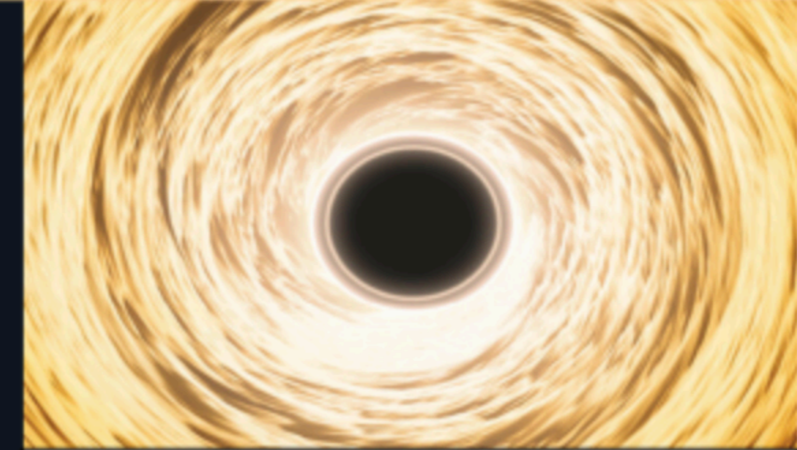
Workshop on Cosmology and Quantum Space Time (CQUeST 2023)

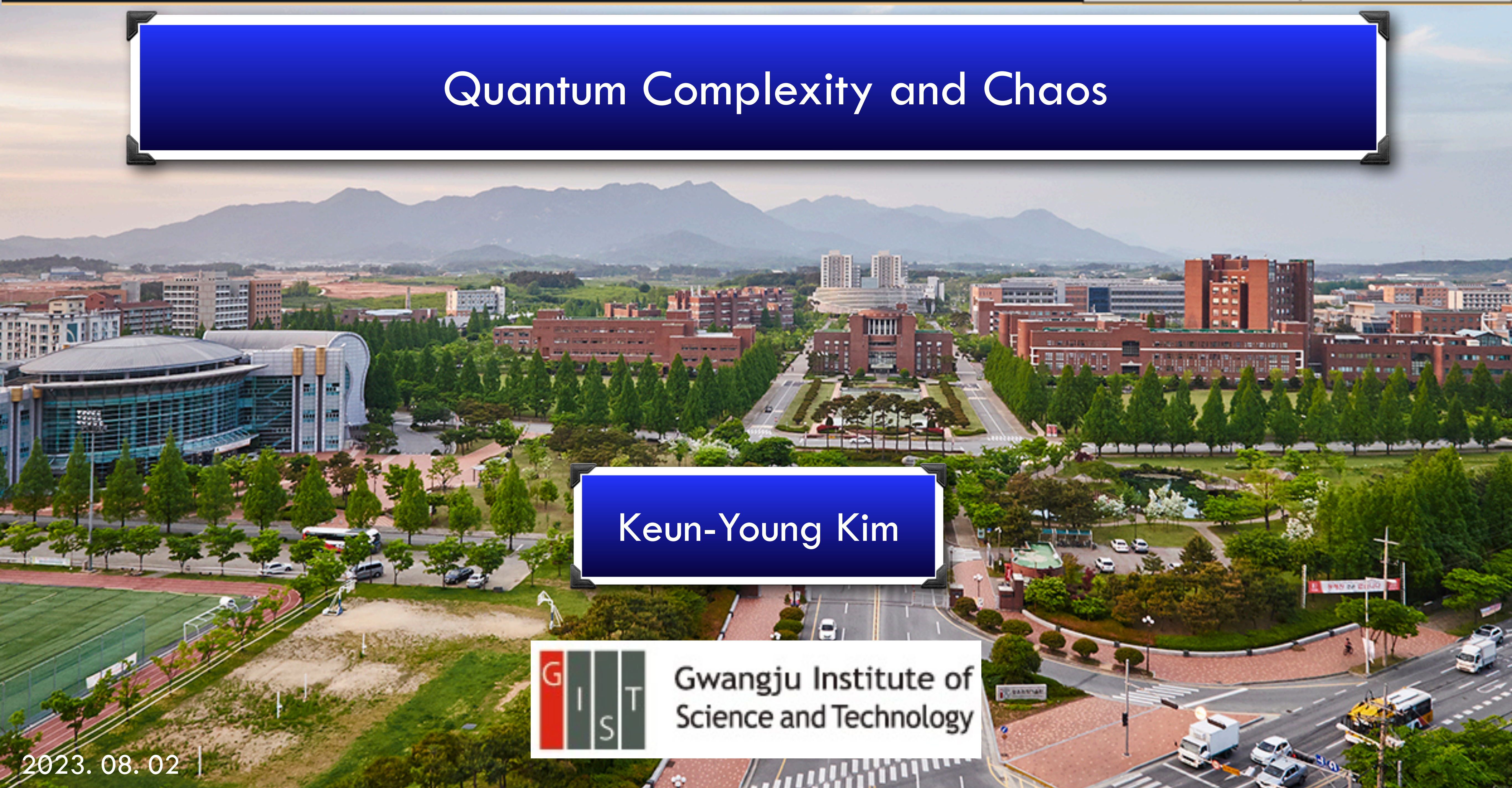
임채호 교수님 추모 학회

JULY 31 (MON) ~ AUGUST 04 (FRI), 2023

OFFLINE ONLY AT BEST WESTERN PLUS HOTEL, JEONJU.



CQUeST
Center for Quantum SpaceTime



Quantum Complexity and Chaos

Keun-Young Kim

Gwangju Institute of Science and Technology

2023. 08. 02

Research Directions in Quantum Field Theory and String Theory 2020

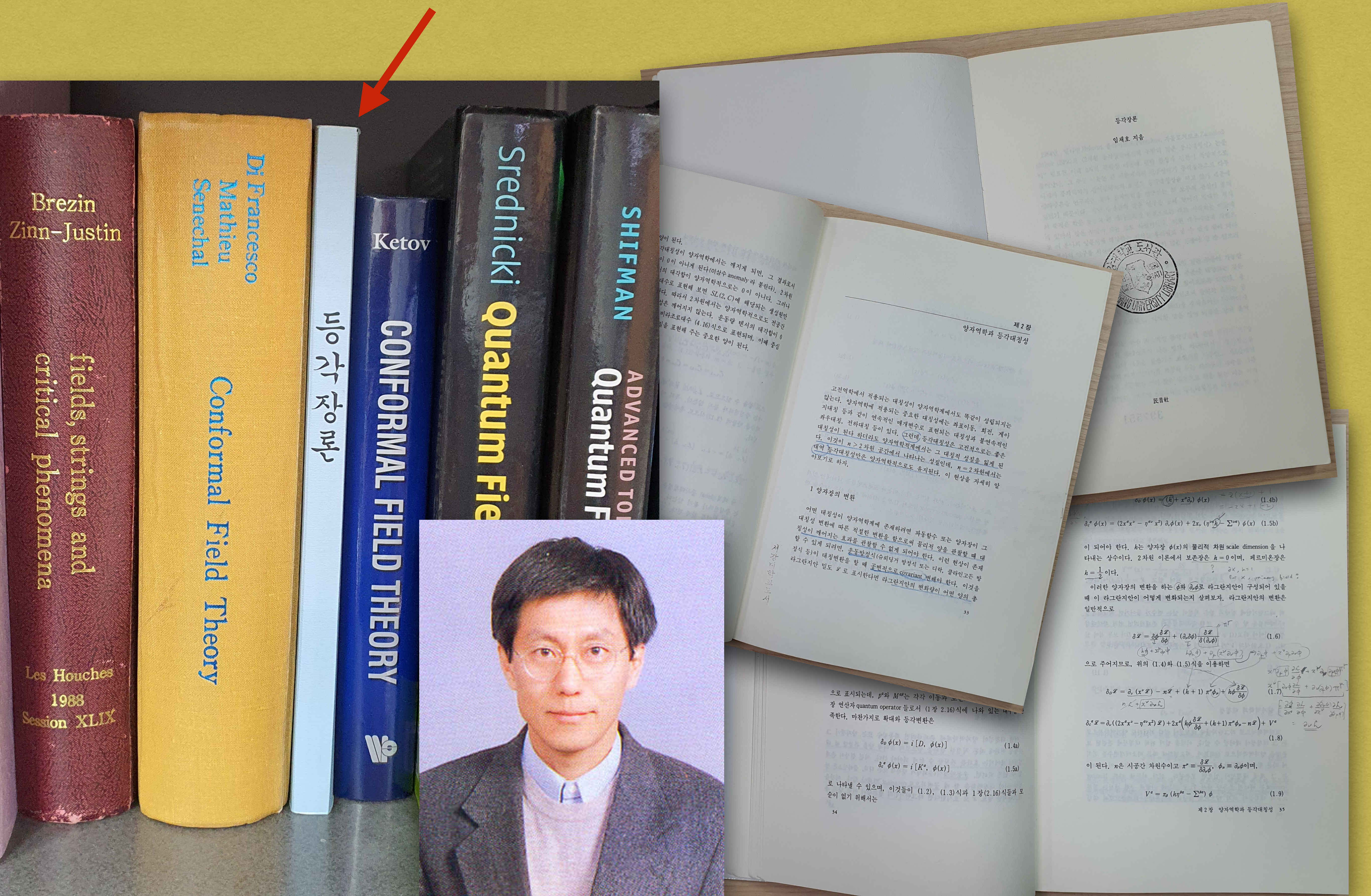
The workshop for Prof. *Chaiho Rim*

Applications of AdS/CFT
or Holographic duality

Keun-Young Kim

at GIST

1998년부터 저와 함께 세계 이곳 저곳을 돌아서
지금은 제 책꽂이에 자리 잡은 책이 하나 있습니다.
바로 임채호 교수님의 “등각장론” 입니다.



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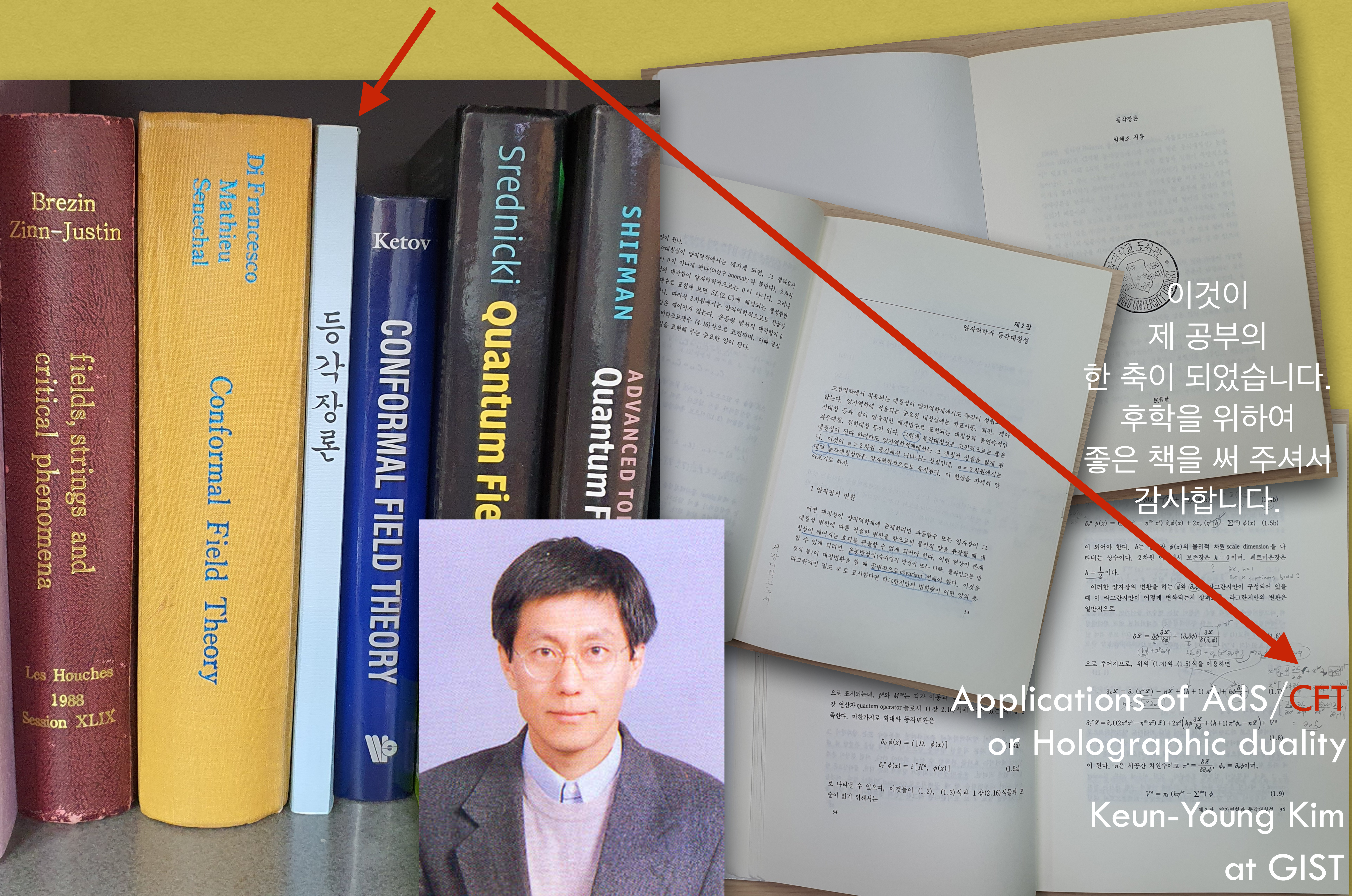

이것이
제 공부의
한 축이 되었습니다.
후학을 위하여
좋은 책을 써 주셔서
감사합니다.

Applications of AdS/CFT
or Holographic duality

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at GIST

Research Directions in Quantum Field Theory and String Theory 2020

The workshop for Prof. *Chaiho Rim*

Applications of AdS/CFT
or Holographic duality

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- Considering a broad range of participants, I would like to take this opportunity to introduce very interesting topics (to me) but boring looking subjects (complexity?, chaos?) in hep-th community. In the end, however, I hope you get to like them like I did.
- I will try to convey motivations, history, basic ideas instead of too much technical details.
- I think I am not so an expert on quantum information and its holographic avatar, so some introductory part of my talk may be based on premature thoughts. However, I hope my story is mature enough to make you get curious on the topics.
- I will try to go slowly, so I may stop my talk in 30 minutes in the middle without finishing up all of my slides. I think it is ok, because the conclusion of my story is open anyway.

Quantum Complexity and Quantum Chaos

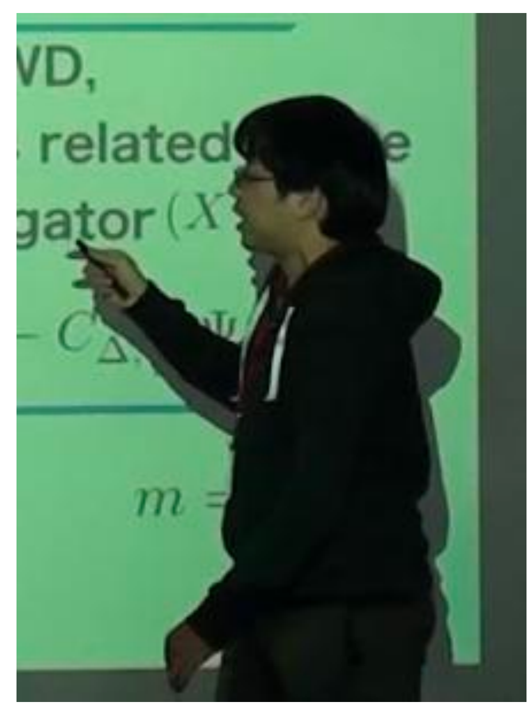
arXiv > hep-th > arXiv:2212.14702

High Energy Physics – Theory

[Submitted on 30 Dec 2022 (v1), last revised 25 Jan 2023 (this version, v2)]

Krylov Complexity in Free and Interacting Scalar Field Theories with Bounded Power Spectrum

Hugo A. Camargo, Viktor Jahnke, Keun-Young Kim, Mitsuhiro Nishida



Quantum Complexity and Quantum Chaos

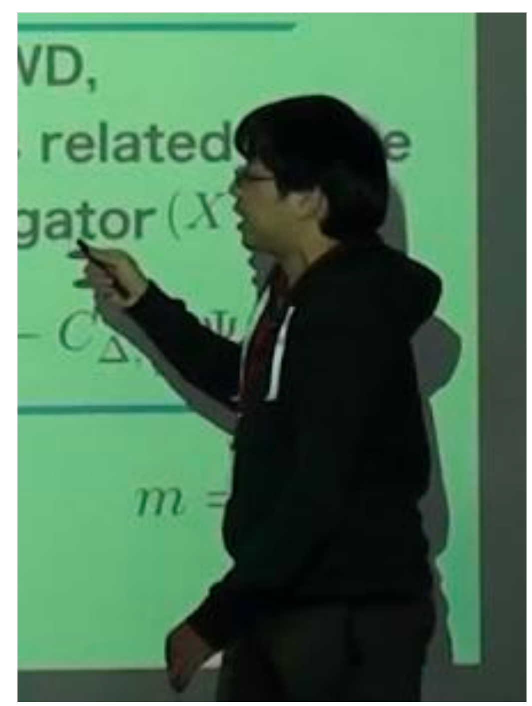
arXiv > hep-th > arXiv:2306.11632

High Energy Physics – Theory

[Submitted on 20 Jun 2023]

Spectral and Krylov Complexity in Billiard Systems

Hugo A. Camargo, Viktor Jahnke, Hyun-Sik Jeong, Keun-Young Kim, Mitsuhiro Nishida



Quantum Complexity and Quantum Chaos

arXiv > hep-th > arXiv:2212.14702

High Energy Physics – Theory

[Submitted on 30 Dec 2022 (v1), last revised 25 Jan 2023 (this version, v2)]

Krylov Complexity in Free and Interacting Scalar Field Theories with Bounded Power Spectrum

Hugo A. Camargo, Viktor Jahnke, Keun-Young Kim, Mitsuhiro Nishida

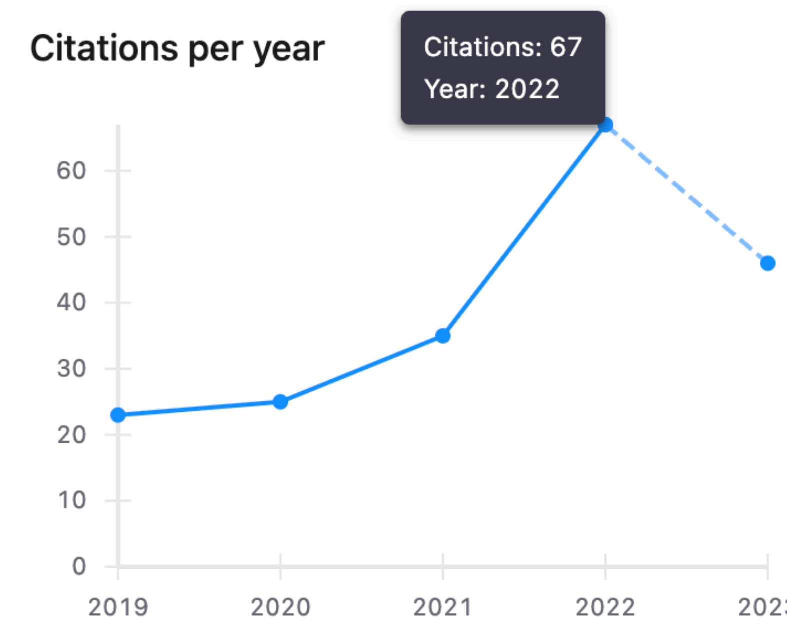
arXiv > hep-th > arXiv:2212.14429

High Energy Physics – Theory

[Submitted on 29 Dec 2022]

Krylov complexity in quantum field theory, and beyond

Alexander Avdoshkin, Anatoly Dymarsky, Michael Smolkin



A Universal Operator Growth Hypothesis

Daniel E. Parker (UC, Berkeley), Xiangyu Cao (UC, Berkeley), Alexander Avdoshkin (UC, Berkeley), Thomas Scaffidi (UC, Berkeley), Ehud Altman (UC, Berkeley) (Dec 20, 2018)

Published in: *Phys.Rev.X* 9 (2019) 4, 041017 • e-Print: 1812.08657 [cond-mat.stat-mech]

pdf DOI cite claim reference search 196 citations

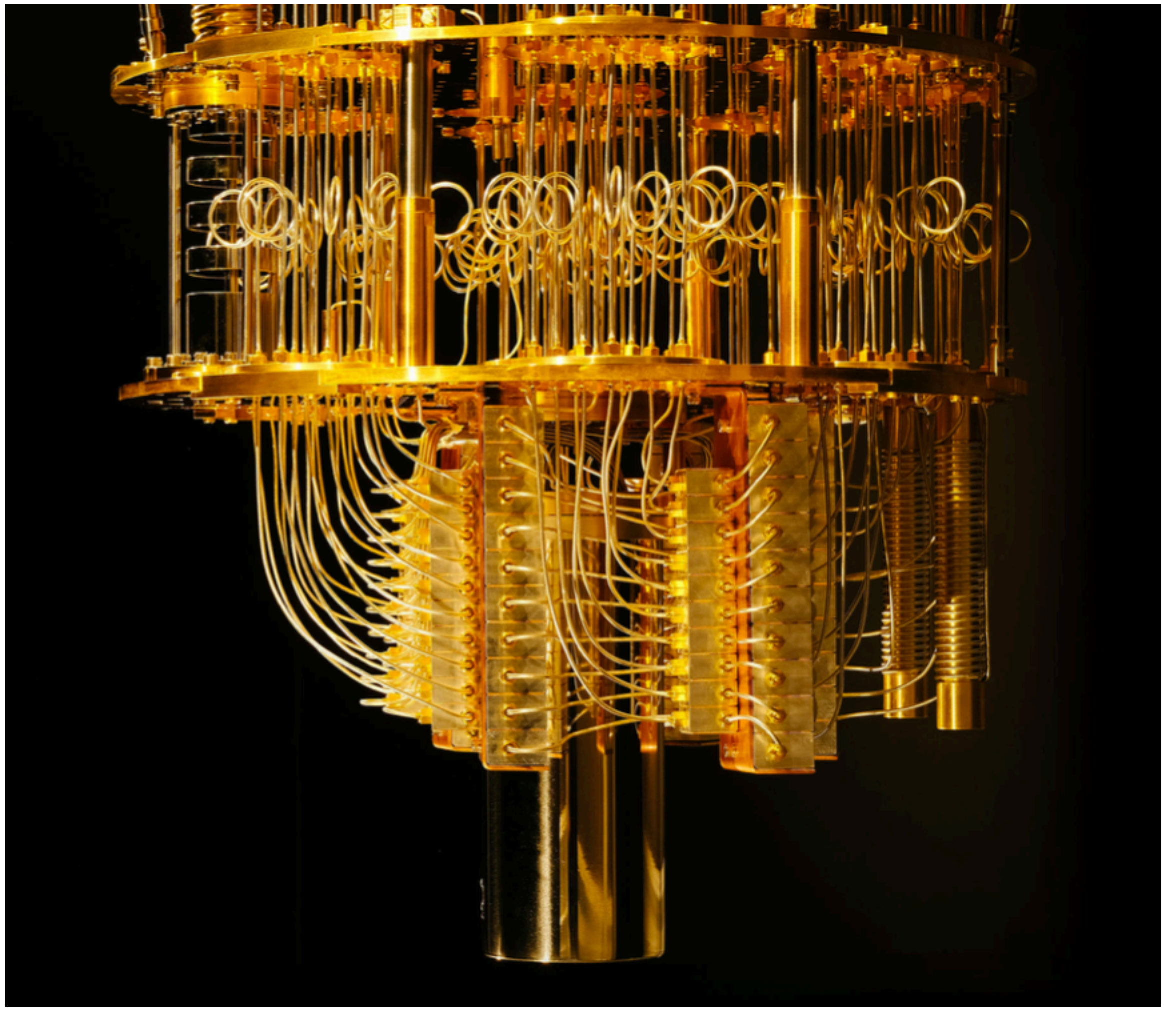
Quantum complexity
why?

Complexity

(Computational) complexity [Computer science] quantifying the difficulty of carrying out a task.

Quantum Computer

Input state →



→ output state

Complexity

(Computational) complexity [Computer science] quantifying the difficulty of carrying out a task.

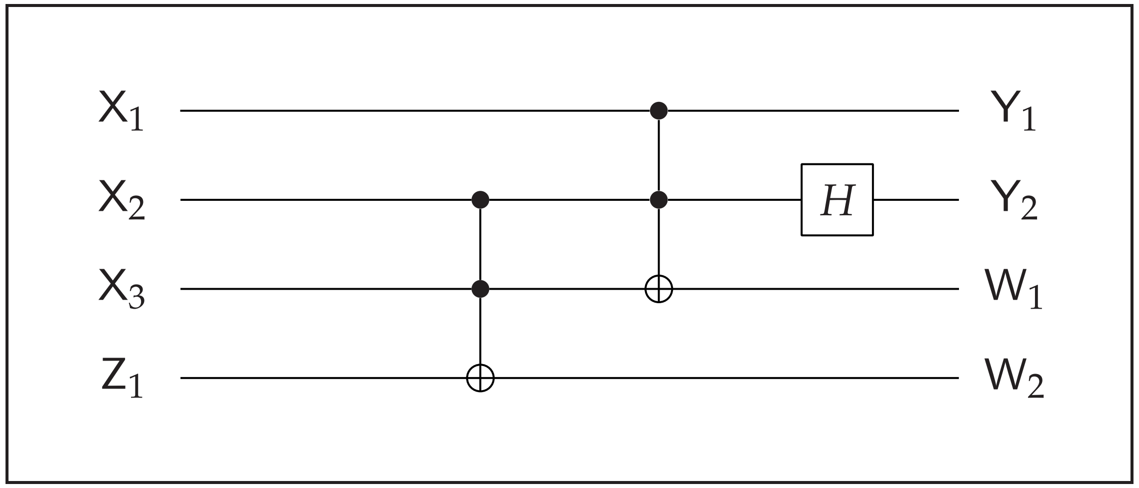
(Circuit) complexity

Quantum Computer ~ Quantum Circuit

Minimal number of gates for the transformation from the reference to target state

$$|\psi_T\rangle = U|\psi_R\rangle = g_n g_{n-1} \cdots g_2 g_1 |\psi_R\rangle$$

ex)



Ambiguity

Universal gate sets = {a,b,c,d,e,f}

$$G = dbe$$
$$G = ceab$$
$$G = abefa$$

complexity = 3

Complexity

“Distance” between two sates?

