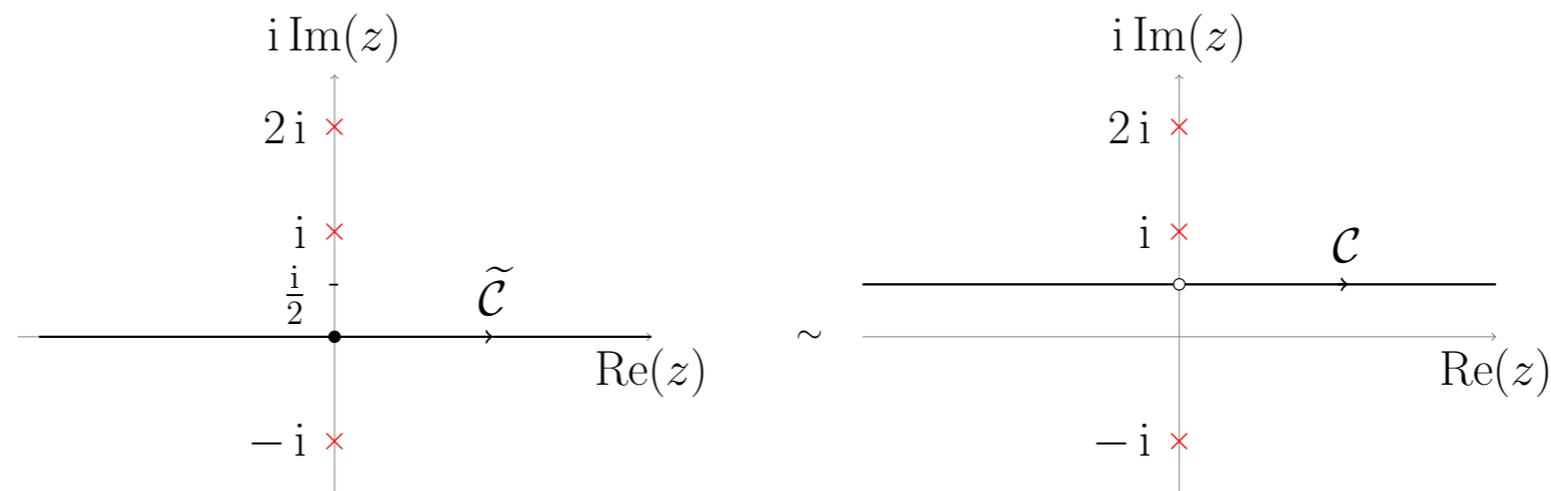
$$Z_{1\text{-loop}} = Z_{\text{zm}} Z'_{1\text{-loop}}$$

- Use the heat kernel method.

$$\begin{aligned} \log Z_{\psi} &= \frac{1}{2} \int_{\epsilon/L^2}^{\infty} \frac{d\bar{s}}{\bar{s}} K_{\psi}(\bar{s}) = -\frac{1}{2} \int_{\epsilon/L^2}^{\infty} \frac{d\bar{s}}{\bar{s}} \text{Tr} \exp \left[-\bar{s} (iL(\not{D} + M_{\psi}))^2 \right], \\ &= \int_{\epsilon/L^2}^{\infty} \frac{d\bar{s}}{\bar{s}} \int d^2x \sqrt{g} \int_{\mathcal{C}} dz \mu_{\psi}(z) \exp \left[-\bar{s} (z + iLM_{\psi})^2 \right]. \end{aligned}$$

1-loop in SUSY Hilbert space

- Standard basis vs. SUSY basis



The shift of the contour does not cross any pole and thus does not change the result of the heat kernel.

- Local contribution of the heat kernel is unchanged, whereas the global contribution (zero mode contribution) can be different differ.

1-loop in SUSY Hilbert space

- How can the 1-loop study using standard basis for black hole entropy matches with supersymmetric result?
 - Black hole near horizon geometry has additional geometry to AdS_2 . e.g. $\text{AdS}_2 \times S^2$.
 - Dirac operator along S^2 does not give zero mass in the Kaluza-Klein tower, thus there is no zero mode even in the standard basis.

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Concluding remarks

- We have constructed supersymmetric basis for scalar and spinor field by complexifying the spectrum of Dirac operator.
- We expect: complexified spectrum is pervasively necessary:
 - For higher dimensions, $i / 2$ shift
 - Vector, graviton multiplets should have complexified spectrum.
- SUSY Hilbert space for vector, graviton multiplet is an interesting problem to study the supersymmetric black hole entropy.
- Quantum fluctuation of SUSY theory should reside in a SUSY Hilbert space. Therefore, our construction provides a basic foundation for quantum studies of SUSY theory on AdS_2 .

Thank you.

Speculation

- Even not considering the supersymmetry, complexified spectrum seems to be a ‘correct’ basis.
 - Free from fermionic zero mode
 - Elliptic boundary condition [[Witten '18](#)]:
“This condition leads to a satisfactory perturbation theory of Euclidean gravity on Euclidean space with boundary”
 - Fermion modes survive at the boundary → Chiral boundary condition. Anomaly inflow and η -invariant? [[Yonekura, Witten '19](#)]
 - Global (zero mode) contribution can differ. Can we elaborate the difference in detail?