(inner-product) distance: $d_{AB} = \arccos |\langle B|A\rangle|$ (closest) $0 \sim \pi/2$ (farthest)

Are these close or far? $|0000000000\rangle \longrightarrow |0000000001\rangle$

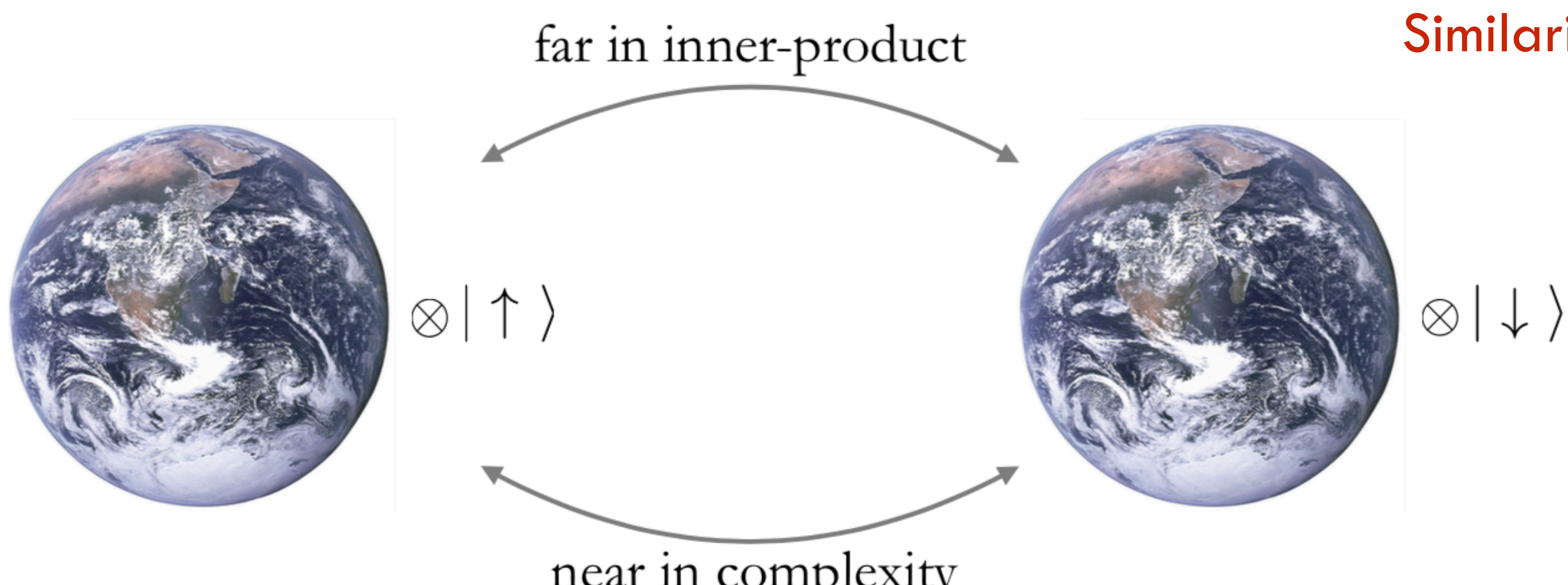
Far in the inner-product sense

However, in some sense they are close

“easy” or “difficult” transform

Need a new distance reflecting this sense: “Complexity distance?”

Similarity vs Distance?



1903.12621 Brown and Susskind

Complexity

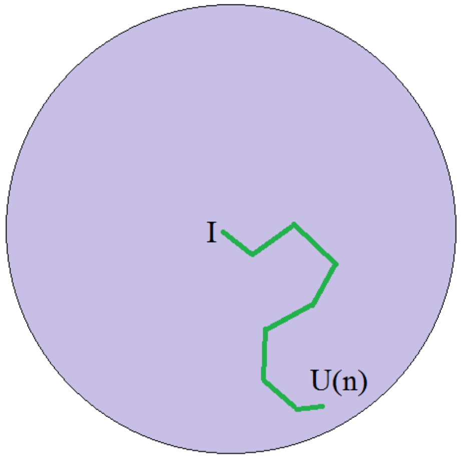
Complexity of quantum states New distance in Hilbert space Spread complexity

For given states $|\psi_T\rangle = U|\psi_R\rangle$ ~How hard (minimal number of gates) from the reference to target state

Complexity of operator (unitary transformation) New distance in Unitary group

For a given operator $U = g_n g_{n-1} \cdots g_2 g_1$ ~ minimum number of gates

$$\mathbb{I} \longrightarrow U$$



Krylov complexity

Relation between two

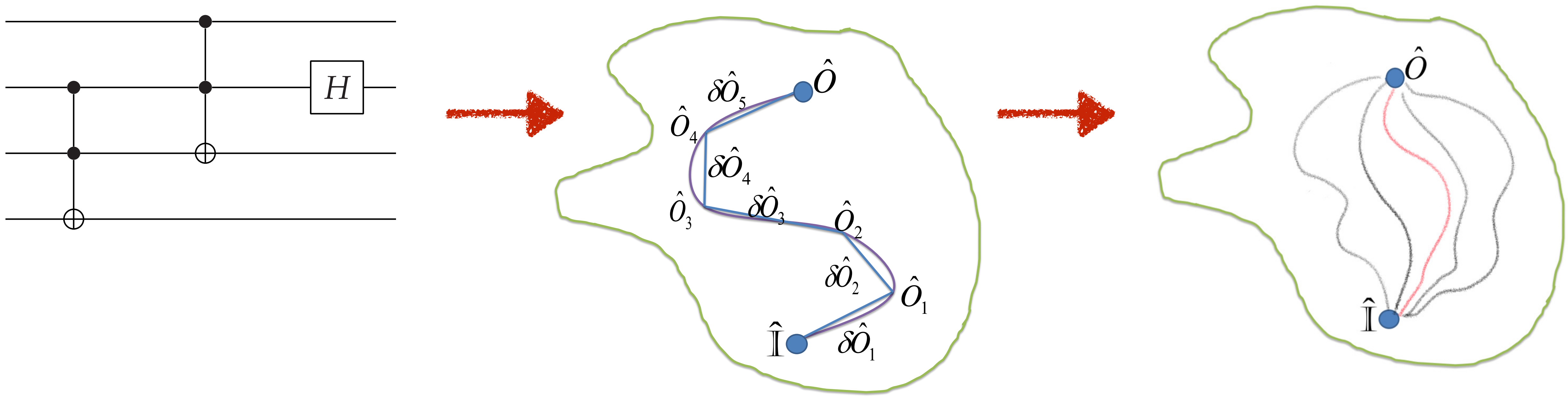
$$\mathcal{C}(|\psi_1\rangle, |\psi_2\rangle) = \min\left\{\mathcal{C}(U) \mid \forall\, \hat{U} \in \mathcal{O},\ \ |\psi_2\rangle = \hat{U}|\psi_1\rangle\right\}$$

Continuous version

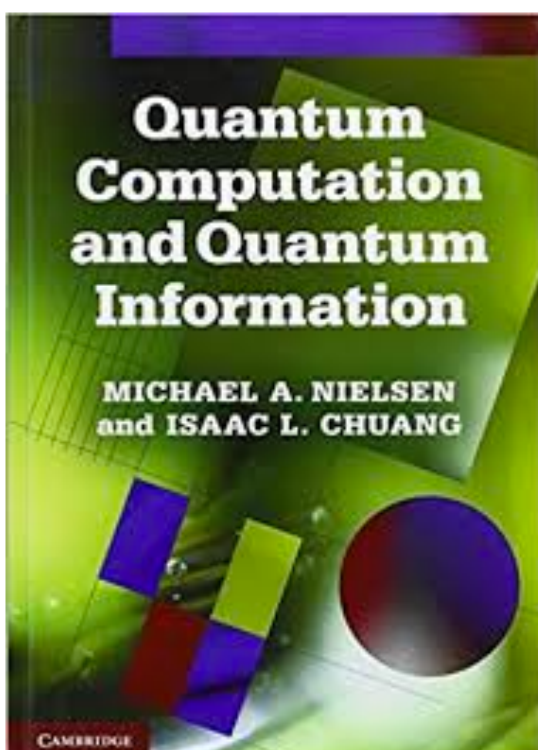
$U = g_n g_{n-1} \cdots g_2 g_1$

$\hat{O}_n = \delta\hat{O}_n^{(r)} \delta\hat{O}_{n-1}^{(r)} \cdots \delta\hat{O}_2^{(r)} \delta\hat{O}_1^{(r)}$

Complexity = geodesic length?



A geometric approach to quantum circuit lower bounds (2008)



Susskind and collaborators

- introduced Nielsen's idea to hep-th community in 2014
- have been developing the theory of complexity in QFT based on intuitions from circuit complexity

Complexity

Complexity of quantum states New distance in Hilbert space Spread complexity

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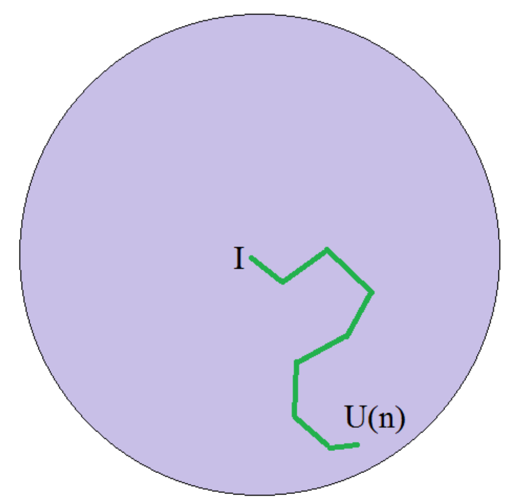

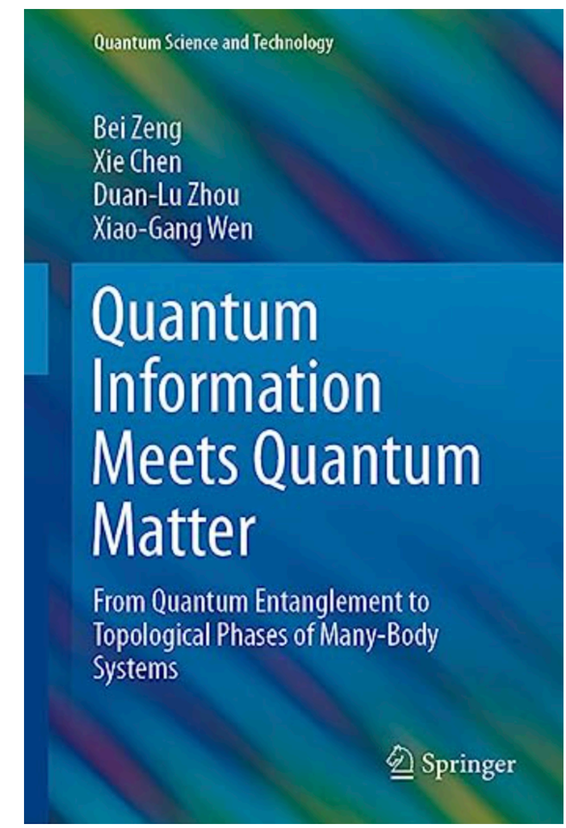

Relation between two

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Now, where is physics?
Entanglement entropy vs Complexity?
Is quantum theory explored enough?
Mainly about superposition and symmetry?
What about entanglement or something else?

QUANTUM ENTANGLEMENT IN CONDENSED MATTER SYSTEMS

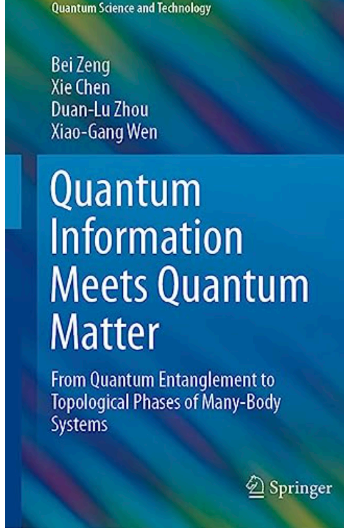
Nicolas Laflorencie

Laboratoire de Physique Théorique, Université de Toulouse, CNRS, UPS, France

QUANTUM ENTANGLEMENT IN CONDENSED MATTER SYSTEMS

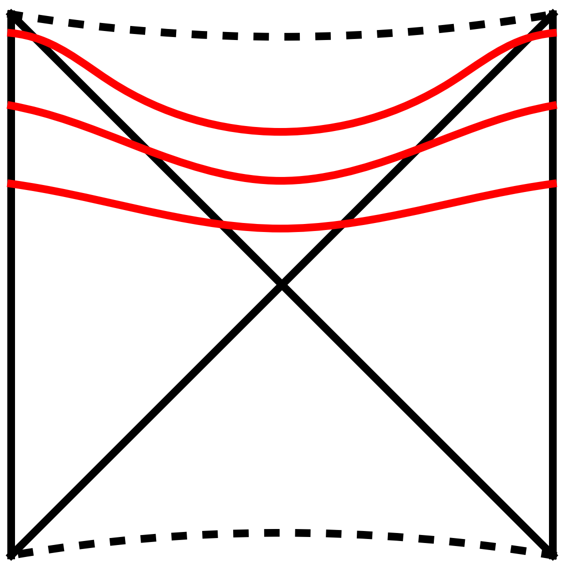
Nicolas Laflorencie
Laboratoire de Physique Théorique, Université de Toulouse, CNRS, UPS, France

This review focuses on the field of quantum entanglement applied to condensed matter physics systems with strong correlations, a domain which has rapidly grown over the last decade. By tracing out part of the degrees of freedom of correlated quantum systems, useful and non-trivial informations can be obtained through the study of the reduced density matrix, whose eigenvalue spectrum (the entanglement spectrum) and the associated Rényi entropies are now well recognized to contains key features. In particular, the celebrated area law for the entanglement entropy of ground-states will be discussed from the perspective of its subleading corrections which encode universal details of various quantum states of matter, *e.g.* symmetry breaking states or topological order. Going beyond entropies, the study of the low-lying part of the entanglement spectrum also allows to diagnose topological properties or give a direct access to the excitation spectrum of the edges, and may also raise significant questions about the underlying entanglement Hamiltonian. All these powerful tools can be further applied to shed some light on disordered quantum systems where impurity/disorder can conspire with quantum fluctuations to induce non-trivial effects. Disordered quantum spin systems, the Kondo effect, or the many-body localization problem, which have all been successfully (re)visited through the prism of quantum entanglement, will be discussed in details. Finally, the issue of experimental access to entanglement measurement will be addressed, together with its most recent developments.



Complexity

[Susskind: 1402.5674
Stanford and Susskind: 1406.2678]



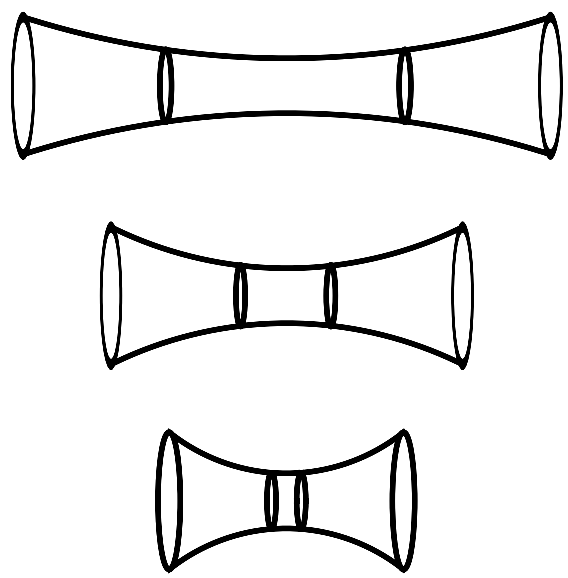
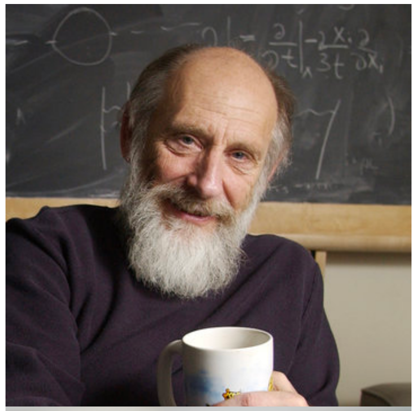
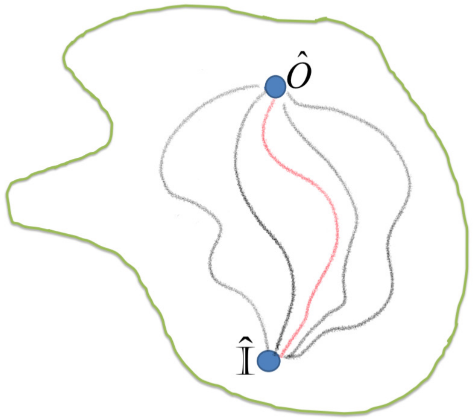


Fig. from [Koji, Norihiro, Sotaro: 1707.03840]



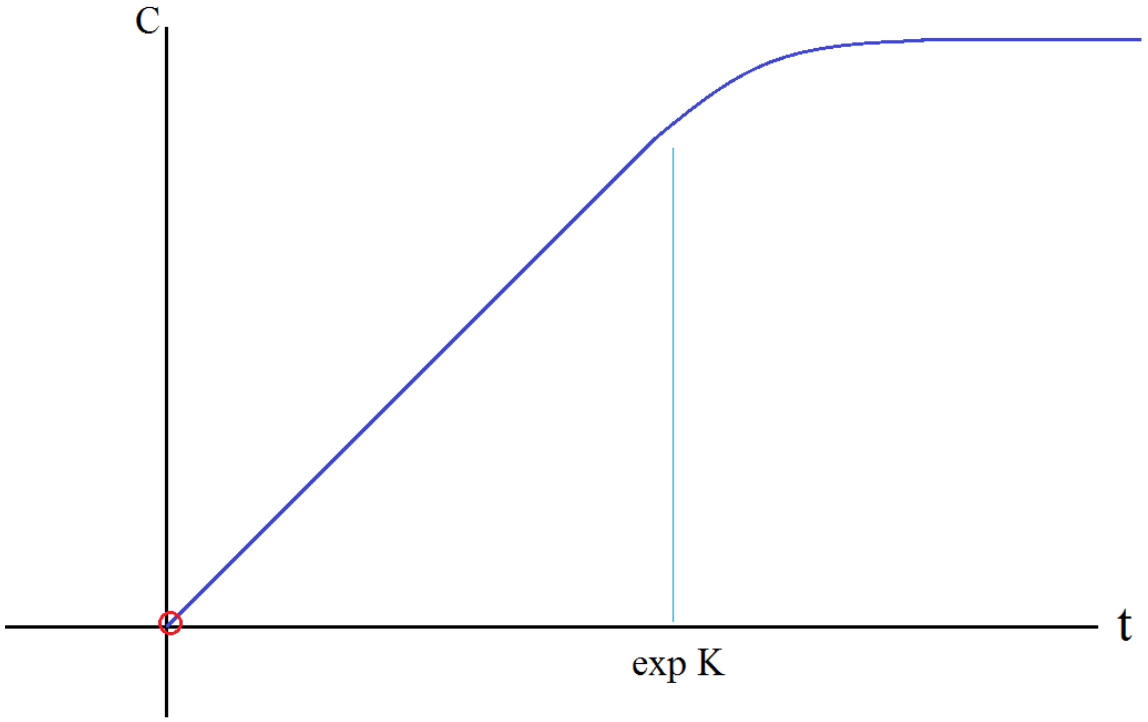
2014:
Entanglement is not enough!
Black hole interior?

2017
Complexity in Field theory?
Complexity = geodesic length?



Entanglement is not enough?

1. Einstein-Rosen bridge increases even after thermalization
2. The field theory meaning of this?
3. Physics inside black hole?



Quantum Chaos
Why?

Quantum Chaos
Why?

Classical Physics $\underset{\not\longrightarrow}{\xleftarrow{\hbar \to 0}}$ Quantum physics

Classical Chaos $\underset{\not\longrightarrow}{\xleftarrow{\hbar \to 0}}$ Quantum chaos

We do not need to stick to classical concept,
which may be sometimes obstruction

Quantum Chaos

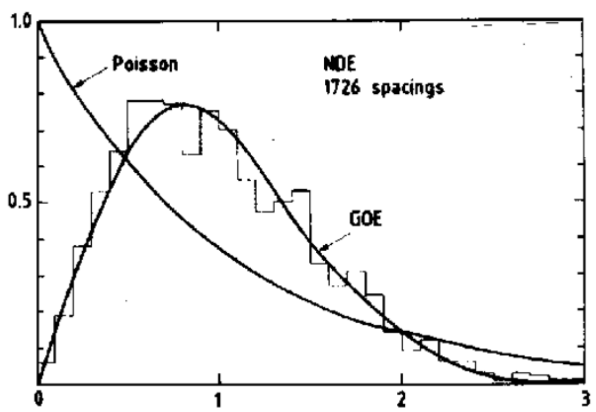
$$\left|\{q^i(t), p^j(0)\}_{PB} = \left|\frac{\partial q^i(t)}{\partial q^j(0)}\right| \sim e^{\lambda t}\right.$$

$$-\langle [q^i(t), p^j(0)]^2 \rangle_\beta,$$

$$-\langle [V(t), W(0)]^2 \rangle_\beta \quad \sim e^{\lambda t}$$

Out-of-time-order correlator (OTOC)

Level spacing statistics



Random Matrix Theory

Thermalization
(ETH, Quantum device)
Quantum black holes
Quantum gravity

FO What is quantum chaos?

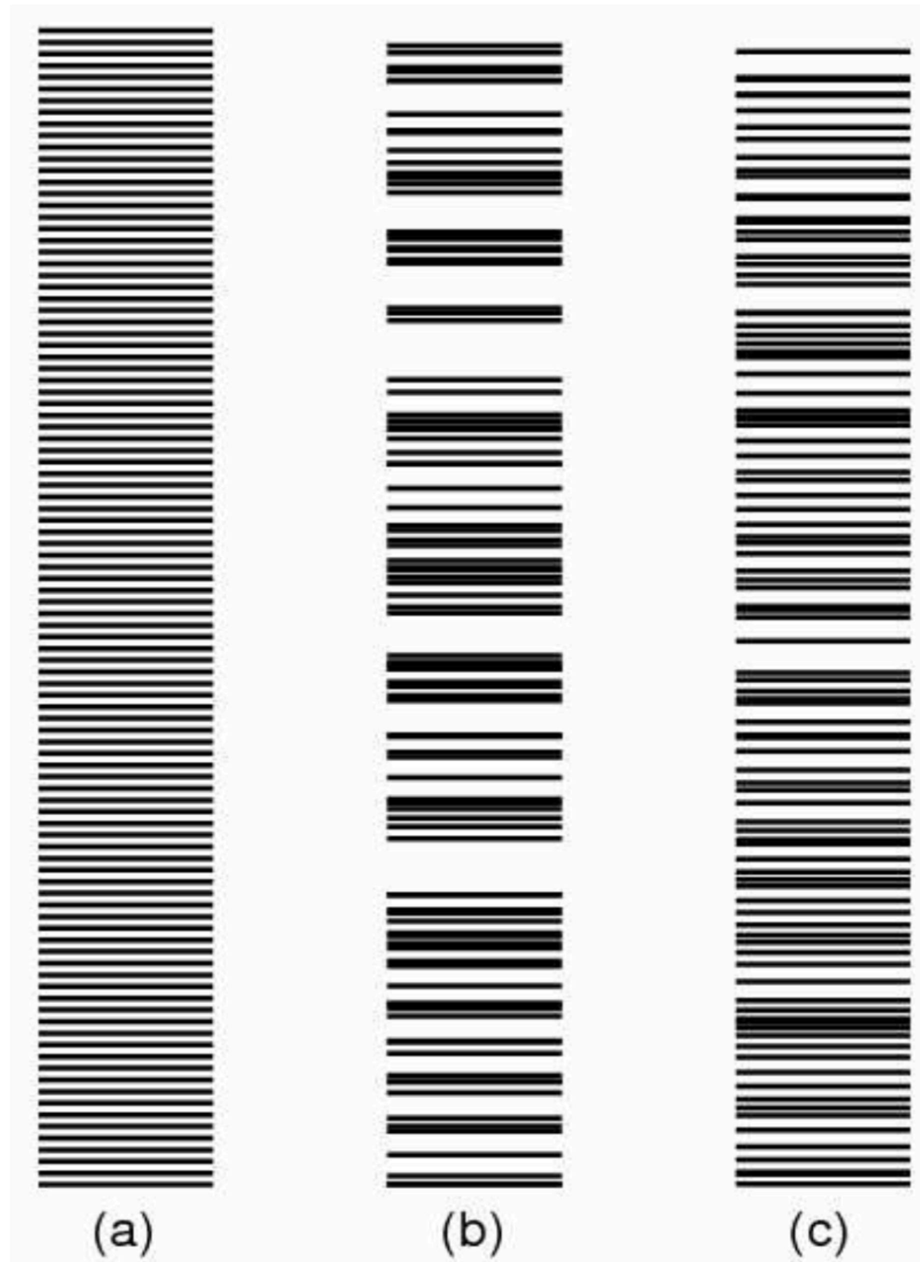
ChatGPT

Quantum chaos is a field of physics that studies the behavior of quantum systems that are classically chaotic. Classically chaotic systems are those that exhibit sensitive dependence on initial conditions, meaning small changes in the initial conditions lead to vastly different outcomes over time.

In quantum mechanics, chaotic behavior can manifest in the statistical properties of the system's energy spectrum or in the time evolution of its wave function. The study of quantum chaos is concerned with understanding how the behavior of the system changes as classical chaos is introduced or increased, and how this behavior can be observed and measured experimentally.

Quantum chaos has applications in various fields, including solid-state physics, quantum computing, and quantum information theory. It has also led to new insights into the fundamental nature of quantum mechanics and the connections between classical and quantum physics.

Krylov complexity?


(a)
(b)
(c)

Quantum chaos and complexity

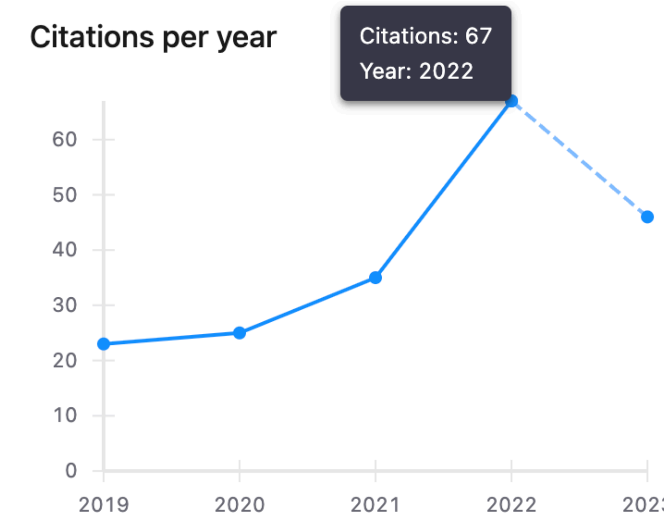
Midjourney

A Universal Operator Growth Hypothesis #1

Daniel E. Parker (UC, Berkeley), Xiangyu Cao (UC, Berkeley), Alexander Avdoshkin (UC, Berkeley), Thomas Scaffidi (UC, Berkeley), Ehud Altman (UC, Berkeley) (Dec 20, 2018)

Published in: *Phys.Rev.X* 9 (2019) 4, 041017 • e-Print: 1812.08657 [cond-mat.stat-mech]

pdf DOI cite claim reference search 196 citations



Comments on Krylov Complexity in Field Theory

Complexity: how much things are complex

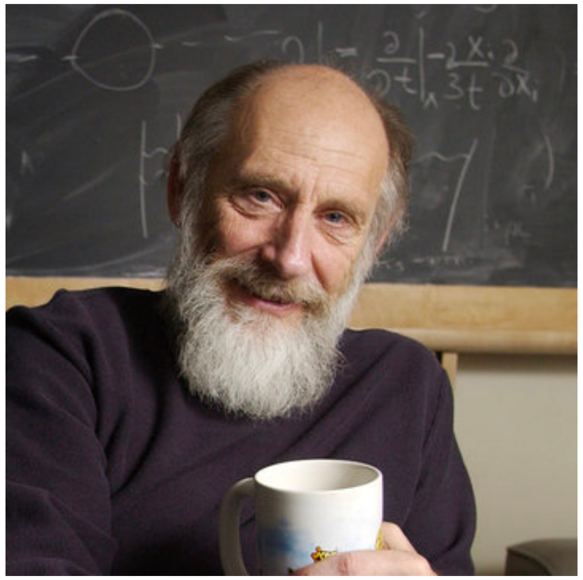
Chaos: how fast things get complex

~ fast increase of complexity

*Circuit complexity is not well-defined

"Krylov complexity" is a well-defined concept
proposed as a diagnose of quantum chaos (which is not-well defined)

Entanglement is not enough!
Black hole interior?



Krylov complexity in conformal field theory

Anatoly Dymarsky (Kentucky U. and Skoltech), Michael Smolkin (Hebrew U.) (Apr 19, 2021)

Published in: *Phys.Rev.D* 104 (2021) 8, L081702 • e-Print: 2104.09514 [hep-th]

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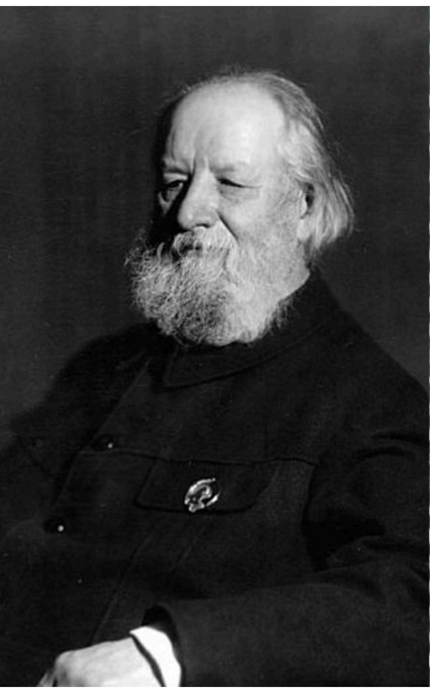
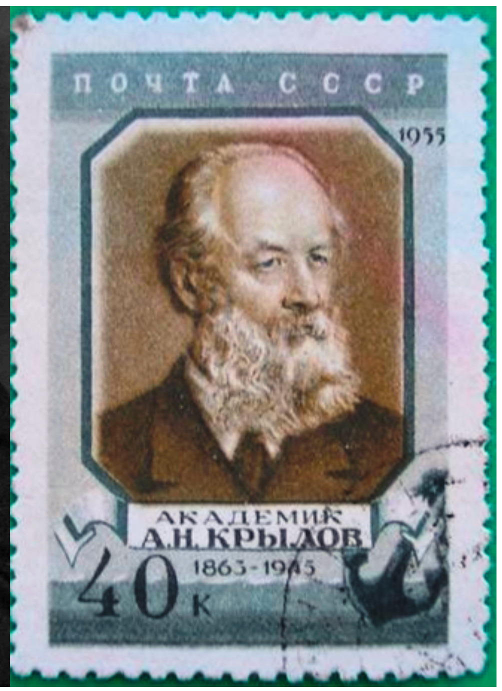
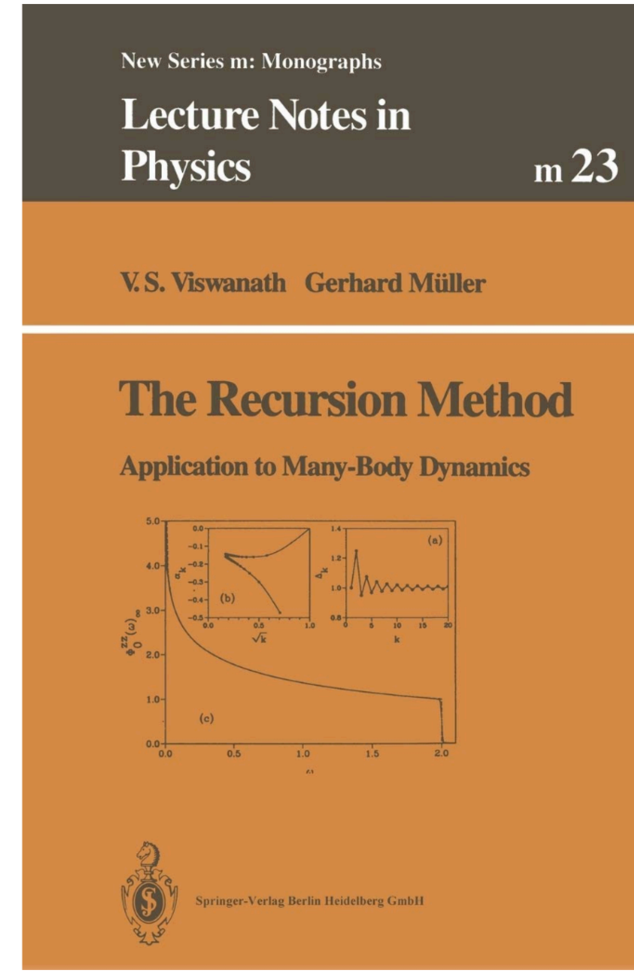
Contents

Aleksey Nikolaevich Krylov (1863 –1945)
Russian naval engineer, applied mathematician
and memoirist.





- Short Review on Krylov Complexity
  - Operator growth
  - Krylov space
  - Lanczos coefficient
  - Krylov complexity
- Success in lattice systems
- Towards field theory
  - Too good to be true
  - How to extract info from the power spectrum
    (IR/UV cutoff effect)



1994

Cornelius (Cornel) Lanczos (1893-1974):
a Hungarian-American and later Hungarian-Irish
mathematician and physicist.

Short Review on Krylov Complexity

Operator growth

The time evolution of an operator O by a time independent Hamiltonian *H*

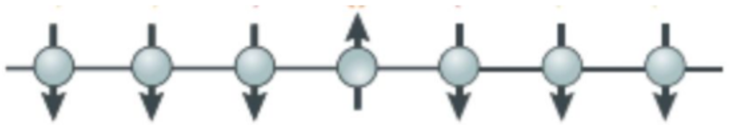
$$\partial_t \mathcal{O}(t) = i\,[H, \mathcal{O}(t)]$$

$$\mathcal{O}(t) = e^{itH}\,\mathcal{O}(0)\,e^{-itH}$$

Baker-Campbell-Hausdorff (BCH) formula $e^X Y e^{-X} = \sum_{n=0}^{\infty} \frac{\mathcal{L}_X^n Y}{n!}$

$$\mathcal{O}(t) = \mathcal{O}_0 + it[H,\mathcal{O}] + \frac{(it)^2}{2!}[H,[H,\mathcal{O}]] + \frac{(it)^3}{3!}[H,[H,[H,\mathcal{O}]]] + \cdots \quad .$$

ex) 1D spin chain



$$H = -\sum (Z_i Z_{i+1} + gX_i + hZ_i)$$

$$Z_1(t) = Z_1 + it[H, Z_1] - \frac{t^2}{2!}[H,[H,Z_1]] - \frac{it^3}{3!}[H,[H,[H,Z_1]]] + \ldots$$

$$\begin{aligned}
[H, Z_1] &\sim Y_1 \\
[H,[H,Z_1]] &\sim Y_1 + X_1 Z_2 \\
[H,[H,[H,Z_1]]] &\sim Y_1 + Y_2 X_1 + Y_1 Z_2 \\
[H,[H,[H,[H,Z_1]]]] &\sim X_1 + Y_1 + Z_1 + X_1 X_2 + Y_1 Y_2 + Z_1 Z_2 + X_1 Z_2 + \\
&\quad + Z_3 Y_1 + Y_1 Z_2 Y_2 + Z_1 X_2 X_1 + X_2 Z_3 X_1
\end{aligned}$$

Operator growth

1812.08657: Parker et al.

The time evolution of an operator O by a time independent Hamiltonian H

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$$\mathcal{O}(t) = \boxed{\mathcal{O}_0} + it\boxed{[H,\mathcal{O}]} + \frac{(it)^2}{2!}\boxed{[H,[H,\mathcal{O}]]} + \frac{(it)^3}{3!}\boxed{[H,[H,[H,\mathcal{O}]]]} + \cdots \quad .$$

$$\mathcal{O}(t) = \sum_{n=0}^{\infty} \frac{(i\,t)^n}{n!}\boxed{\tilde{\mathcal{O}}_n} \qquad \tilde{\mathcal{O}}_n = \mathcal{L}^n \mathcal{O}(0) \qquad \mathcal{L} := [H, \cdot\,]$$

ex) 1D spin chain

$$H = -\sum (Z_i Z_{i+1} + g X_i + h Z_i)$$

$$Z_1(t) = Z_1 + it[H, Z_1] - \frac{t^2}{2!}[H,[H,Z_1]] - \frac{it^3}{3!}[H,[H,[H,Z_1]]] + \ldots$$

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[H, Z_1] &\sim Y_1 \\
[H,[H,Z_1]] &\sim Y_1 + X_1 Z_2 \\
[H,[H,[H,Z_1]]] &\sim Y_1 + Y_2 X_1 + Y_1 Z_2 \\
[H,[H,[H,[H,Z_1]]]] &\sim X_1 + Y_1 + Z_1 + X_1 X_2 + Y_1 Y_2 + Z_1 Z_2 + X_1 Z_2 + \\
&\quad + Z_3 Y_1 + Y_1 Z_2 Y_2 + Z_1 X_2 X_1 + X_2 Z_3 X_1
\end{aligned}$$

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1812.08657: Parker et al.

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$$\mathcal{O}(t) = \sum_{n=0}^{\infty} \frac{(i\,t)^n}{n!}\tilde{\mathcal{O}}_n \qquad \tilde{\mathcal{O}}_n = \mathcal{L}^n \mathcal{O}(0) \qquad \mathcal{L} := [H, \cdot\,]$$

- The set of operators $\{\tilde{\mathcal{O}}_n\}$ defines a basis of the so-called *Krylov space* associated to the operator $\mathcal{O}$
- Regard the operator as a state $\mathcal{O} \to |\mathcal{O})$ in the Hilbert space of operators

Inner product: Wightman inner product

$$(A|B) := \langle e^{\beta H/2} A^\dagger e^{-\beta H/2} B\rangle_\beta = \frac{1}{\mathcal{Z}_\beta}\mathrm{Tr}(e^{-\beta H/2}A^\dagger e^{-\beta H/2}B) \qquad \mathcal{Z}_\beta := \mathrm{Tr}\left(e^{-\beta H}\right)$$

Krylov basis

$(\mathcal{O}_m|\mathcal{O}_n) = \delta_{mn}$ (Lanczos algorithm: Gram–Schmidt procedure)

$$|\mathcal{O}_0) := |\tilde{\mathcal{O}}_0) := |\mathcal{O}(0))$$

$\{b_n\}$: Lanczos coefficients

$$|\mathcal{O}_1) := b_1^{-1}\mathcal{L}|\tilde{\mathcal{O}}_0) \qquad b_1 := (\tilde{\mathcal{O}}_0\mathcal{L}|\mathcal{L}\tilde{\mathcal{O}}_0)^{1/2}$$

$$|\mathcal{O}_n) := b_n^{-1}|A_n) \qquad b_n := (A_n|A_n)^{1/2}$$

$$|A_n) := \mathcal{L}|\mathcal{O}_{n-1}) - b_{n-1}|\mathcal{O}_{n-2})$$

Operator growth

1812.08657: Parker et al.

The time evolution of an operator O by a time independent Hamiltonian *H*

$$\partial_t \mathcal{O}(t) = i\,[H, \mathcal{O}(t)]$$

$$\mathcal{O}(t) = e^{itH}\,\mathcal{O}(0)\,e^{-itH}$$

$$\partial_t |\mathcal{O}(t)) = i\mathcal{L}|\mathcal{O}(t)) \qquad |\mathcal{O}(t)) = \sum_{n=0}^{\infty} i^n \varphi_n(t)|\mathcal{O}_n) \qquad \sum_{n=0}^{\infty} |\varphi_n(t)|^2 = 1$$

$$\mathcal{O}(t) = e^{i\mathcal{L}t}\,\mathcal{O}(0)$$

"probability amplitudes"

$$\mathcal{O}(t) = \mathcal{O}_0 + it[H,\mathcal{O}] + \frac{(it)^2}{2!}[H,[H,\mathcal{O}]] + \frac{(it)^3}{3!}[H,[H,[H,\mathcal{O}]]] + \cdots \quad .$$

$$\mathcal{O}(t) = \sum_{n=0}^{\infty} \frac{(i\,t)^n}{n!}\tilde{\mathcal{O}}_n \qquad \tilde{\mathcal{O}}_n = \mathcal{L}^n \mathcal{O}(0) \qquad \mathcal{L} := [H, \cdot\,]$$

- The set of operators $\{\tilde{\mathcal{O}}_n\}$ defines a basis of the so-called *Krylov space* associated to the operator $\mathcal{O}$
- Regard the operator as a state $\mathcal{O} \to |\mathcal{O})$ in the Hilbert space of operators

Inner product: Wightman inner product

$$(A|B) := \langle e^{\beta H/2} A^\dagger e^{-\beta H/2} B\rangle_\beta = \frac{1}{\mathcal{Z}_\beta}\mathrm{Tr}(e^{-\beta H/2} A^\dagger e^{-\beta H/2} B) \qquad \mathcal{Z}_\beta := \mathrm{Tr}\left(e^{-\beta H}\right)$$

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$$|\mathcal{O}_n) := b_n^{-1}|A_n) \qquad b_n := (A_n|A_n)^{1/2}$$

$$|A_n) := \mathcal{L}|\mathcal{O}_{n-1}) - b_{n-1}|\mathcal{O}_{n-2})$$

Discrete "Schrodinger equation"

$$\partial_t \mathcal{O}(t) = i\,[H, \mathcal{O}(t)]$$

$$\partial_t |\mathcal{O}(t)) = i\mathcal{L}|\mathcal{O}(t))$$

"probability amplitudes" $\sum_{n=0}^{\infty} |\varphi_n(t)|^2 = 1$

$$|\mathcal{O}(t)) = \sum_{n=0}^{\infty} i^n \varphi_n(t) |\mathcal{O}_n) \qquad \varphi_n(t) := i^{-n}(\mathcal{O}_n|\mathcal{O}(t))$$

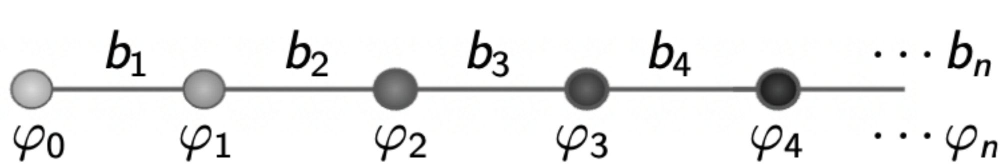
$$L_{nm} := (\mathcal{O}_n|\mathcal{L}|\mathcal{O}_m) = \begin{pmatrix} 0 & b_1 & 0 & 0 & \cdots \\ b_1 & 0 & b_2 & 0 & \cdots \\ 0 & b_2 & 0 & b_3 & \cdots \\ 0 & 0 & b_3 & 0 & \ddots \\ \vdots & \vdots & \vdots & \ddots & \ddots \end{pmatrix} = b_n\delta_{m,n-1} + b_{n+1}\delta_{m,n+1}$$

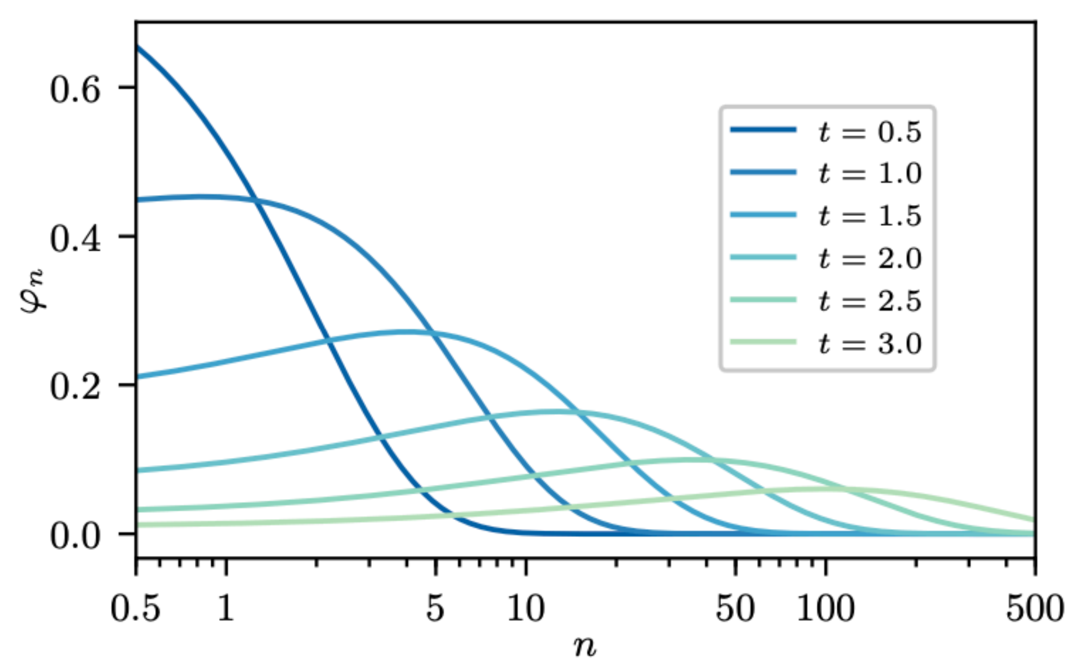
$$\frac{\mathrm{d}\,\varphi_n(t)}{\mathrm{d}t} = b_n\varphi_{n-1}(t) - b_{n+1}\varphi_{n+1}(t) \qquad \varphi_n(0) = \delta_{n,0} \qquad \varphi_{-1}(t) \equiv 0 \equiv b_0$$

$$\dot\varphi_0(t) = b_0 \overbrace{\varphi_{-1}(t)}^{=0} - b_1\varphi_1(t)$$
$$\dot\varphi_1(t) = b_1\varphi_0(t) - b_2\varphi_2(t)$$
$$\vdots$$
$$\dot\varphi_n(t) = b_n\varphi_{n-1}(t) - b_{n+1}\varphi_{n+1}(t)$$

a quantum-mechanical particle on a 1- dimensional chain.

b_n = hopping amplitudes





Krylov complexity

average position over the chain

$$K_{\mathcal{O}}(t) := (\mathcal{O}(t)|n|\mathcal{O}(t)) = \sum_{n=0}^{\infty} n|\varphi_n(t)|^2$$

K-complexity from the auto-correlation function

1812.08657: Parker et al.

Auto-correlation function

$C(t) = \Pi^W(t) = \varphi_0(t)$

$$C(t) := (\mathcal{O}(t)|\mathcal{O}(0)) = \varphi_0(t)$$
$$= \langle e^{i\,(t-i\,\beta/2)H}\mathcal{O}^\dagger(0)e^{-i\,(t-i\,\beta/2)H}\mathcal{O}(0)\rangle_\beta$$
$$= \langle \mathcal{O}^\dagger(t-i\beta/2)\mathcal{O}(0)\rangle_\beta =: \Pi^W(t)\ .$$

$$\langle\cdots\rangle_\beta = \mathrm{Tr}(e^{-\beta H}\cdots)/\mathrm{Tr}(e^{-\beta H})$$

Power spectrum

$f^W(\omega)$

$$\Pi^W(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\, e^{-i\omega t} f^W(\omega)$$

Moments

μ_{2n}

$$\Pi^W(t) := \sum_{n=0}^{\infty}\mu_{2n}\frac{(it)^{2n}}{(2n)!} \qquad \mu_{2n} := \frac{1}{i^{2n}}\frac{\mathrm{d}^{2n}\Pi^W(t)}{\mathrm{d}t^{2n}}\Big|_{t=0}$$

$$\mu_{2n} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}\omega\,\omega^{2n} f^W(\omega)$$

Lanczos coefficients from moments

$$b_1^{2n}\cdots b_n^2 = \det\left(\mu_{i+j}\right)_{0\le i,j\le n}$$

$$\mu_2 = b_1^2\,, \quad \mu_4 = b_1^4 + b_1^2 b_2^2\,, \quad \cdots\,.$$

Hankel matrix constructed from the moments.

$$H_n = \begin{bmatrix} a_1 & a_2 & \dots & a_n \\ a_2 & a_3 & \dots & a_{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ a_n & a_{n+1} & \dots & a_{2n-1} \end{bmatrix}$$

$$b_n = \sqrt{M_{2n}^{(n)}}\,, \qquad M_{2l}^{(j)} = \frac{M_{2l}^{(j-1)}}{b_{j-1}^2} - \frac{M_{2l-2}^{(j-2)}}{b_{j-2}^2} \quad \text{with} \quad l = j,\dots,n\,,$$
$$M_{2l}^{(0)} = \mu_{2l} \quad , \quad b_{-1}\equiv b_0 := 1 \quad , \quad M_{2l}^{(-1)} = 0\,.$$

Computation method

Lanczos coefficients

$$\mu_{2n} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}\omega\, \omega^{2n} f^W(\omega)$$

$$\begin{tikzcd}
{\color{red} f^W(\omega)} \arrow[rr] \arrow[d, "{\Pi^W(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}\omega\, e^{-i\omega t} f^W(\omega)}"'] & & \mu_{2n} \arrow[d, "{b_1^{2n}\cdots b_n^2 = \det\left(\mu_{i+j}\right)_{0\le i,j\le n}}"] \\
C(t) = \Pi^W(t) = \varphi_0(t) & & b_n
\end{tikzcd}$$

K-complexity

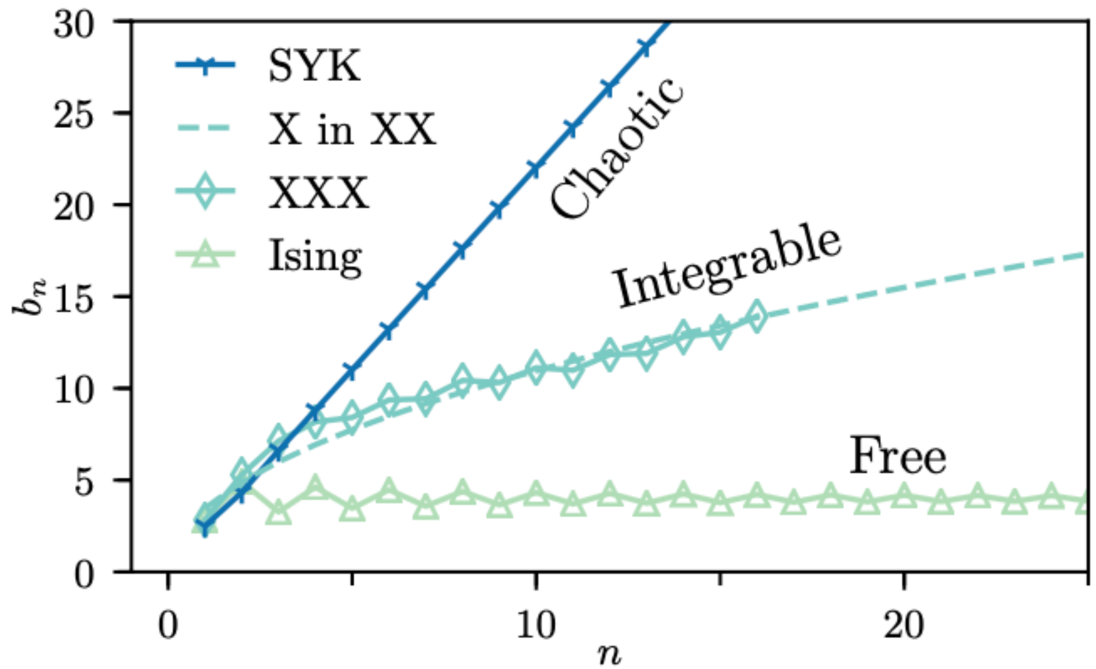
$$\dot{\varphi}_0(t) = b_0 \overbrace{\varphi_{-1}(t)}^{=0} - b_1\varphi_1(t)$$
$$\dot{\varphi}_1(t) = b_1\varphi_0(t) - b_2\varphi_2(t)$$
$$\vdots$$
$$\dot{\varphi}_n(t) = b_n\varphi_{n-1}(t) - b_{n+1}\varphi_{n+1}(t)$$

$$K_{\mathcal{O}}(t) = \sum_{n=1}^{n_{\max}} n\,|\varphi_n(t)|^2\,, \quad n_{\max} = 200.$$

Success in lattice systems

Universal operator growth hypothesis

$$b_n \sim n^\delta \Longleftrightarrow f^W(\omega) \sim \exp(-|\omega/\omega_0|^{1/\delta})$$
$$\delta \leq 1$$



D. S. Lubinsky, "A survey of general orthogonal polynomials for weights on finite and infinite intervals," Acta Applicandae Mathematica **10**, 237–296 (1987).
A. Magnus, "The recursion method and its applications: Proceedings of the conference, imperial college, london, england september 13–14, 1984," (Springer Science & Business Media, 2012) Chap. 2, pp. 22–45.

Signatures of chaos in time series generated by many-spin systems at high temperatures

Tarek A. Elsayed, Benjamin Hess, and Boris V. Fine
Phys. Rev. E **90**, 022910 – Published 20 August 2014

$f^W(\omega) \sim e^{-\frac{\omega}{\omega_0}}$ Is a signature of classical chaos

Universal operator growth hypothesis

In a chaotic quantum system

Lanczos coefficients $\{b_n\}$ grow as fast as possible

$$b_n \sim \alpha n$$

the slowest possible decay of the power spectrum

$$f^W(\omega) \sim e^{-\frac{\pi|\omega|}{2\alpha}}$$

Krylov complexity grows exponentially

$$K_{\mathcal{O}}(t) \sim e^{2\alpha t}$$

$$f^W(\omega) \sim e^{-\frac{\pi|\omega|}{2\alpha}} \quad \Longleftrightarrow \quad b_n \sim \alpha n \quad \Longleftrightarrow \quad K_{\mathcal{O}}(t) \sim e^{2\alpha t}$$

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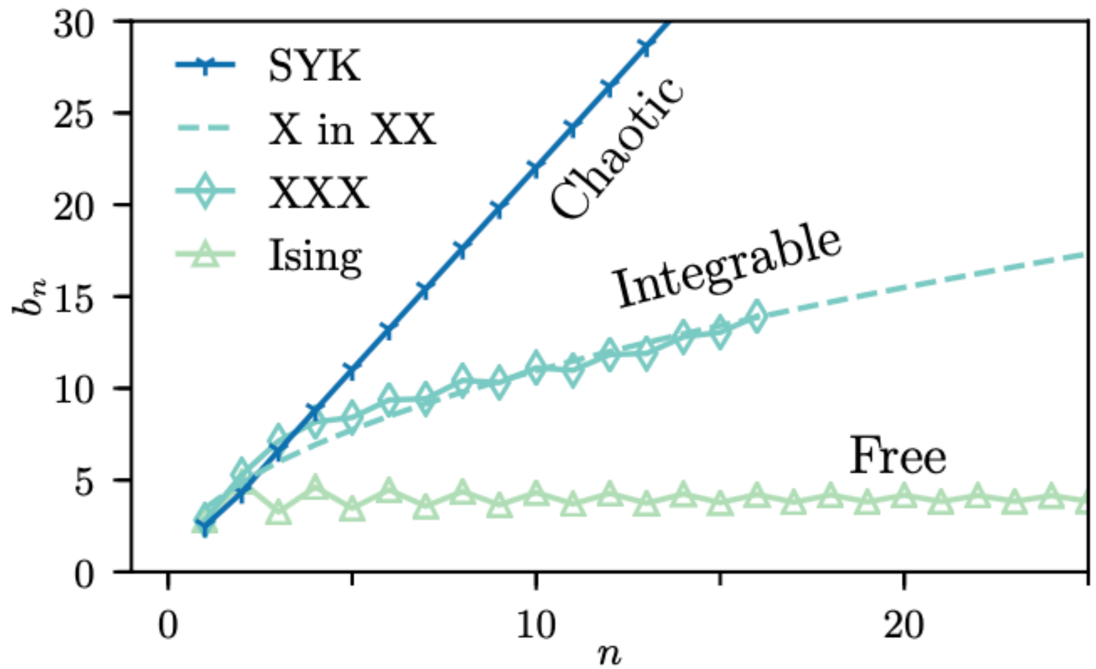

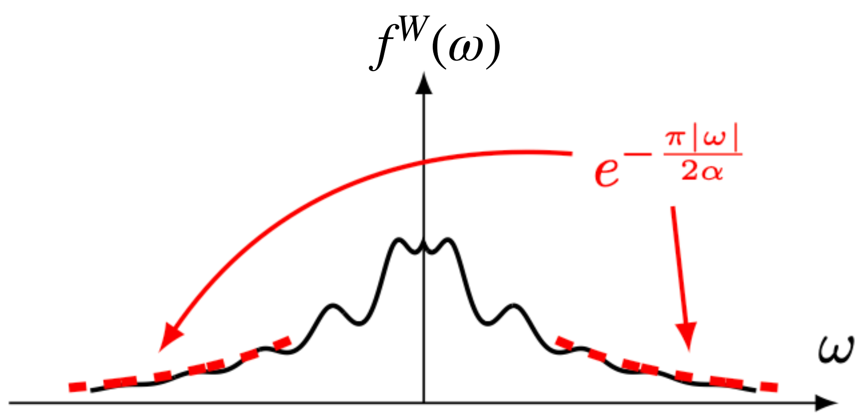

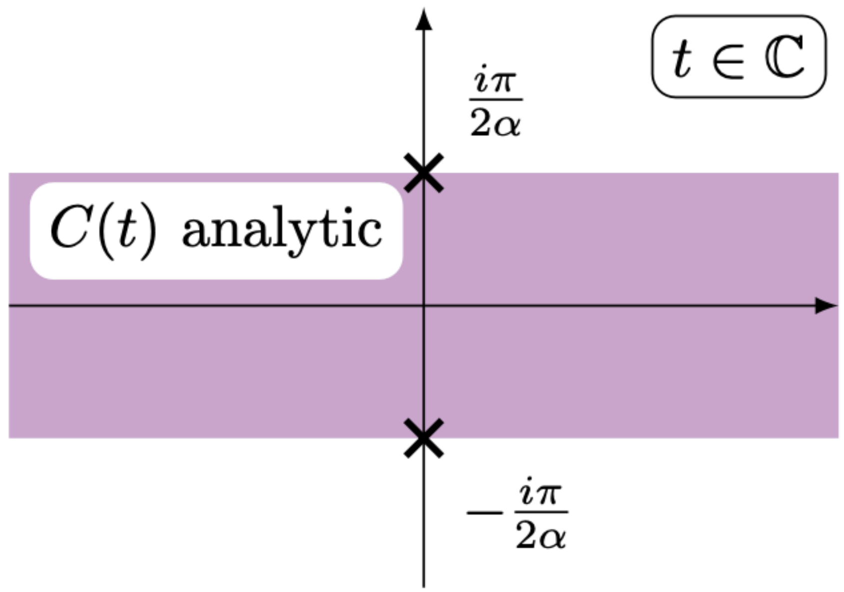

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Towards Field theory

Free massive scalar in d-dimensions

2212.14702: Camargo, Jahnke, KYK, Nishida

$$\mathcal{L}_E^{\text{free}} = \frac{1}{2}(\partial\phi)^2 + \frac{1}{2}m^2\phi^2$$

Wightman 2-point function

$$\Pi^W(t,\mathbf{x}) := \langle \phi(t - i\beta/2, \mathbf{x})\phi(0,\mathbf{0})\rangle_\beta \ ,$$

$$\Pi^W(\omega,\mathbf{k}) := \int \mathrm{d}t \int \mathrm{d}^{d-1}\mathbf{x}\, e^{i\omega t - i\mathbf{k}\cdot\mathbf{x}}\, \Pi^W(t,\mathbf{x})$$

$$\Pi^W(\omega,\mathbf{k}) = \frac{1}{\sinh[\beta\omega/2]}\rho(\omega,\mathbf{k}).$$

$$\rho(\omega,\mathbf{k}) = \frac{N}{\epsilon_k}[\delta(\omega-\epsilon_k) - \delta(\omega+\epsilon_k)],$$

$$\epsilon_k := \sqrt{|\mathbf{k}|^2 + m^2}.$$

Power spectrum

$$C(t) = \Pi^W(t,\mathbf{0})$$

$$f^W(\omega) := \int \mathrm{d}t\, C(t) e^{i\omega t} = \int \mathrm{d}t\, \Pi^W(t,\mathbf{0}) e^{i\omega t} = \int \frac{\mathrm{d}^{d-1}\mathbf{k}}{(2\pi)^{d-1}} \Pi^W(\omega,\mathbf{k}).$$

$$f^W(\omega) = N(m,\beta,d)\, \frac{(\omega^2 - m^2)^{(d-3)/2}}{|\sinh\left(\frac{\beta\omega}{2}\right)|}\Theta(|\omega| - m)$$

$$\int \frac{\mathrm{d}\omega}{2\pi} f^W(\omega) = 1$$

m=0, d=4

$$f^W(\omega) = \frac{\beta^2\omega}{\pi\sinh(\frac{\beta\omega}{2})}$$

$$f^W(\omega) \sim e^{-\frac{\beta|\omega|}{2}} \sim e^{-\frac{\pi|\omega|}{2\alpha}}$$

$$f^W(\omega) \longrightarrow \mu_{2n} \longrightarrow b_n$$

$$\mu_{2n} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}\omega\, \omega^{2n} f^W(\omega) \qquad b_1^{2n}\cdots b_n^2 = \det\left(\mu_{i+j}\right)_{0\le i,j\le n}$$

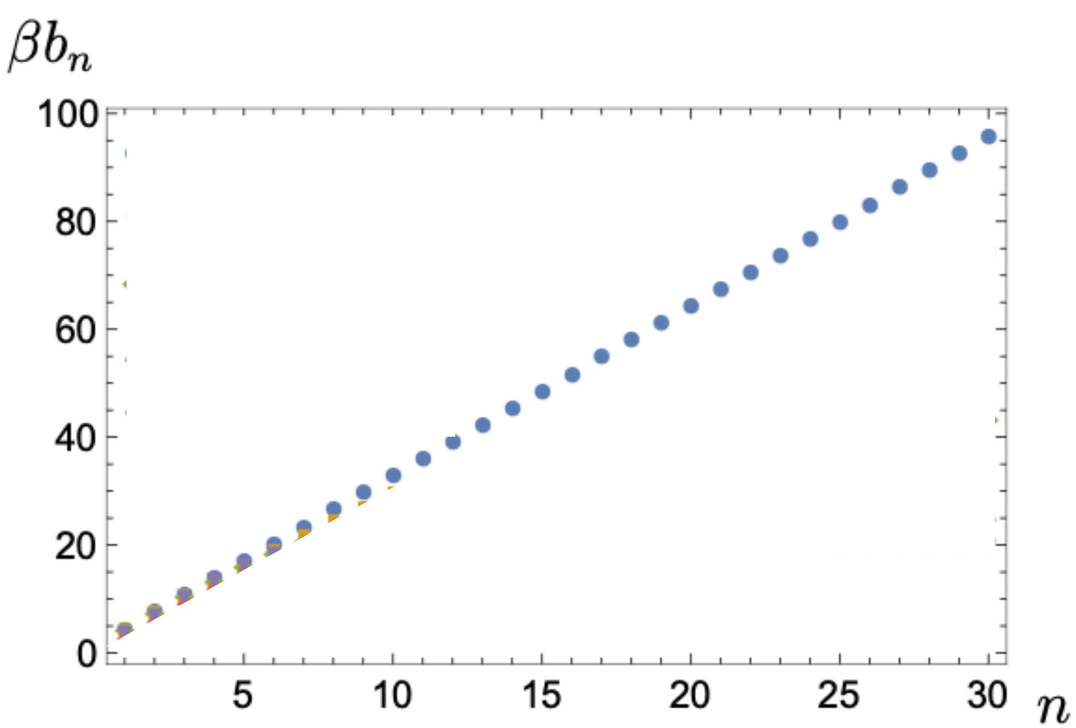
2212.14429: Avdoshikin, Dymarsky, Smolkin
2212.14702: Camargo, Jahnke, KYK, Nishida

$$\mathcal{L}_E^{\text{free}} = \frac{1}{2}(\partial\phi)^2 + \frac{1}{2}m^2\phi^2$$

Power spectrum (m=0, d=4)

$$f^W(\omega) = \frac{\beta^2\omega}{\pi \sinh(\frac{\beta\omega}{2})}$$

$$f^W(\omega) \sim e^{-\frac{\beta|\omega|}{2}} \sim e^{-\frac{\pi|\omega|}{2\alpha}} \qquad \left(\alpha = \frac{\pi}{\beta}\right)$$



Free theory is chaotic?

~~In a chaotic quantum system~~ In free QFT

Lanczos coefficients $\{b_n\}$ grow as fast as possible??

$$b_n \sim \alpha n \sim \frac{\pi}{\beta} n$$

?

$$f^W(\omega) \sim e^{-\frac{\pi|\omega|}{2\alpha}} \iff b_n \sim \alpha n \iff K_{\mathcal{O}}(t) \sim e^{2\alpha t}$$

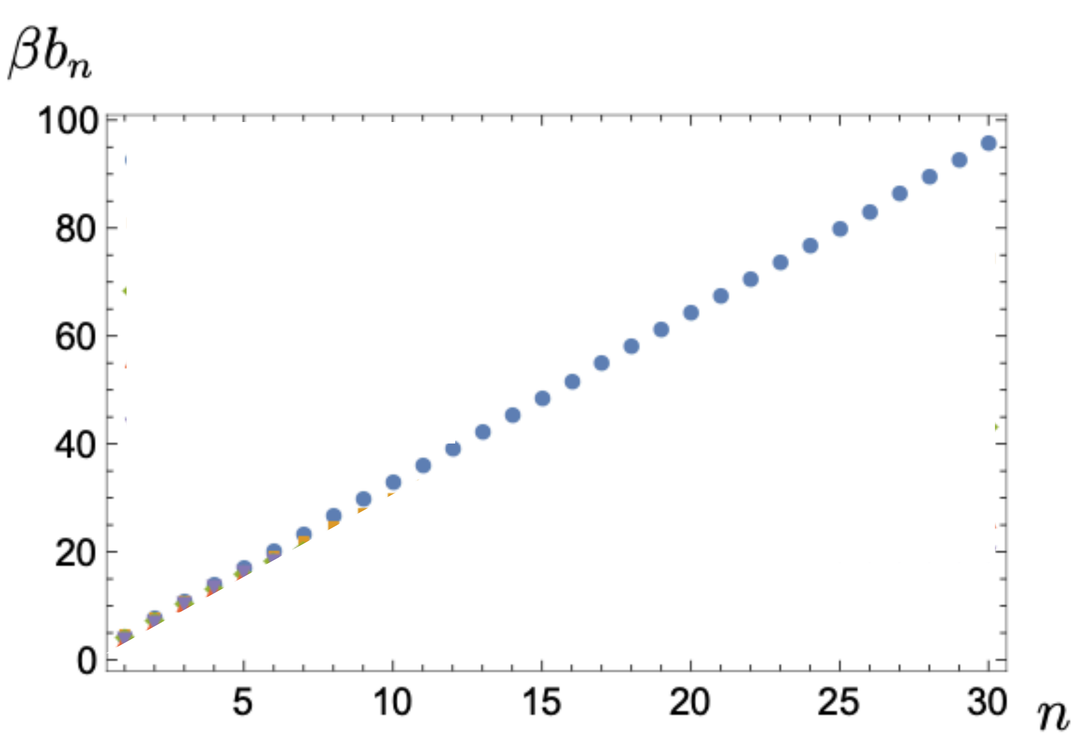
Subtlety in QFT

$$\mathcal{L}_E^{\text{free}} = \frac{1}{2}(\partial\phi)^2 + \frac{1}{2}m^2\phi^2$$

Power spectrum (m=0, d=4)

$$f^W(\omega) = \frac{\beta^2\omega}{\pi\sinh(\frac{\beta\omega}{2})}$$

$$f^W(\omega) \sim e^{-\frac{\beta|\omega|}{2}} \sim e^{-\frac{\pi|\omega|}{2\alpha}} \qquad \left(\alpha = \frac{\pi}{\beta}\right)$$



Wightman 2-point function

$$\Pi^W(t,\mathbf{x}) := \langle\phi(t - i\beta/2,\mathbf{x})\phi(0,\mathbf{0})\rangle_\beta \qquad \left(t = \frac{i\beta}{2}\right)$$

Power spectrum $\quad C(t) = \Pi^W(t,\mathbf{0})$

$$f^W(\omega) := \int \mathrm{d}t\, C(t)e^{i\omega t} = \int \mathrm{d}t\, \Pi^W(t,\mathbf{0})e^{i\omega t}$$

$$f^W(\omega) \sim e^{-\frac{\beta|\omega|}{2}} \sim e^{-\frac{\pi|\omega|}{2\alpha}} \qquad \left(\alpha = \frac{\pi}{\beta}\right)$$

2104.09514: Dymarsky, Smolkin

General QFT is chaotic? No

~~In a chaotic quantum system~~ In general QFT

Lanczos coefficients $\{b_n\}$ grow as fast as possible!

$$b_n \sim \alpha n \sim \frac{\pi}{\beta}n$$

?

$$f^W(\omega) \sim e^{-\frac{\pi|\omega|}{2\alpha}} \iff b_n \sim \alpha n \iff K_{\mathcal{O}}(t) \sim e^{2\alpha t}$$

Too good to be true

Towards Field theory

Counter example:

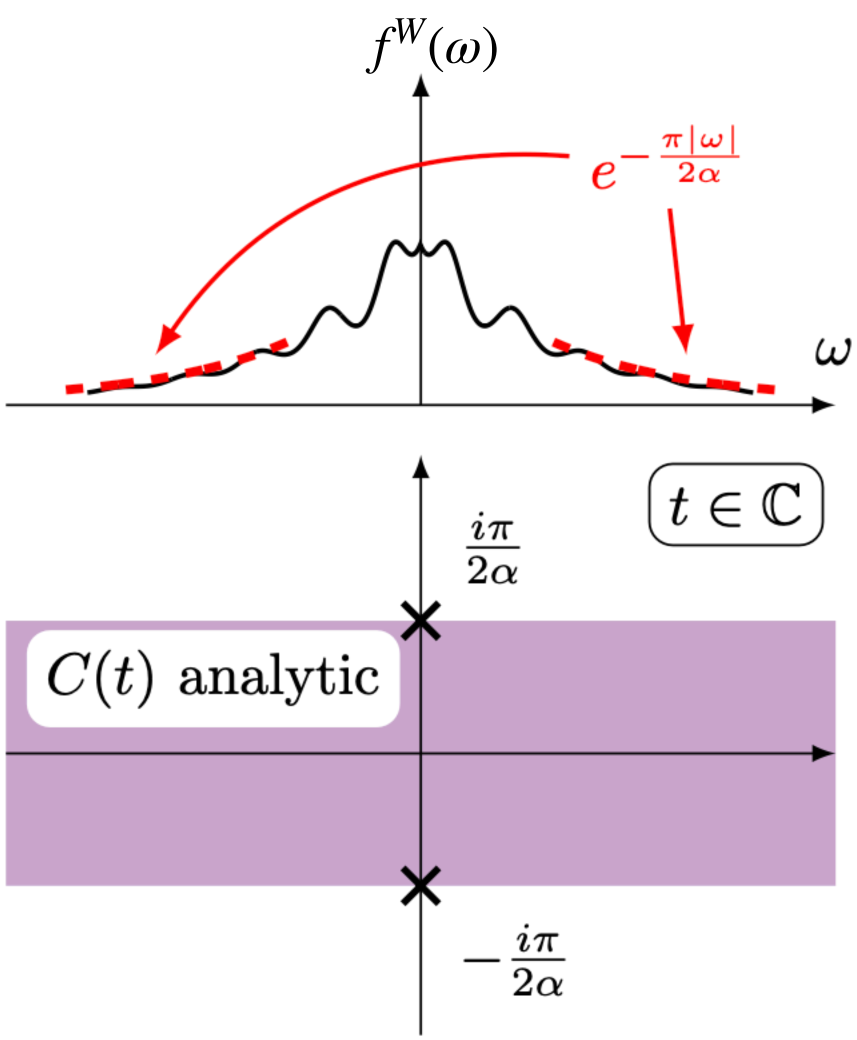
- Field theory
- Krylov complexity in saddle-dominated scrambling (2203.03534: Bhattacharjee, Cao, Nandy, Pathak)

Too good to be true

$$\text{Chaos} \Longleftrightarrow \boxed{f^W(\omega) \sim e^{-\frac{\pi|\omega|}{2\alpha}} \Longleftrightarrow b_n \sim \alpha n \Longleftrightarrow K_{\mathcal{O}}(t) \sim e^{2\alpha t}}$$

Only if b_n is a smooth function of n, Otherwise

$$\text{Chaos} \overset{?}{\Longleftrightarrow} \boxed{f^W(\omega) \sim e^{-\frac{\pi|\omega|}{2\alpha}} \overset{?}{\Longleftrightarrow} b_n \not\sim \alpha n \overset{?}{\Longleftrightarrow} K_{\mathcal{O}}(t) \sim e^{2\alpha t}}$$



Counter example:

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Too good to be true

$$\text{Chaos} \Longleftrightarrow \boxed{f^W(\omega) \sim e^{-\frac{\pi|\omega|}{2\alpha}} \Longleftrightarrow b_n \sim \alpha n \Longleftrightarrow K_{\mathcal{O}}(t) \sim e^{2\alpha t}}$$

Only if b_n is a smooth function of n, Otherwise

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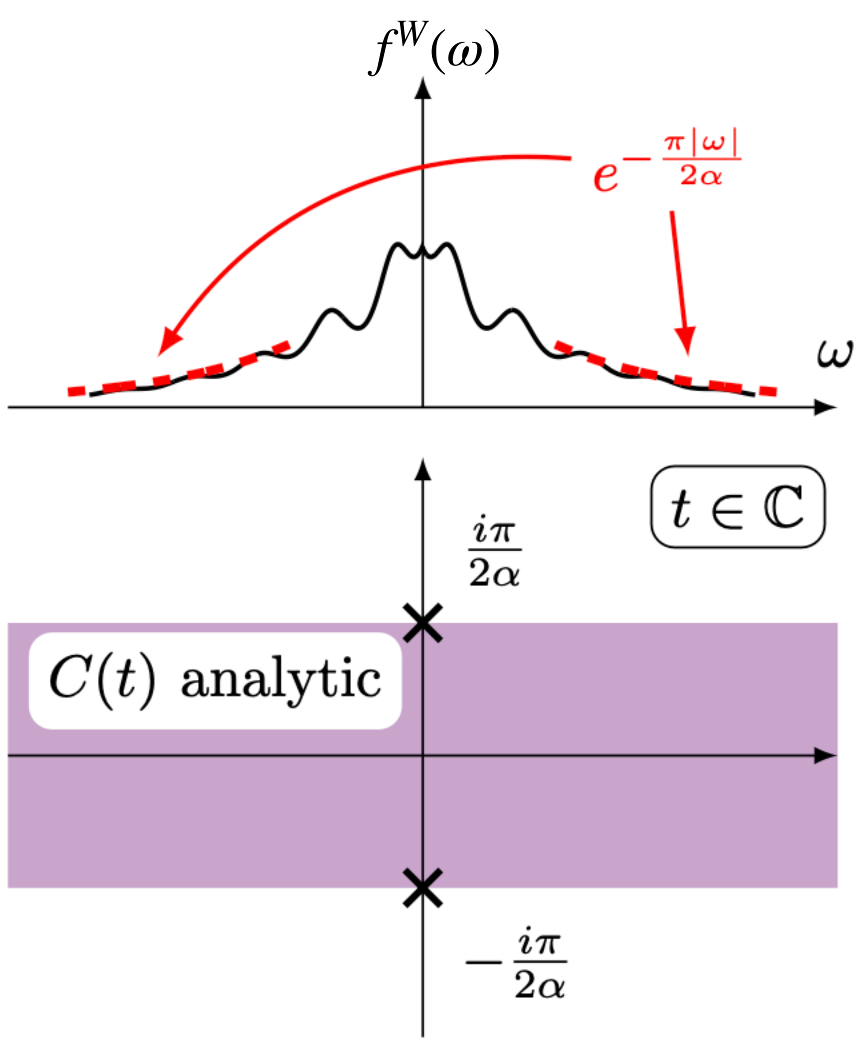
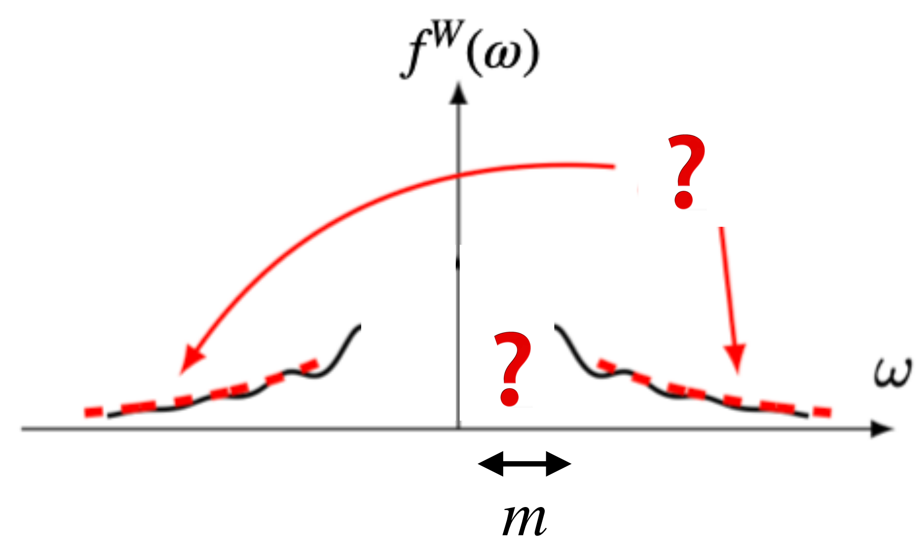
Need to investigate these relations further.
How to extract (chaotic) information from the power spectrum?





Computation method

1812.08657: Parker et al.

Lanczos coefficients

$$
\begin{tikzcd}[column sep=huge, row sep=huge]
{\color{red} f^W(\omega)} \arrow[r, "\mu_{2n} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}\omega\, \omega^{2n} f^W(\omega)"] \arrow[d, swap, "\begin{array}{c}\Pi^W(t) = \\ \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}\omega\, e^{-i\omega t} f^W(\omega)\end{array}"] & \mu_{2n} \arrow[d, "\begin{array}{c}b_1^{2n}\cdots b_n^2 = \\ \det\left(\mu_{i+j}\right)_{0\le i,j\le n}\end{array}"] \\
C(t) = \Pi^W(t) = \varphi_0(t) & b_n
\end{tikzcd}
$$

K-complexity

$$\downarrow$$

$$
\begin{aligned}
\dot{\varphi}_0(t) &= b_0 \overbrace{\varphi_{-1}(t)}^{=0} - b_1\varphi_1(t) \\
\dot{\varphi}_1(t) &= b_1\varphi_0(t) - b_2\varphi_2(t) \\
&\vdots \\
\dot{\varphi}_n(t) &= b_n\varphi_{n-1}(t) - b_{n+1}\varphi_{n+1}(t)
\end{aligned}
$$

$$K_{\mathcal{O}}(t) = \sum_{n=1}^{n_{\max}} n\,|\varphi_n(t)|^2\,, \quad n_{\max} = 200.$$

Non-trivial mass (IR-cutoff) effect: staggering

Power spectrum

$\beta m \gg 1$

$$f^W(\omega) \approx N(m,\beta,d)\, e^{-\beta|\omega|/2} \left(\omega^2 - m^2\right)^{(d-3)/2} \Theta(|\omega| - m) \qquad N(m,\beta,d) = \frac{\pi^{3/2}\beta^{(d-2)/2}}{2^{d-2}\, m^{(d-2)/2} K_{\frac{d-2}{2}}\left(\frac{m\beta}{2}\right)\Gamma\left(\frac{d-1}{2}\right)}$$

$K_n(z)$ is the modified Bessel function of the second kind

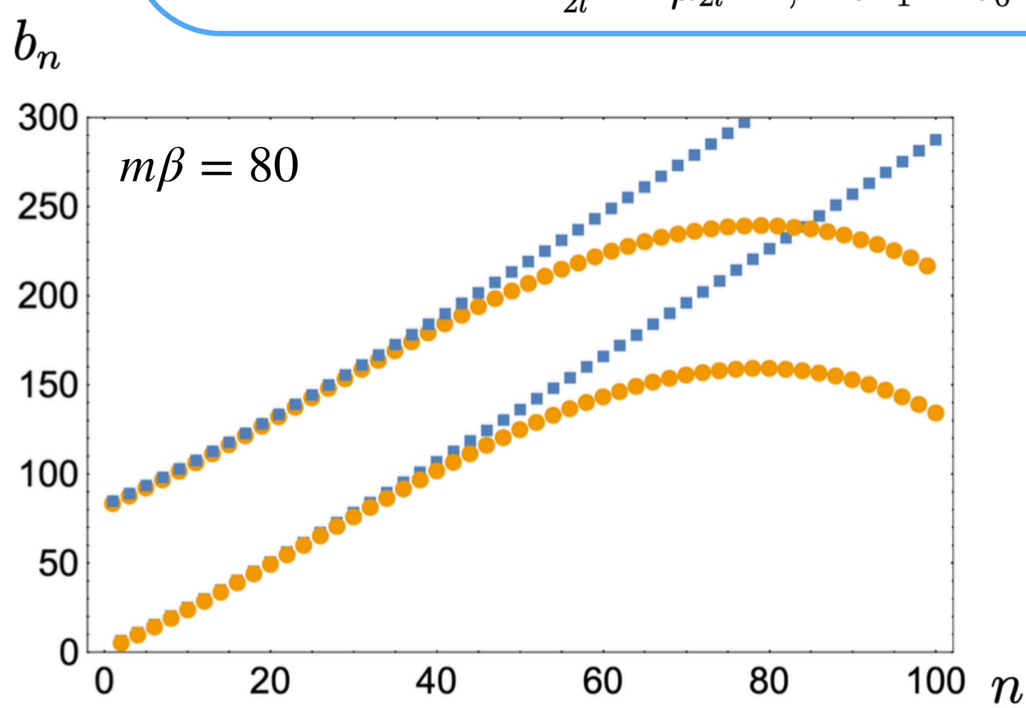
Moments to Lanczos coefficients (d=5)

$\tilde{\Gamma}(n,z)$ is the incomplete Gamma function.

$$\mu_{2n} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}\omega\, \omega^{2n} f^W(\omega) = \frac{2^{-2} e^{\frac{m\beta}{2}}}{2+m\beta}\left(\frac{2}{\beta}\right)^{2n}\left[-m^2\beta^2\,\tilde{\Gamma}\left(2n+1, \frac{m\beta}{2}\right) + 4\tilde{\Gamma}\left(2n+3, \frac{m\beta}{2}\right)\right]$$

$$b_n = \sqrt{M_{2n}^{(n)}}\ , \qquad M_{2l}^{(j)} = \frac{M_{2l}^{(j-1)}}{b_{j-1}^2} - \frac{M_{2l-2}^{(j-2)}}{b_{j-2}^2} \quad \text{with} \quad l = j, \ldots, n\ ,$$

$$M_{2l}^{(0)} = \mu_{2l} \quad , \quad b_{-1} \equiv b_0 := 1 \quad , \quad M_{2l}^{(-1)} = 0\ .$$

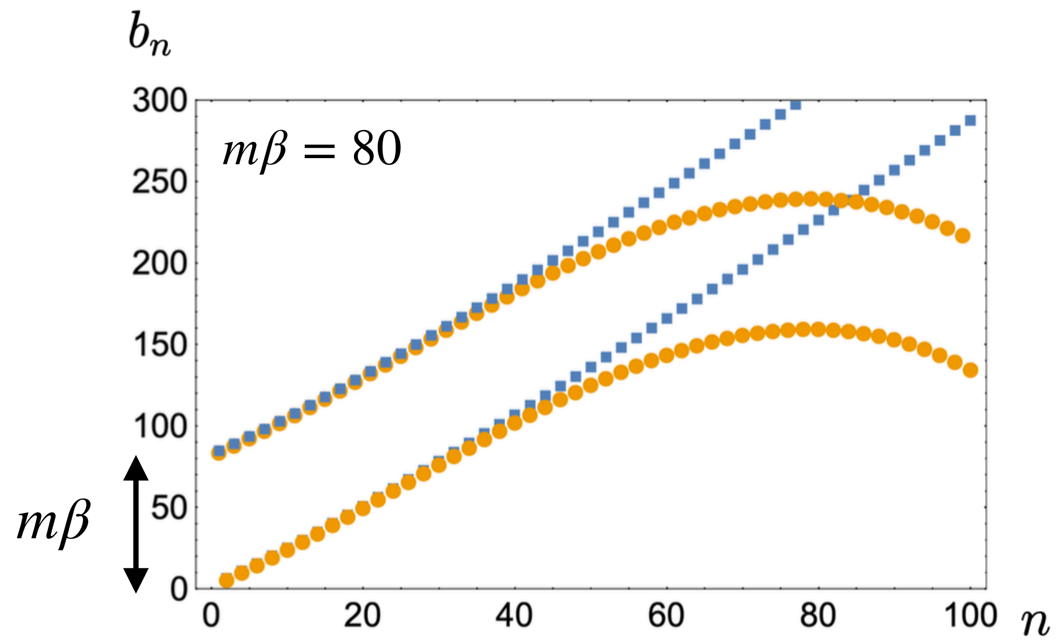


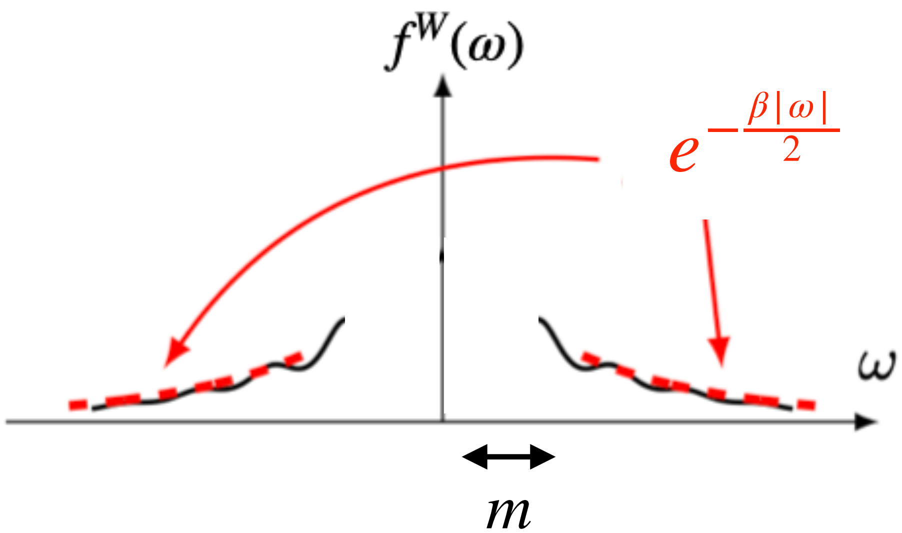
$$\beta^2 b_n^2 = m^2\beta^2 \begin{cases} 1 + 4\frac{1+n}{m\beta} + 8\frac{(n+1)^2}{m^2\beta^2} + 12\frac{(n+1)^3}{m^3\beta^3} + \cdots\ , & \text{for } n \text{ odd}\ , \\ 4\frac{n(n+2)}{m^2\beta^2} + 8\frac{n(n+1)(n+2)}{m^3\beta^3} + \cdots\ , & \text{for } n \text{ even}\ , \end{cases}$$

Staggering: two families for even n and odd n

$$b_n \sim \alpha_{\text{odd}}\, n + \gamma_{\text{odd}} \quad (\text{odd } n)$$
$$b_n \sim \alpha_{\text{even}}\, n + \gamma_{\text{even}} \quad (\text{even } n)$$

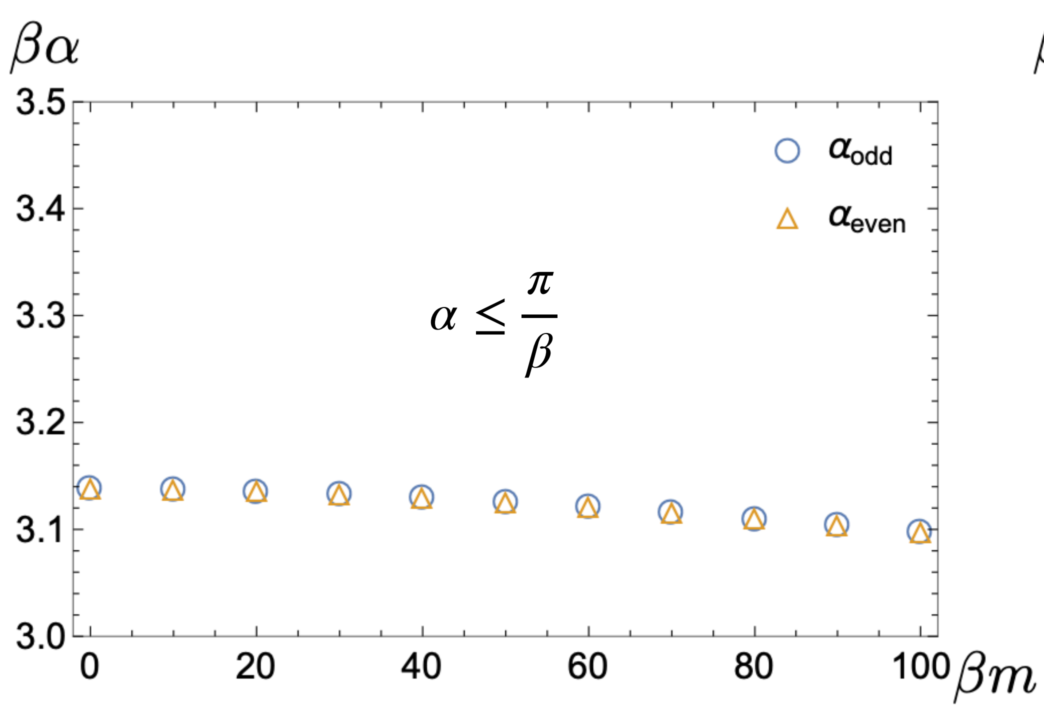
Non-trivial mass (IR-cutoff) effect: staggering



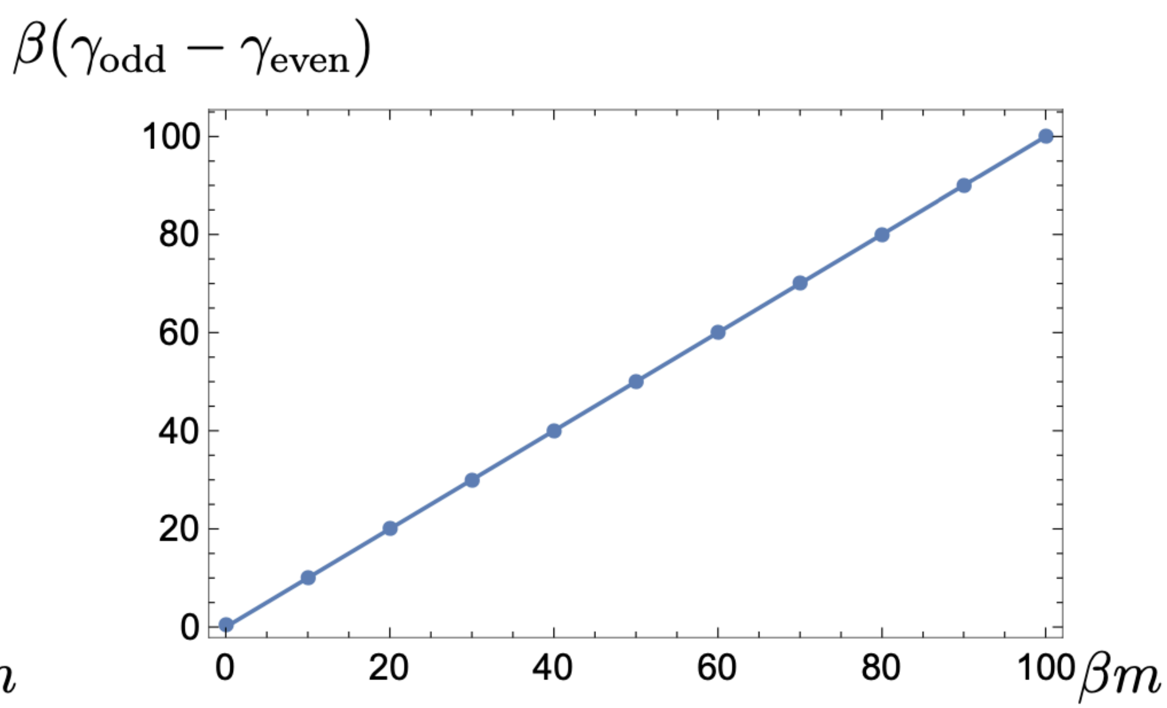


Staggering

$$\boxed{\begin{aligned} b_n &\sim \alpha_{\text{odd}}\, n + \gamma_{\text{odd}} \quad (\text{odd } n) \\ b_n &\sim \alpha_{\text{even}}\, n + \gamma_{\text{even}} \quad (\text{even } n) \end{aligned}}$$



(a) Mass-dependence of α_{odd} and α_{even}



(b) Mass-dependence of $\gamma_{\text{odd}} - \gamma_{\text{even}}$

Non-trivial mass (IR-cutoff) effect: K-complexity

2212.14702: Camargo, Jahnke, KYK, Nishida

Lanczos coefficients

$$\mu_{2n} = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\, \omega^{2n} f^W(\omega)$$

$$f^W(\omega) \longrightarrow \mu_{2n}$$

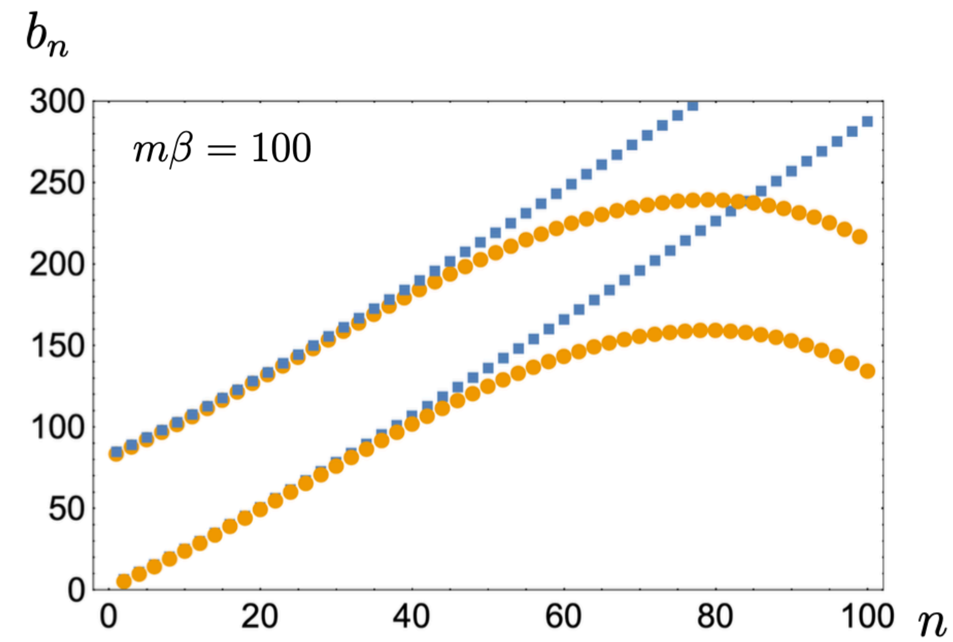
$$\Pi^W(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\, e^{-i\omega t} f^W(\omega)$$

$$b_1^{2n}\cdots b_n^2 = \det\left(\mu_{i+j}\right)_{0\le i,j\le n}$$

$$\mu_{2n} \longrightarrow b_n$$

$$C(t) = \Pi^W(t) = \varphi_0(t)$$

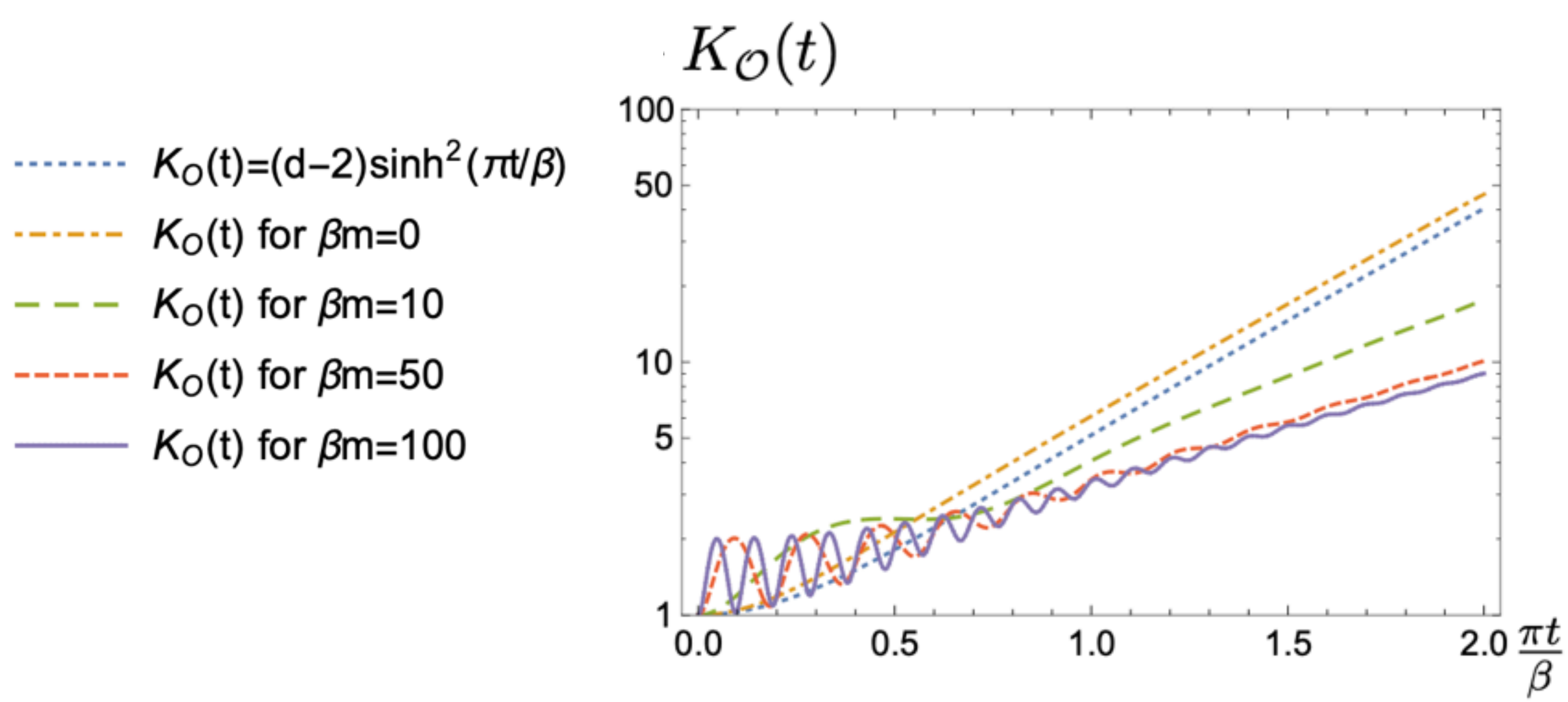
$$C^{(d)}(t) \equiv \varphi_0^{(d)}(t) = c_1^{(d)}(t)\left(c_2^{(d)}(t)\sin(m\,t) + c_3^{(d)}(t)\cos(m\,t)\right)$$



K-complexity

$$\dot\varphi_0(t) = b_0 \overbrace{\varphi_{-1}(t)}^{=0} - b_1\varphi_1(t)$$
$$\dot\varphi_1(t) = b_1\varphi_0(t) - b_2\varphi_2(t)$$
$$\vdots$$
$$\dot\varphi_n(t) = b_n\varphi_{n-1}(t) - b_{n+1}\varphi_{n+1}(t)$$

$$K_{\mathcal{O}}(t) = \sum_{n=1}^{n_{\max}} n\,|\varphi_n(t)|^2\,, \quad n_{\max} = 200.$$



Lanczos coefficients

$$\mu_{2n} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}\omega\, \omega^{2n} f^W(\omega)$$

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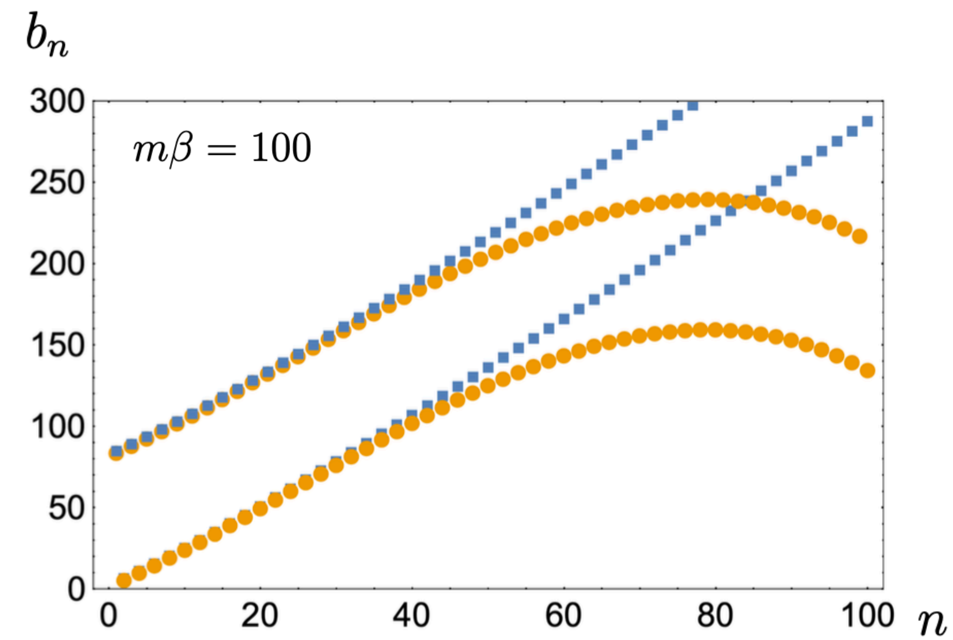
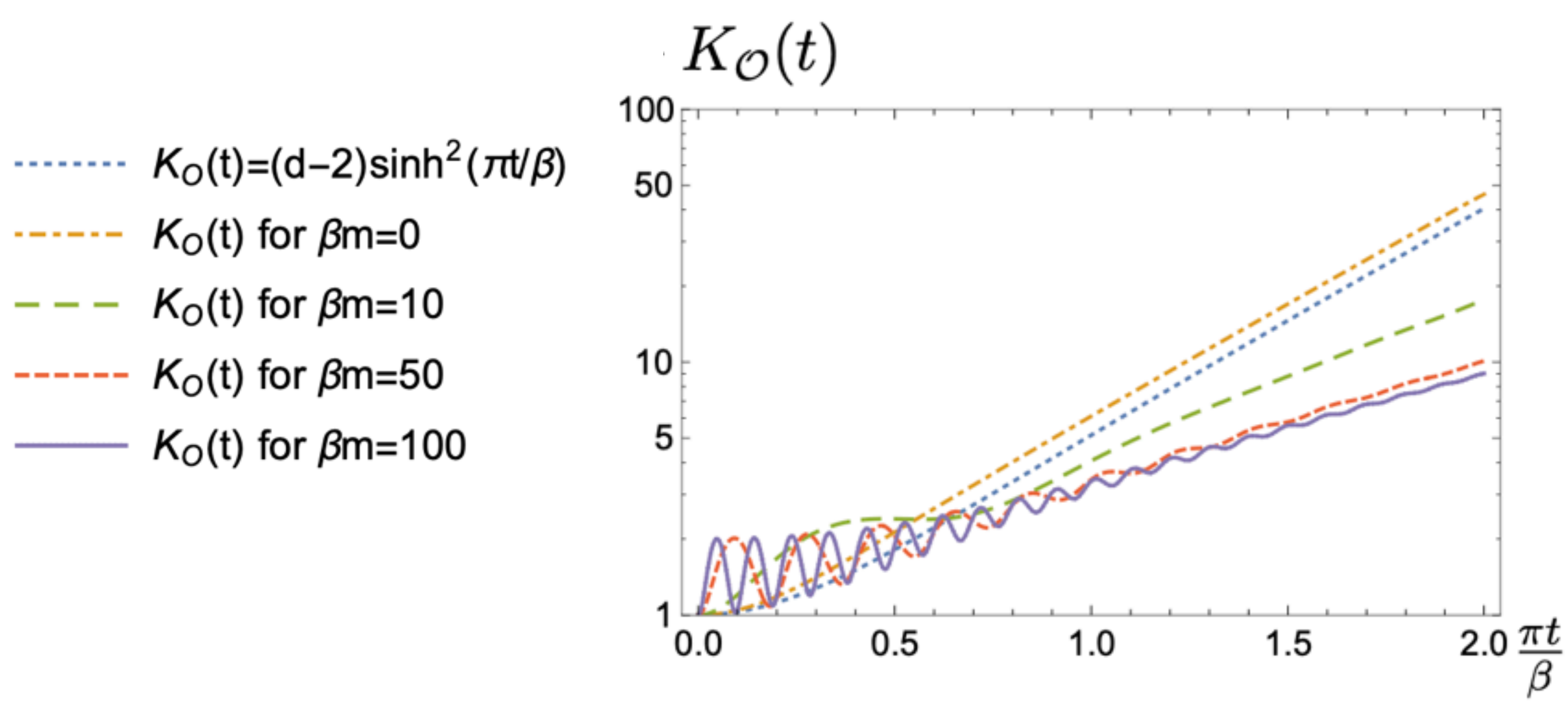


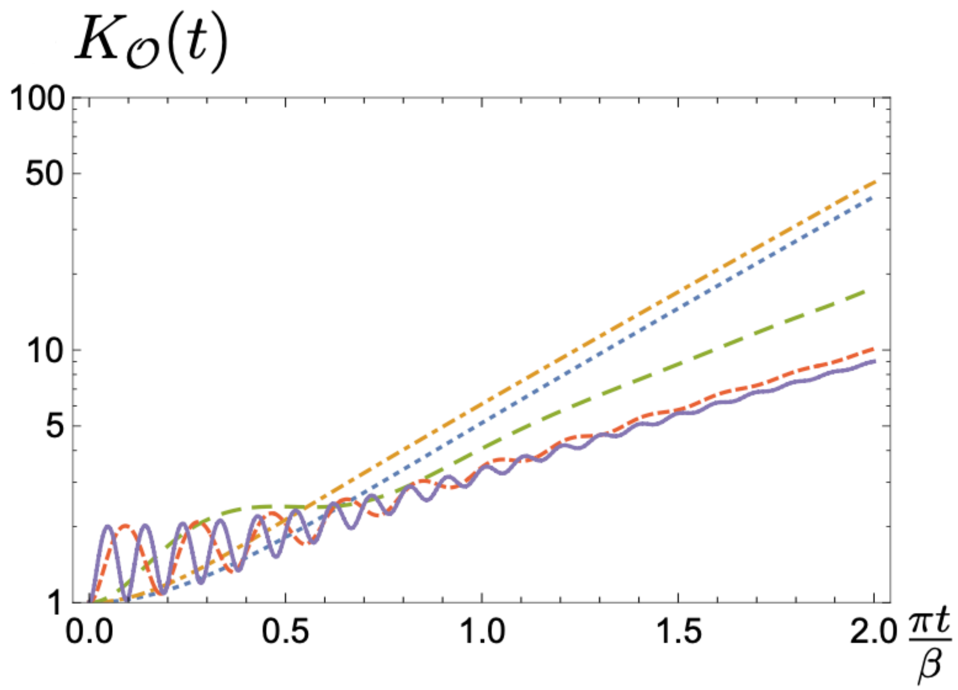
K-complexity

- Early time: oscillation:
  - larger m, shorter period
- Late time: oscillation disappears
  - cancelation due to large n
- Exponential increase
  - larger m, slower increase
  - mass effect

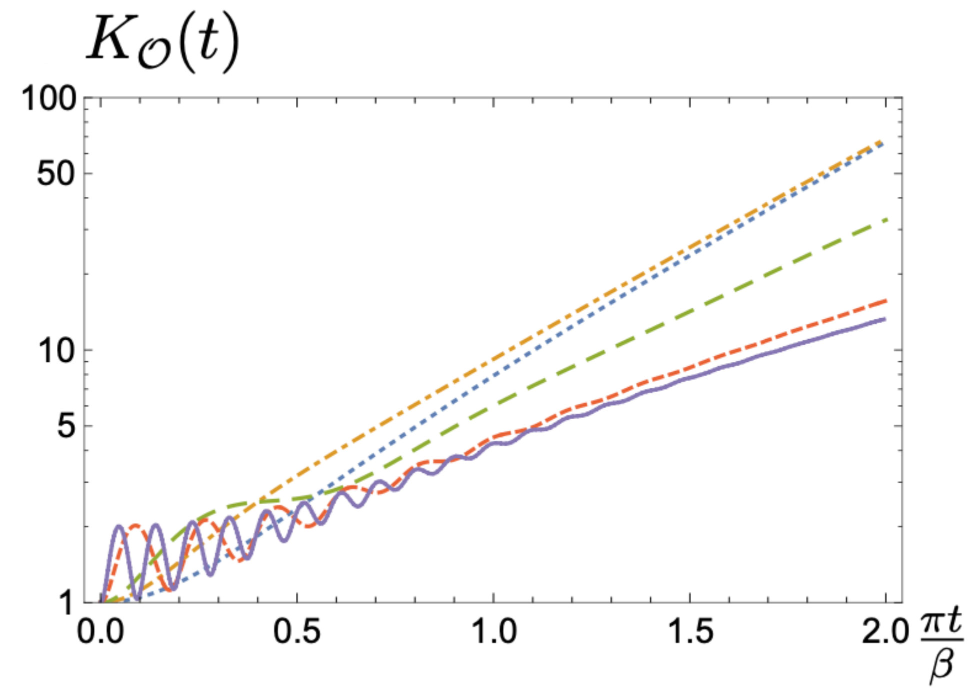


Non-trivial mass (IR-cutoff) effect: K-complexity

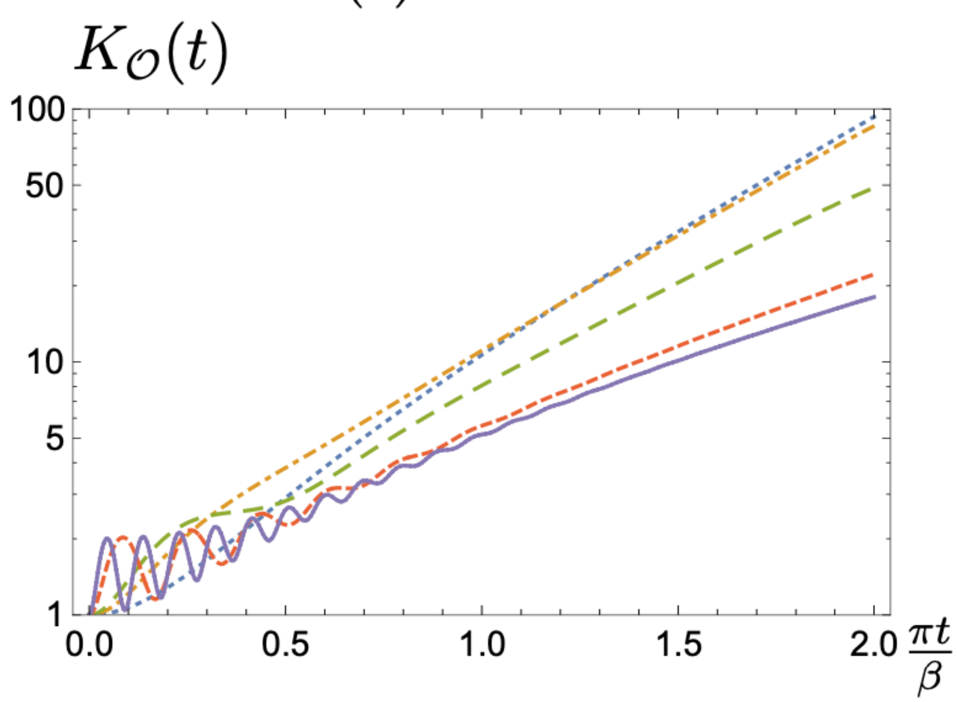
2212.14702: Camargo, Jahnke, KYK, Nishida



(a) $d = 5$



(b) $d = 7$



(c) $d = 9$

$K_O(t)=(d-2)\sinh^2(\pi t/\beta)$

$K_O(t)$ for $\beta m=0$

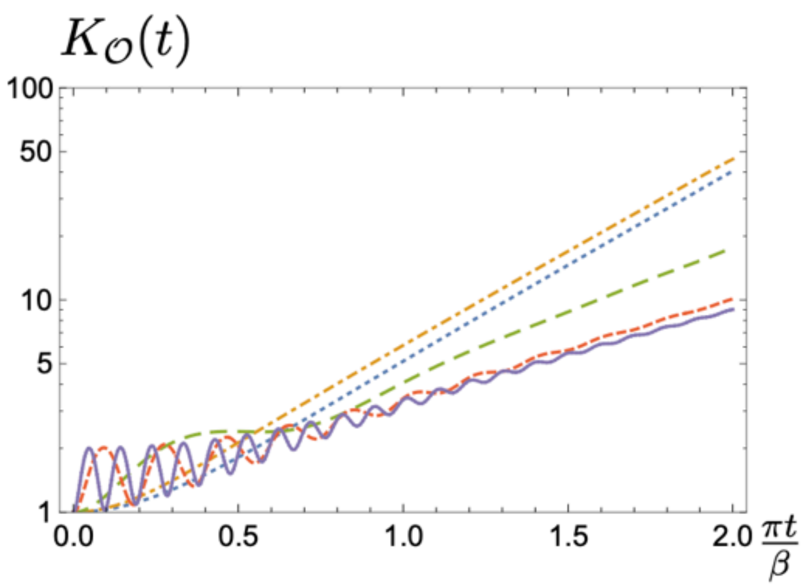
$K_O(t)$ for $\beta m=10$

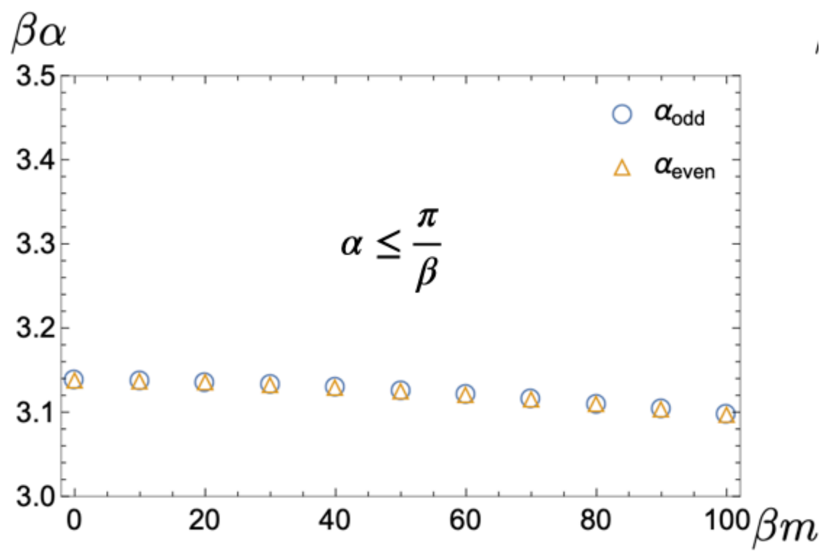
$K_O(t)$ for $\beta m=50$

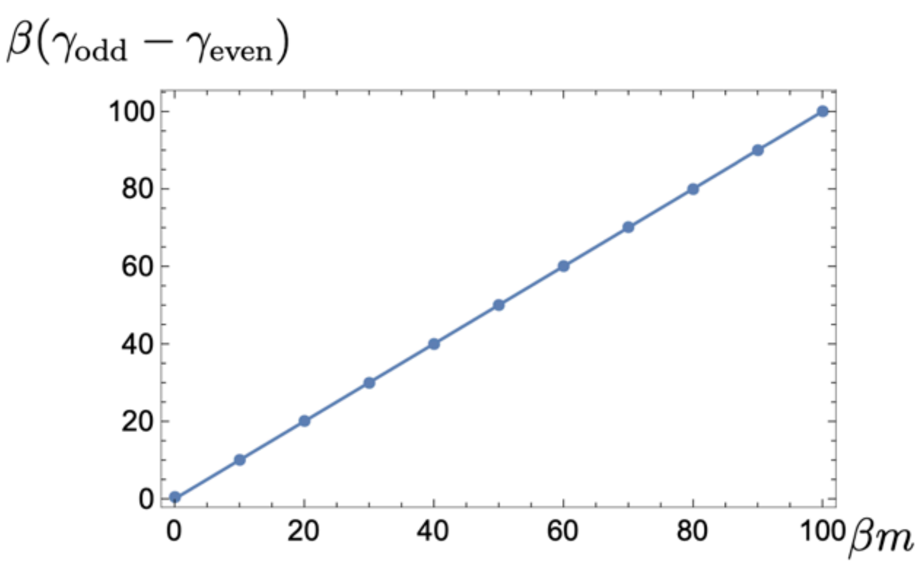
$K_O(t)$ for $\beta m=100$

Non-trivial mass (IR-cutoff) effect: K-complexity

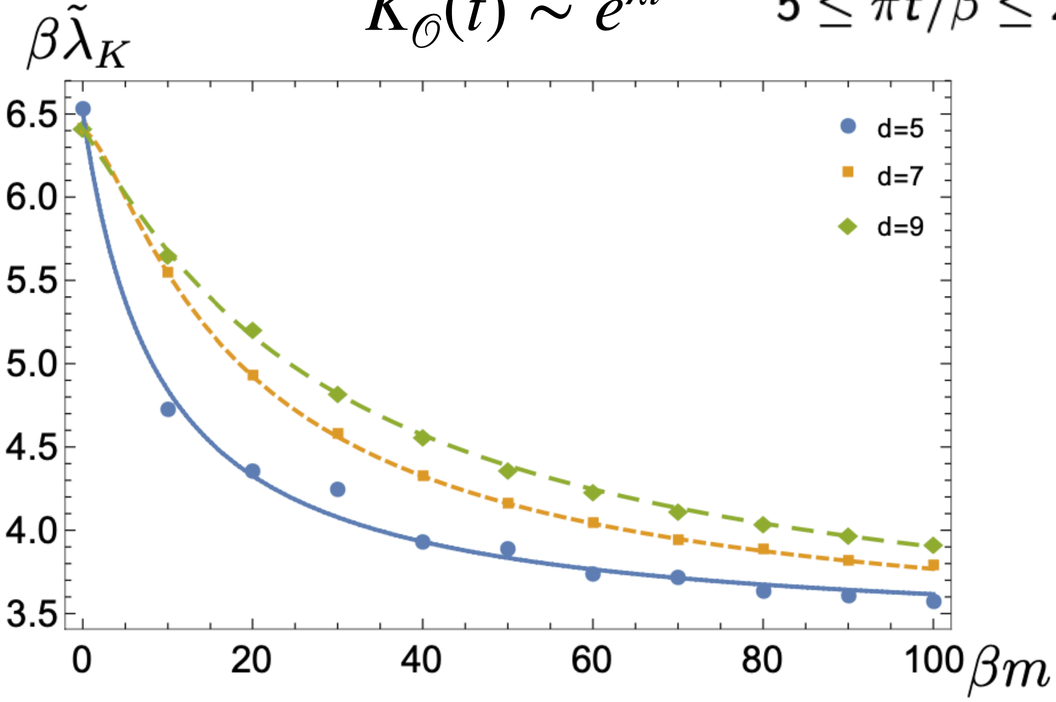
2212.14702: Camargo, Jahnke, KYK, Nishida







$K_{\mathcal{O}}(t) \sim e^{\tilde{\lambda} t}$ $\quad 5 \leq \pi t/\beta \leq 2.$



$$\beta\tilde{\lambda}_K^{(d)} = \beta(\alpha_{\text{odd}} + \alpha_{\text{even}}) + k_2^{(d)}\left(\frac{1}{k_3^{(d)} + \beta|\gamma_{\text{odd}} - \gamma_{\text{even}}|} - \frac{1}{k_3^{(d)}}\right) + k_4^{(d)}\left(\frac{1}{\left(k_3^{(d)} + \beta|\gamma_{\text{odd}} - \gamma_{\text{even}}|\right)^2} - \frac{1}{(k_3^{(d)})^2}\right),$$

Staggering

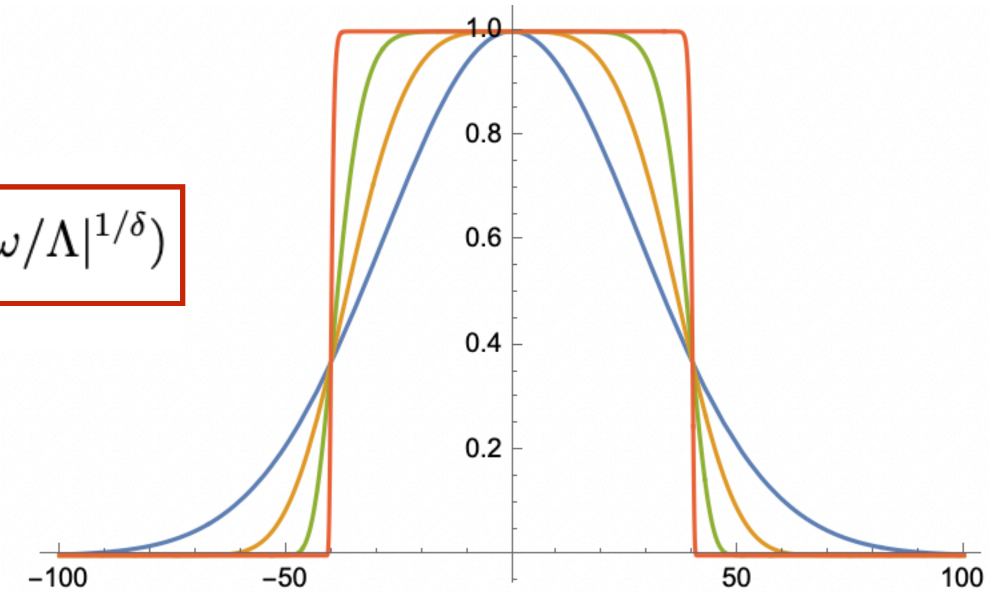
$$\begin{aligned} b_n &\sim \alpha_{\text{odd}}\, n + \gamma_{\text{odd}} \quad (\text{odd } n) \\ b_n &\sim \alpha_{\text{even}}\, n + \gamma_{\text{even}} \quad (\text{even } n) \end{aligned}$$

$$f^W(\omega) \sim e^{-\frac{\pi|\omega|}{2\alpha}} \;\cancel{\Longleftrightarrow}\; b_n \cancel{\sim} \alpha n \;\cancel{\Longleftrightarrow}\; K_{\mathcal{O}}(t) \sim e^{2\alpha t}$$

2212.14702: Camargo, Jahnke, KYK, Nishida

m=0, d=4

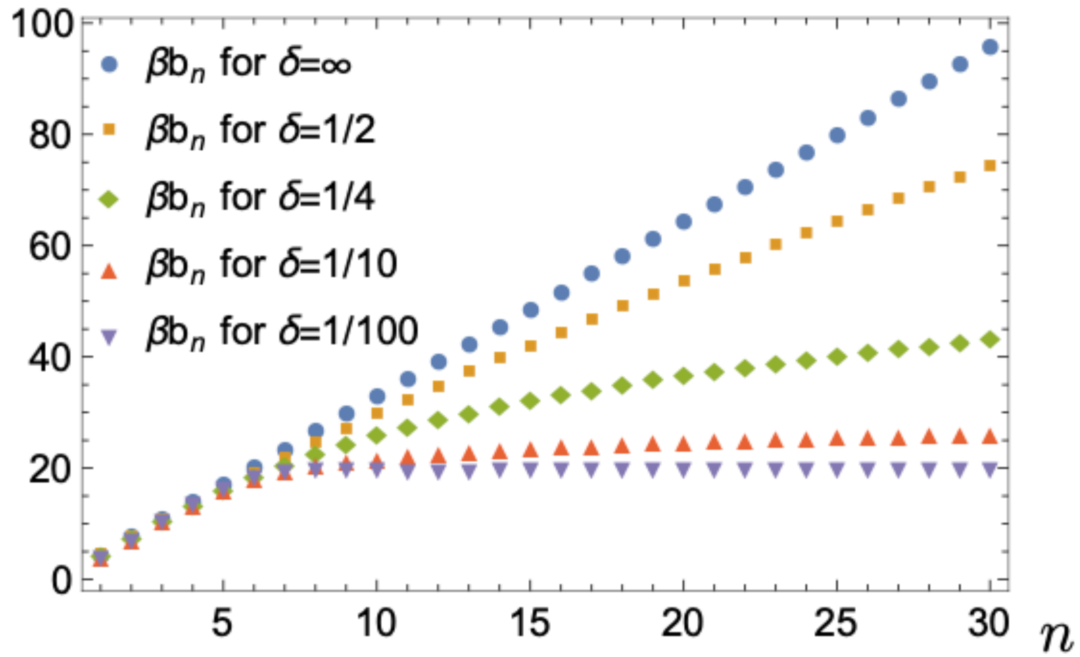
$$f^W(\omega) = N(\beta,\Lambda,\delta)\frac{\omega}{\sinh(\frac{\beta\omega}{2})}\boxed{\exp(-|\omega/\Lambda|^{1/\delta})}$$



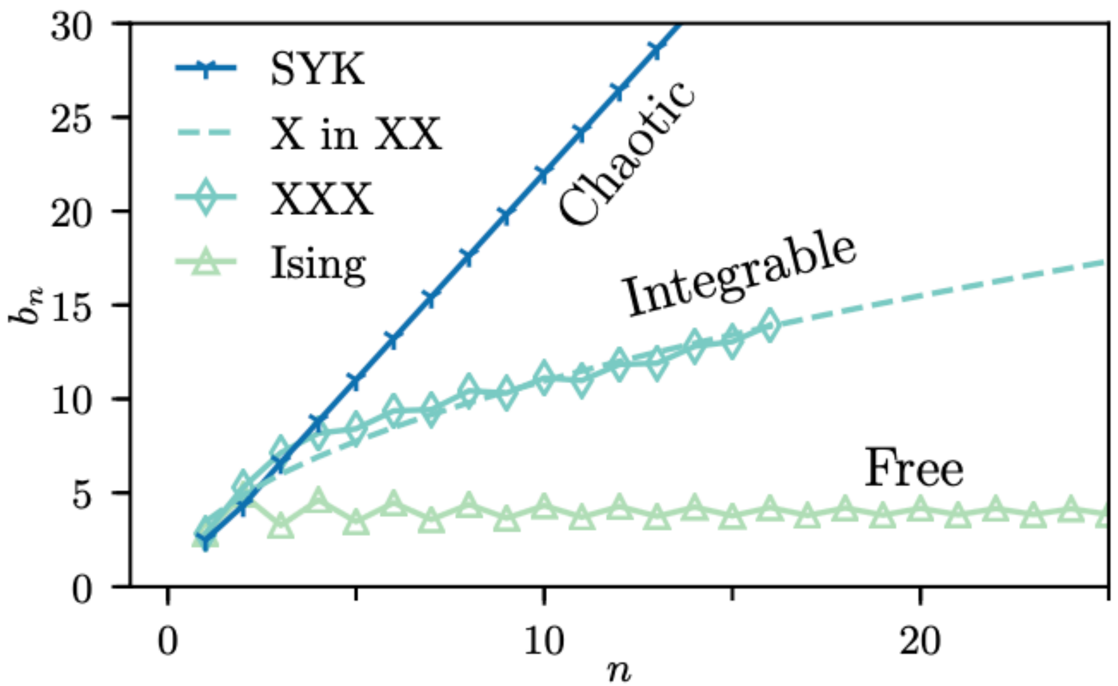
$$b_n \sim n^\delta \Longleftrightarrow f^W(\omega) \sim \exp(-|\omega/\omega_0|^{1/\delta})$$

$\delta \leq 1$

βb_n

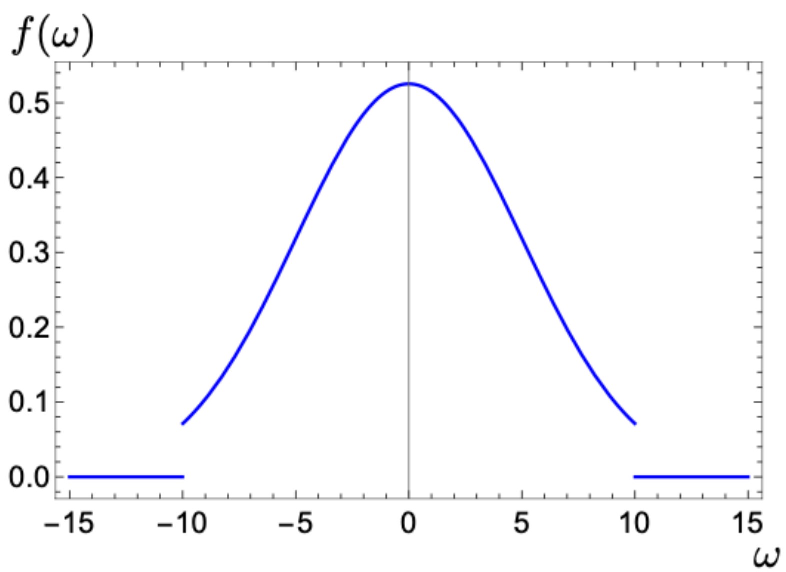


$\beta\Lambda = 40.$

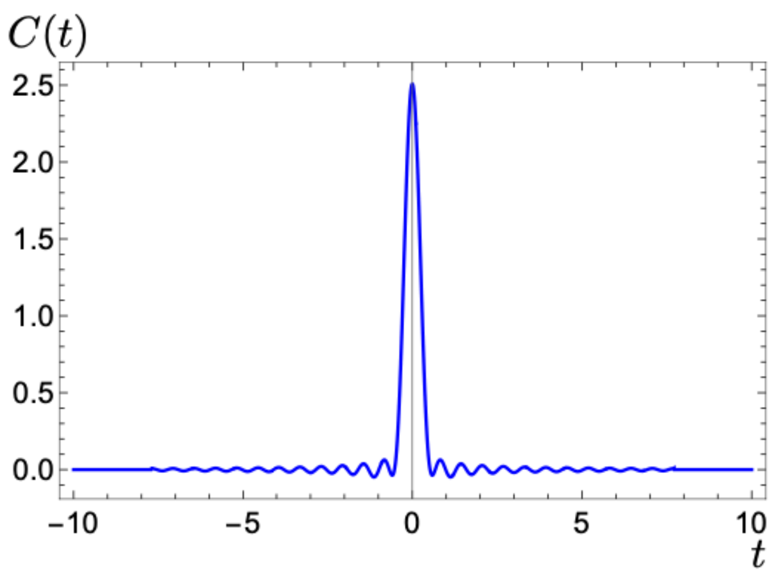


Condition for staggering

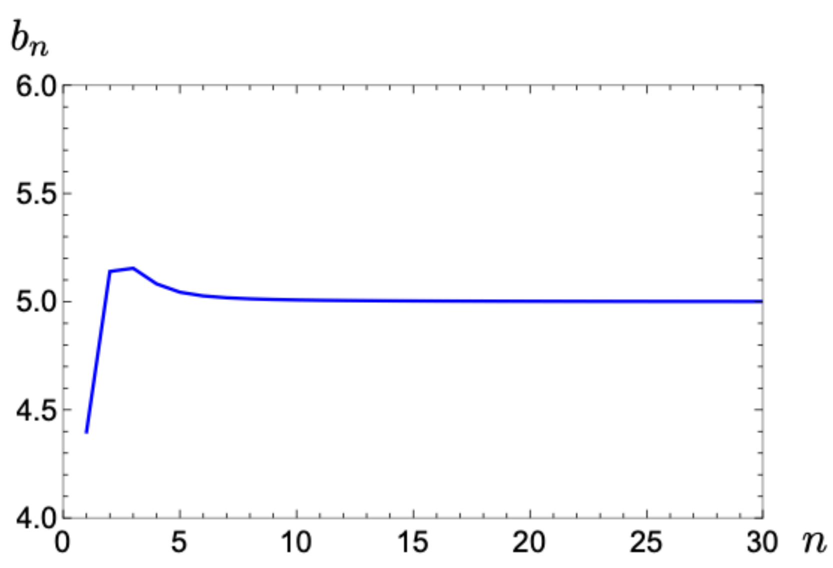
$$f(\omega) = \frac{\sqrt{2\pi}}{\sigma\,\mathrm{Erf}\left(\frac{\Lambda}{\sqrt{2}\sigma}\right)} \begin{cases} e^{-\frac{\omega^2}{2\sigma^2}} & \text{if } |\omega| \leq \Lambda \\ 0 & \text{if } |\omega| > \Lambda \end{cases}$$



(a)



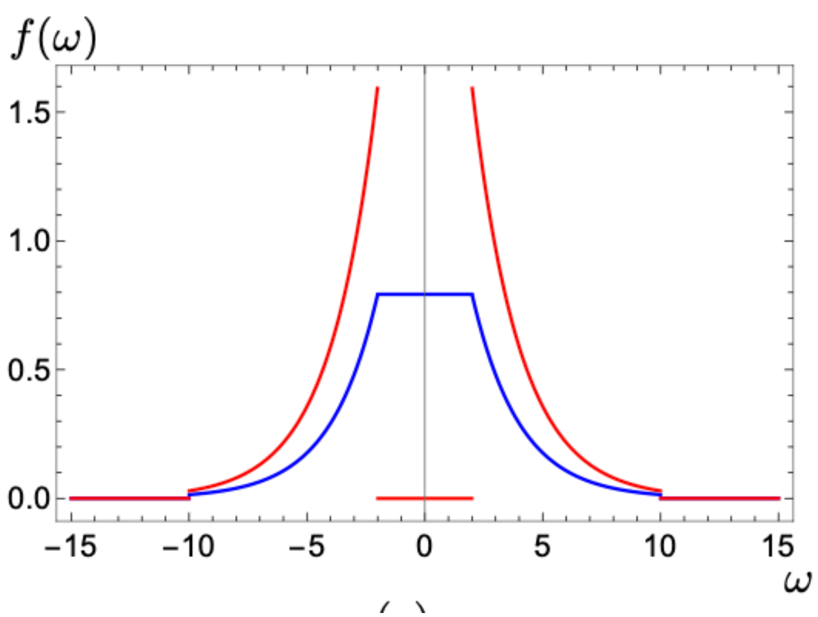
(b)

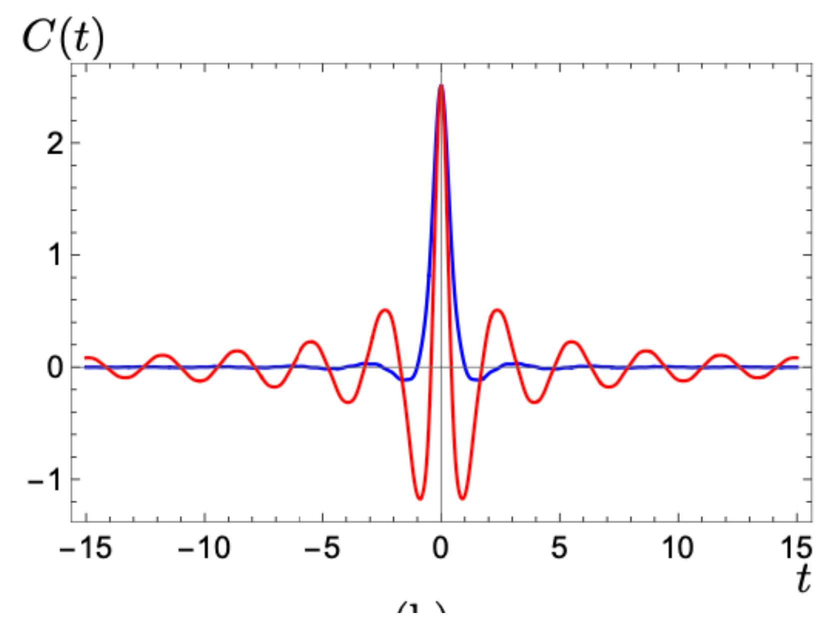


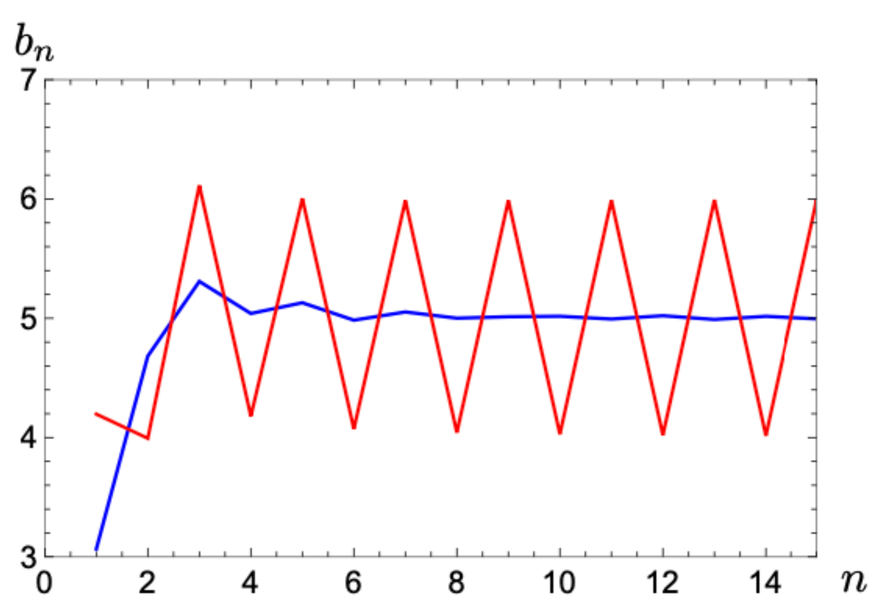
Condition for staggering

2212.14702: Camargo, Jahnke, KYK, Nishida

$$f(\omega)=\frac{\pi a e^{a(\Lambda+m)}}{e^{a\Lambda}(ahm+1)-e^{am}}\begin{cases} h\,e^{-am} & \text{if } |\omega|\le m \\ e^{-a|\omega|} & \text{if } m<|\omega|<\Lambda \\ 0 & \text{if } |\omega|\ge\Lambda \end{cases}$$

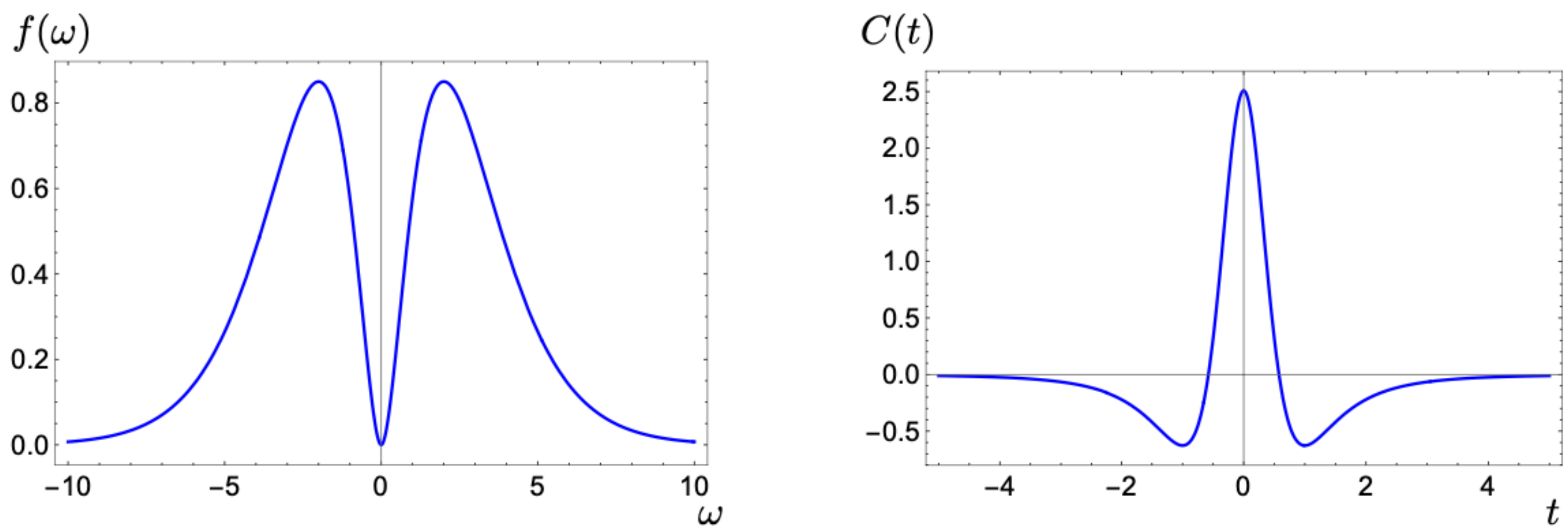




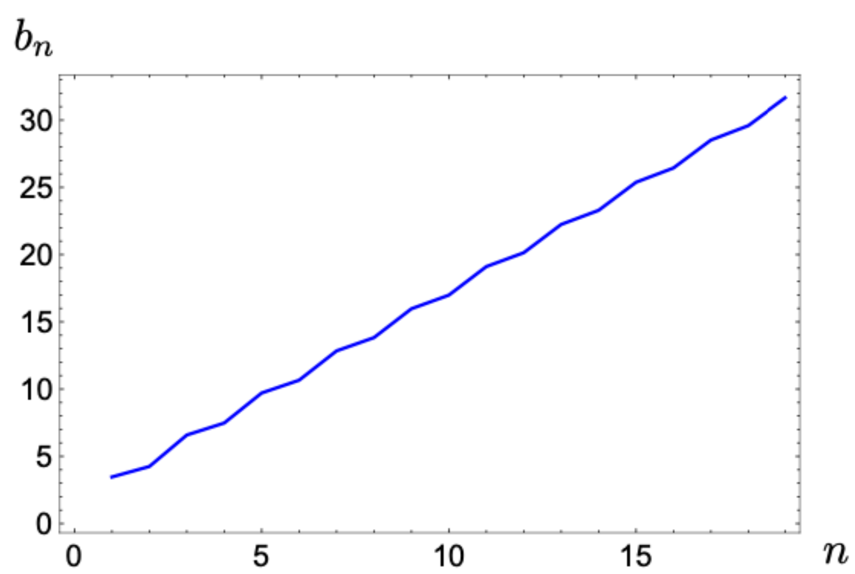


Condition for staggering

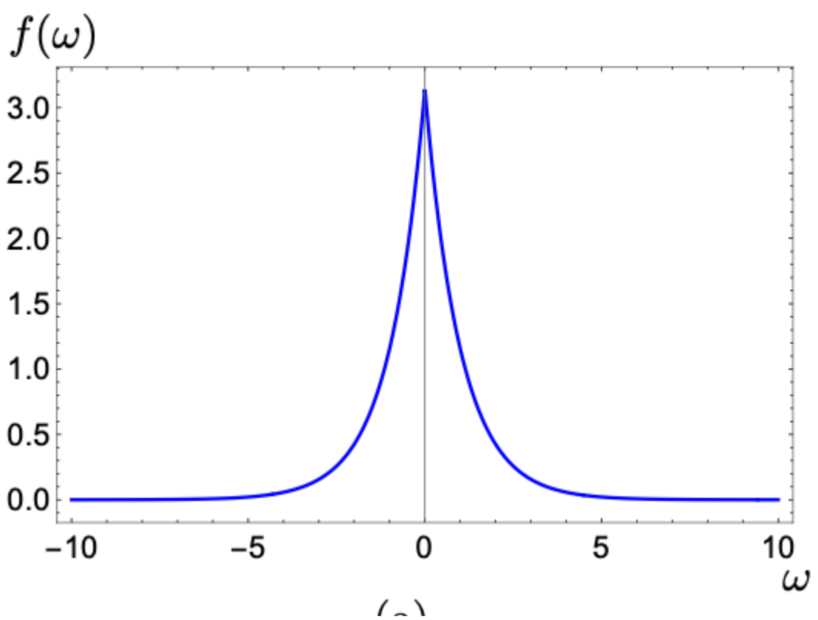
$$f(\omega) = N(\omega_0, \delta, \lambda) \left|\frac{\omega}{\omega_0}\right|^{\lambda} e^{-\left|\frac{\omega}{\omega_0}\right|^{\frac{2}{\delta}}}$$

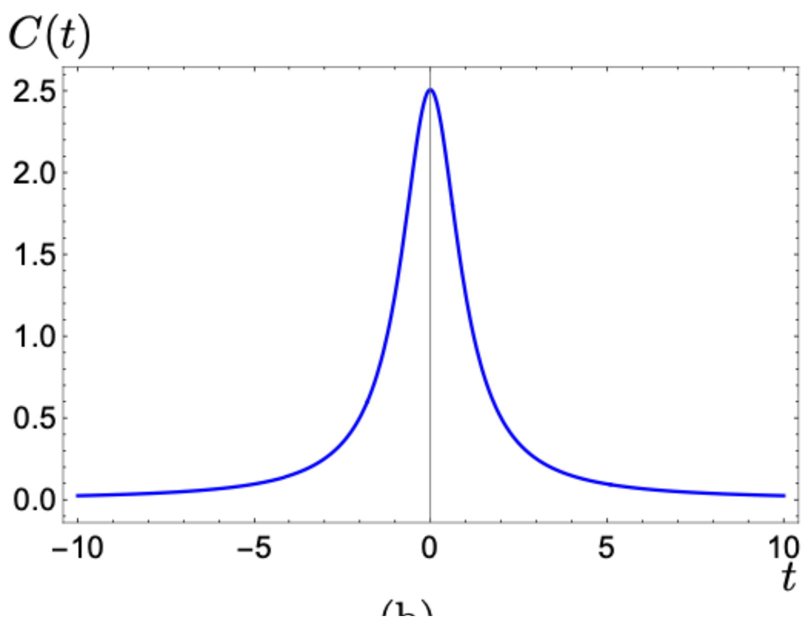


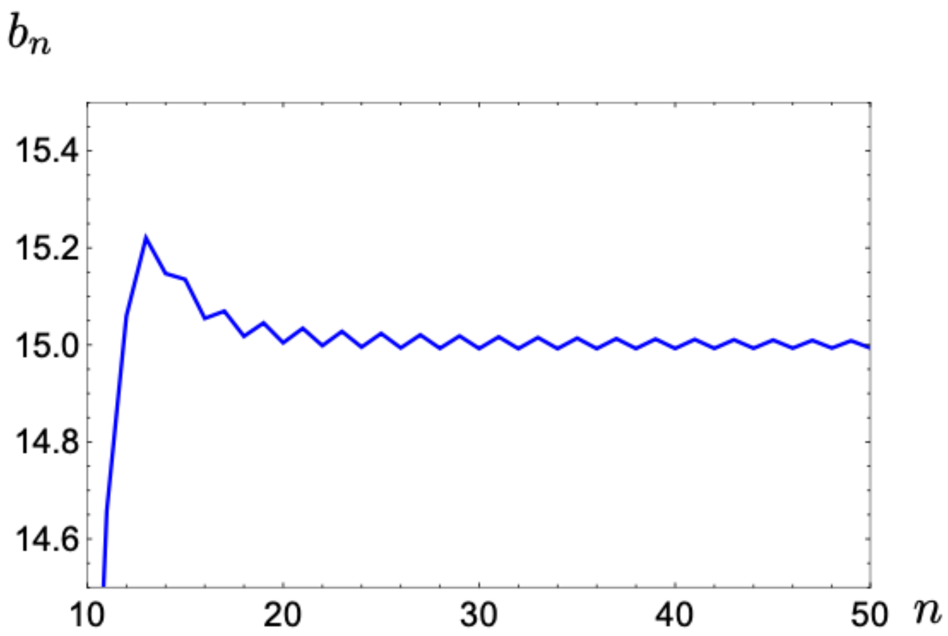
(a) (b)

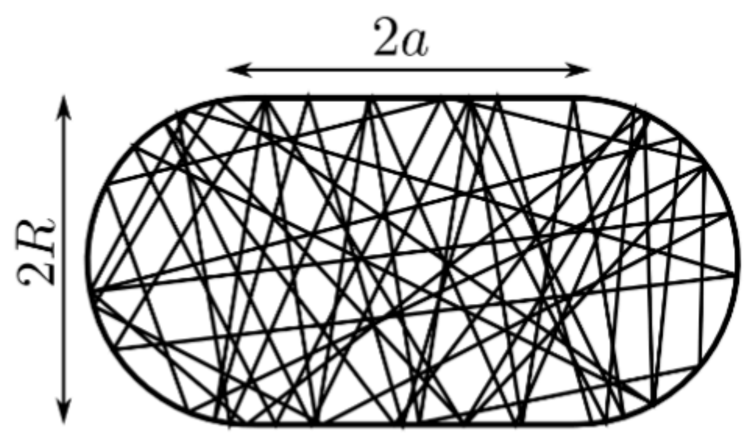


$$f(\omega) = \begin{cases} N(\omega_0, \Lambda) e^{-\left|\frac{\omega}{\omega_0}\right|} & \text{if } |\omega| \leq \Lambda \\ 0 & \text{if } |\omega| > \Lambda \end{cases}$$

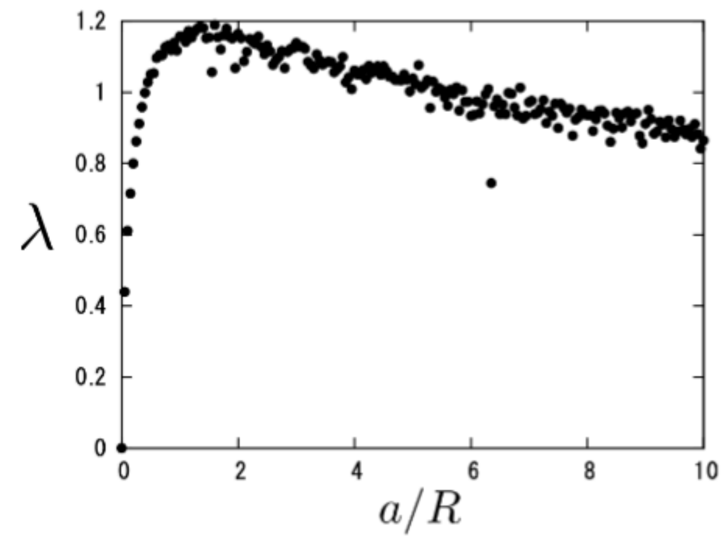




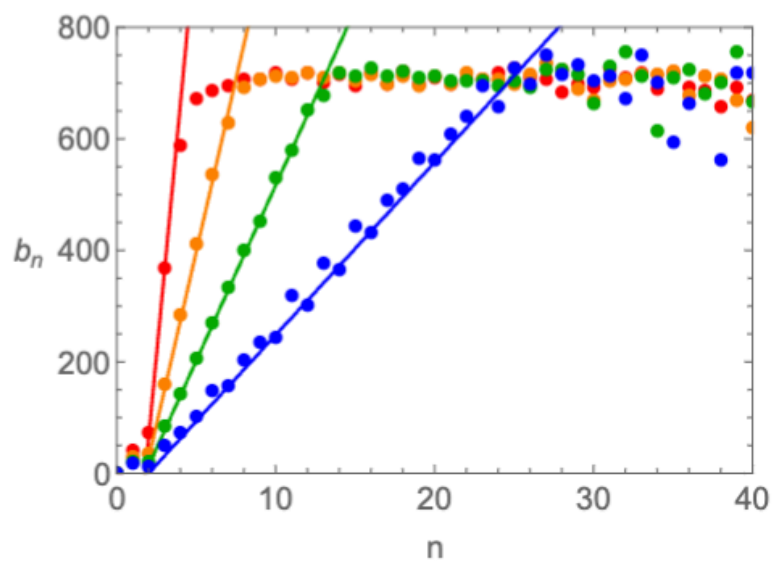




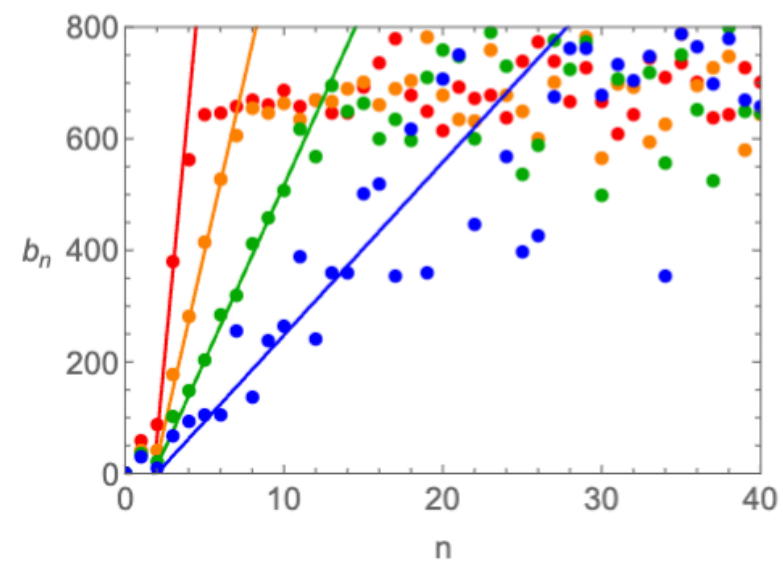
(a) A typical trajectory



(b) Lyapunov exponent



(a) Stadium billiard $(a/R = 1)$



(b) Circle billiard $(a/R = 0)$

2306.11632: Camargo, Jahnke, Jeong, KYK, Nishida

2305.16669: Hashimoto, Murata, Tanahashi, Ryota Watanabe

2112.12128: Rabinovici, Sanchez-Garrido, Shir, Sonner

- Is it possible to extract the chaos-info from a C(t) or the power spectrum?

Lanczos coefficients

$$\mu_{2n} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}\omega\, \omega^{2n} f^{W}(\omega)$$

$$\begin{tikzcd}[column sep=huge, row sep=huge]
f^{W}(\omega) \arrow[r] \arrow[d, "{\Pi^{W}(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}\omega\, e^{-i\omega t} f^{W}(\omega)}"'] & \mu_{2n} \arrow[d, "{b_1^{2n}\cdots b_n^{2} = \det\left(\mu_{i+j}\right)_{0\le i,j\le n}}"] \\
C(t) = \Pi^{W}(t) = \varphi_0(t) & b_n
\end{tikzcd}$$

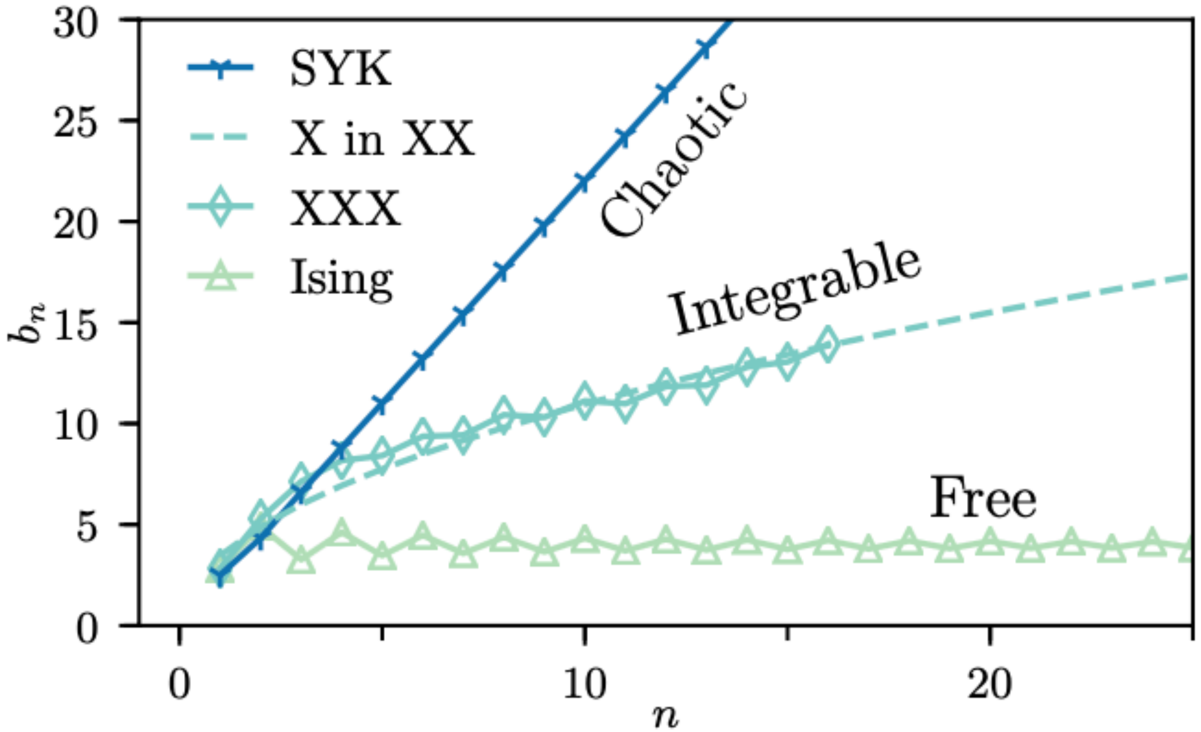
K-complexity

$$\begin{aligned}
\dot{\varphi}_0(t) &= b_0 \overbrace{\varphi_{-1}(t)}^{=0} - b_1\varphi_1(t) \\
\dot{\varphi}_1(t) &= b_1\varphi_0(t) - b_2\varphi_2(t) \\
&\vdots \\
\dot{\varphi}_n(t) &= b_n\varphi_{n-1}(t) - b_{n+1}\varphi_{n+1}(t)
\end{aligned}$$

$$K_{\mathcal{O}}(t) = \sum_{n=1}^{n_{\max}} n\,|\varphi_n(t)|^2, \quad n_{\max} = 200.$$

Summary (Lattice systems)

- Is it possible to extract the chaos-info from a C(t) or a power spectrum?
  - Seems to be possible for Lattice systems.



Universal operator growth hypothesis

In a chaotic quantum system

Lanczos coefficients $\{b_n\}$ grow as fast as possible

$$b_n \sim \alpha n$$

the slowest possible decay of the power spectrum

$$f^W(\omega) \sim e^{-\frac{\pi|\omega|}{2\alpha}}$$

Krylov complexity grows exponentially

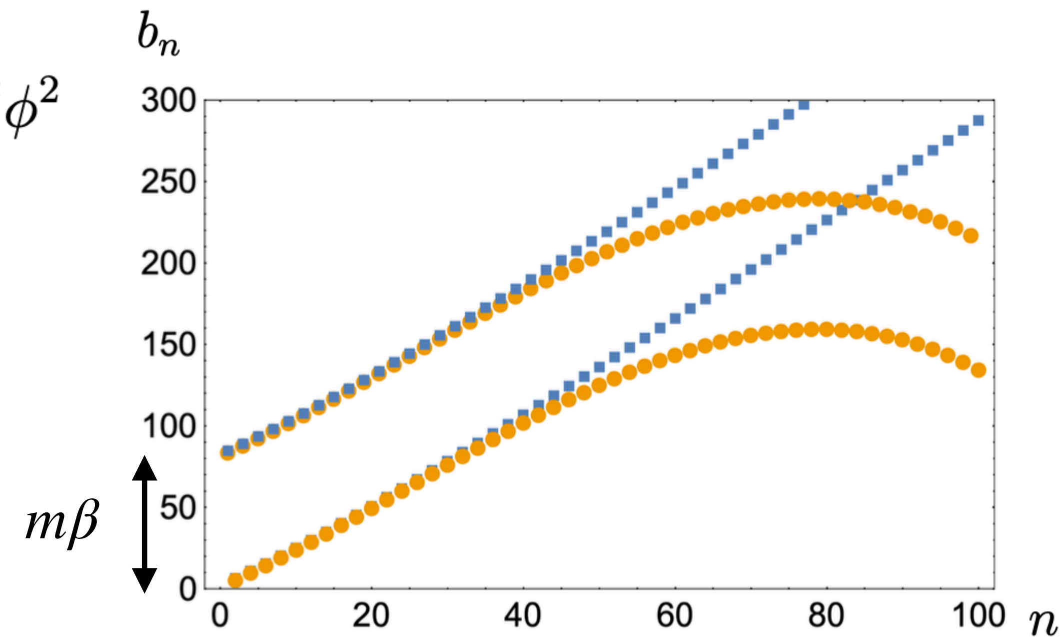
$$K_{\mathcal{O}}(t) \sim e^{2\alpha t}$$

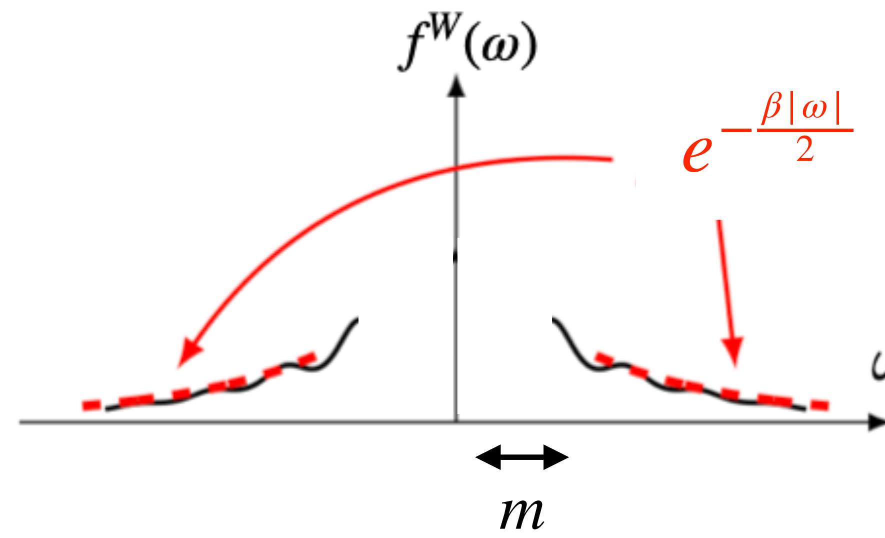
$$f^W(\omega) \sim e^{-\frac{\pi|\omega|}{2\alpha}} \iff b_n \sim \alpha n \iff K_{\mathcal{O}}(t) \sim e^{2\alpha t}$$

- Subtleties in saddle point
- Subtleties in QFT

Summary (QFT)

$$\mathcal{L}_E^{\text{free}} = \frac{1}{2}(\partial\phi)^2 + \frac{1}{2}m^2\phi^2$$

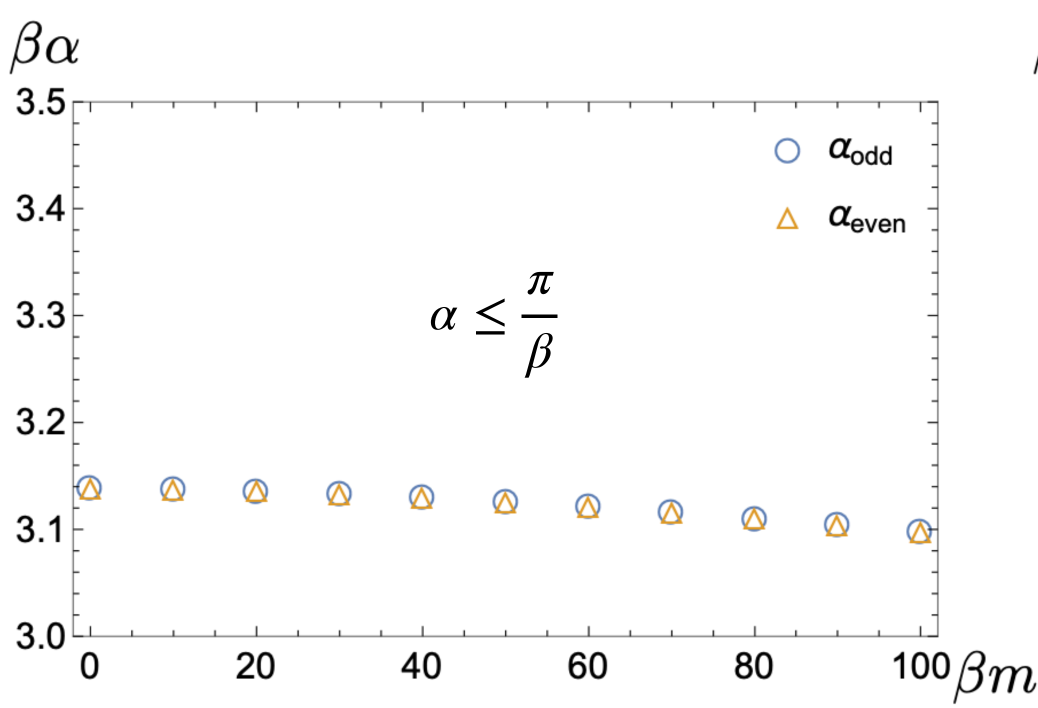




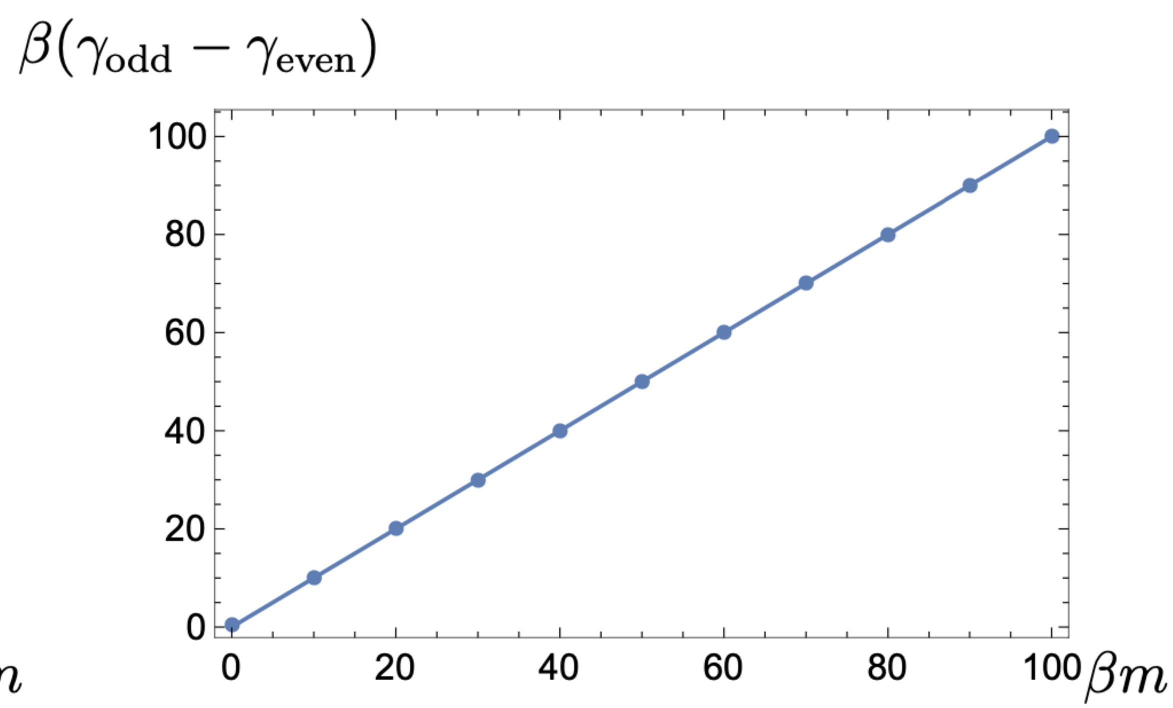
Staggering

$$b_n \sim \alpha_{\text{odd}}\, n + \gamma_{\text{odd}} \quad (\text{odd } n)$$
$$b_n \sim \alpha_{\text{even}}\, n + \gamma_{\text{even}} \quad (\text{even } n)$$

Need to take into account
Low frequency behavior

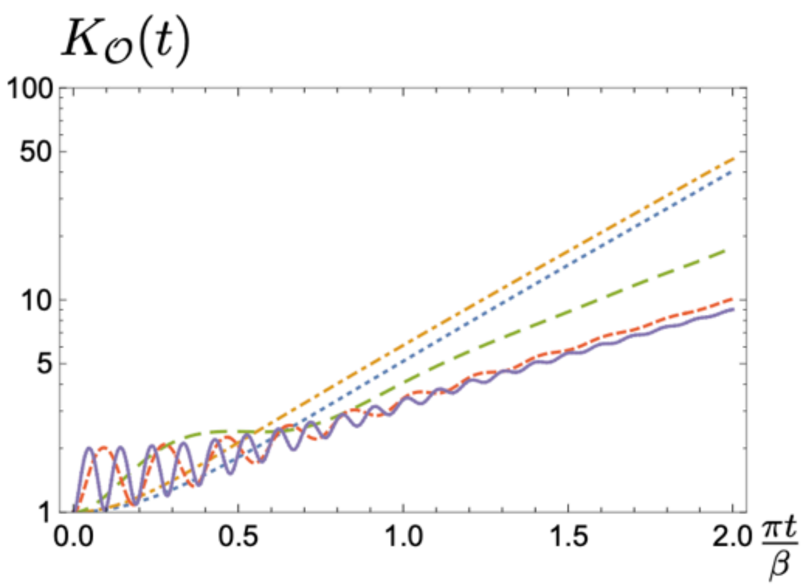


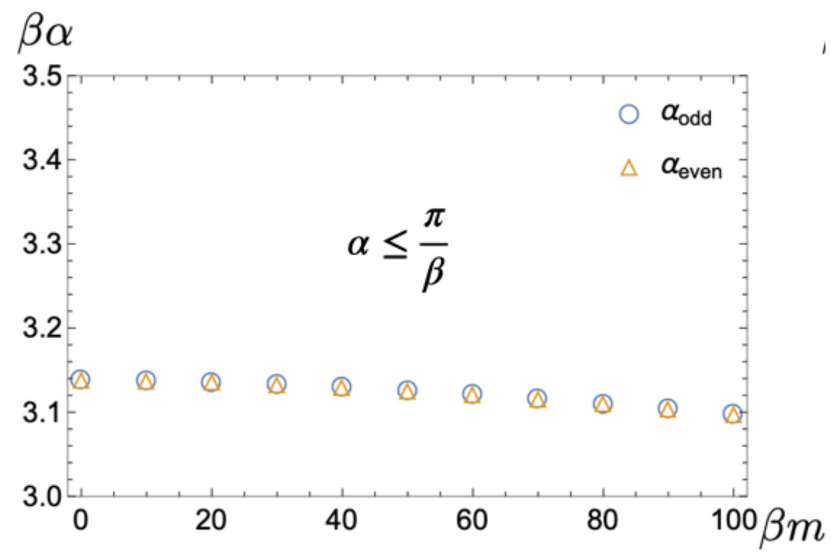
(a) Mass-dependence of α_{odd} and α_{even}

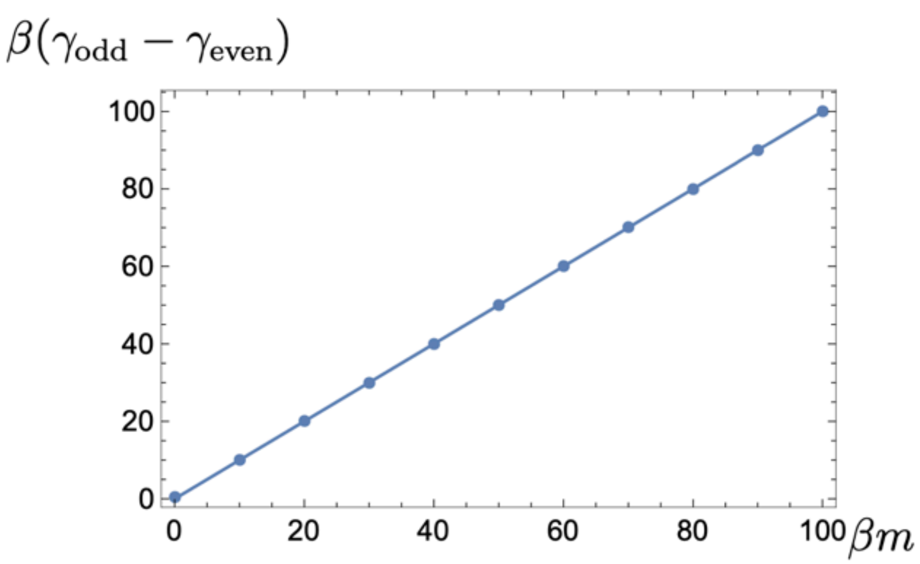


(b) Mass-dependence of $\gamma_{\text{odd}} - \gamma_{\text{even}}$

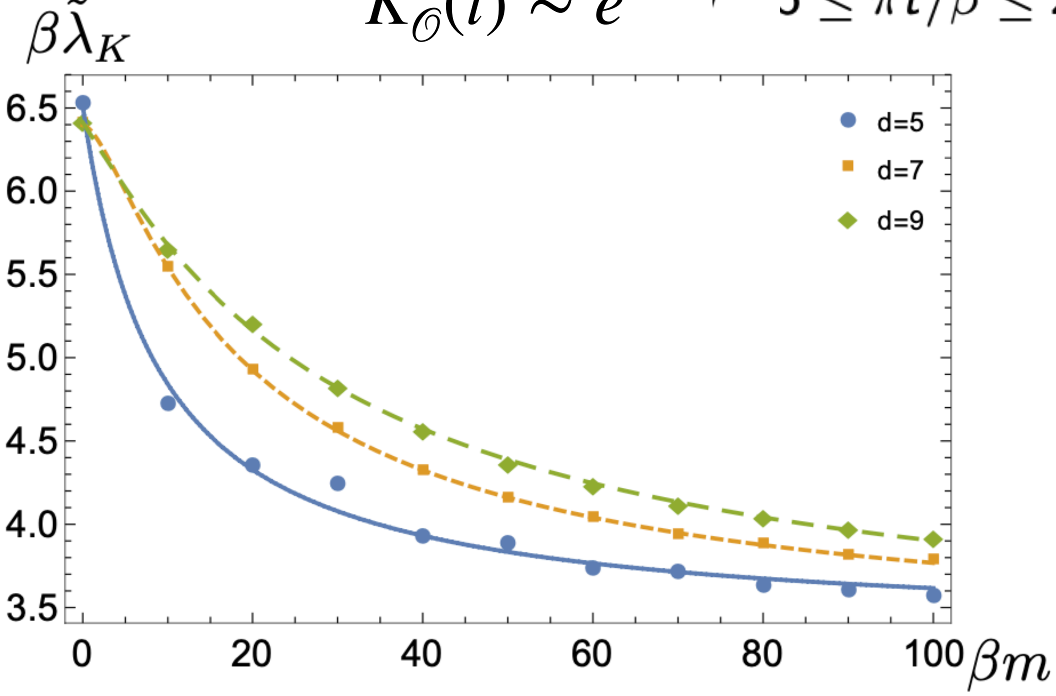
Summary (QFT)







$K_{\mathcal{O}}(t) \sim e^{\tilde{\lambda} t}$, $5 \leq \pi t/\beta \leq 2$.



$$\beta\tilde{\lambda}_K^{(d)} = \beta(\alpha_{\text{odd}} + \alpha_{\text{even}}) + k_2^{(d)}\left(\frac{1}{k_3^{(d)} + \beta|\gamma_{\text{odd}} - \gamma_{\text{even}}|} - \frac{1}{k_3^{(d)}}\right) + k_4^{(d)}\left(\frac{1}{\left(k_3^{(d)} + \beta|\gamma_{\text{odd}} - \gamma_{\text{even}}|\right)^2} - \frac{1}{(k_3^{(d)})^2}\right),$$

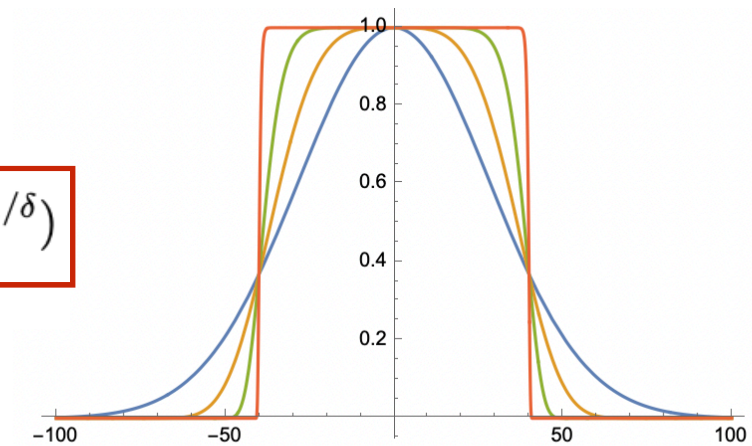
$$f^W(\omega) \sim e^{-\frac{\pi|\omega|}{2\alpha}} \Longleftrightarrow b_n \sim \alpha n \Longleftrightarrow K_{\mathcal{O}}(t) \sim e^{2\alpha t}$$

Only if b_n is a smooth function of n, Otherwise

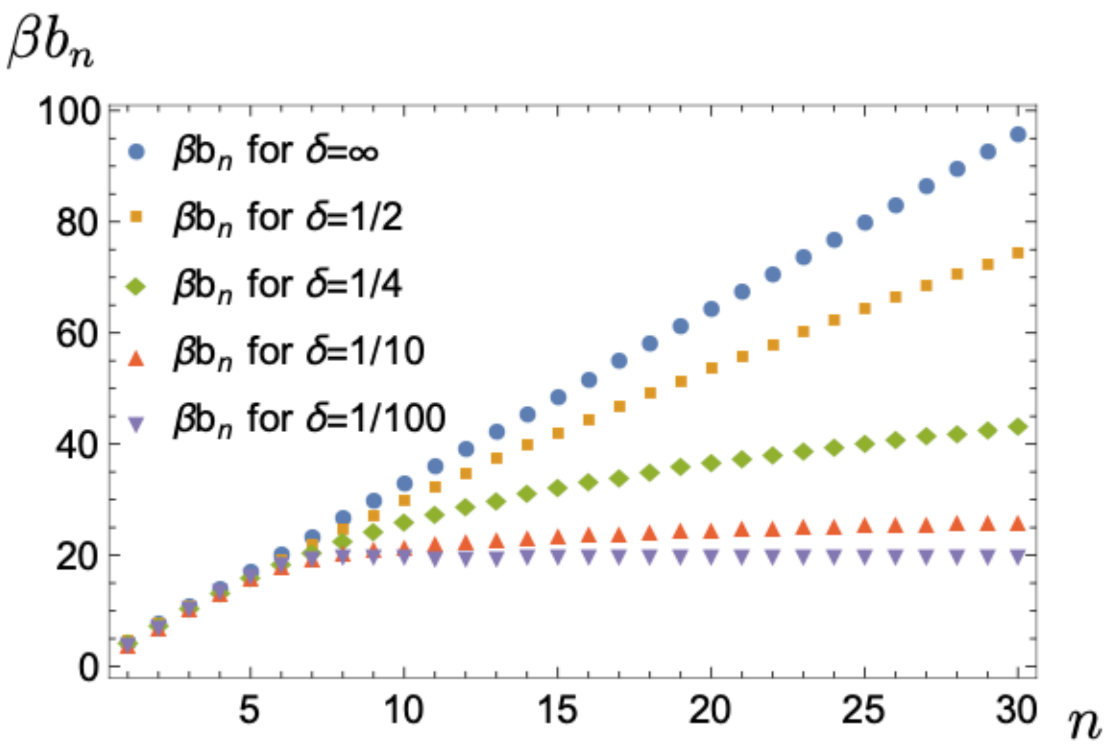
$$f^W(\omega) \sim e^{-\frac{\pi|\omega|}{2\alpha}} \not\Longleftrightarrow b_n \not\sim \alpha n \not\Longleftrightarrow K_{\mathcal{O}}(t) \sim e^{2\alpha t}$$

m=0, d=4

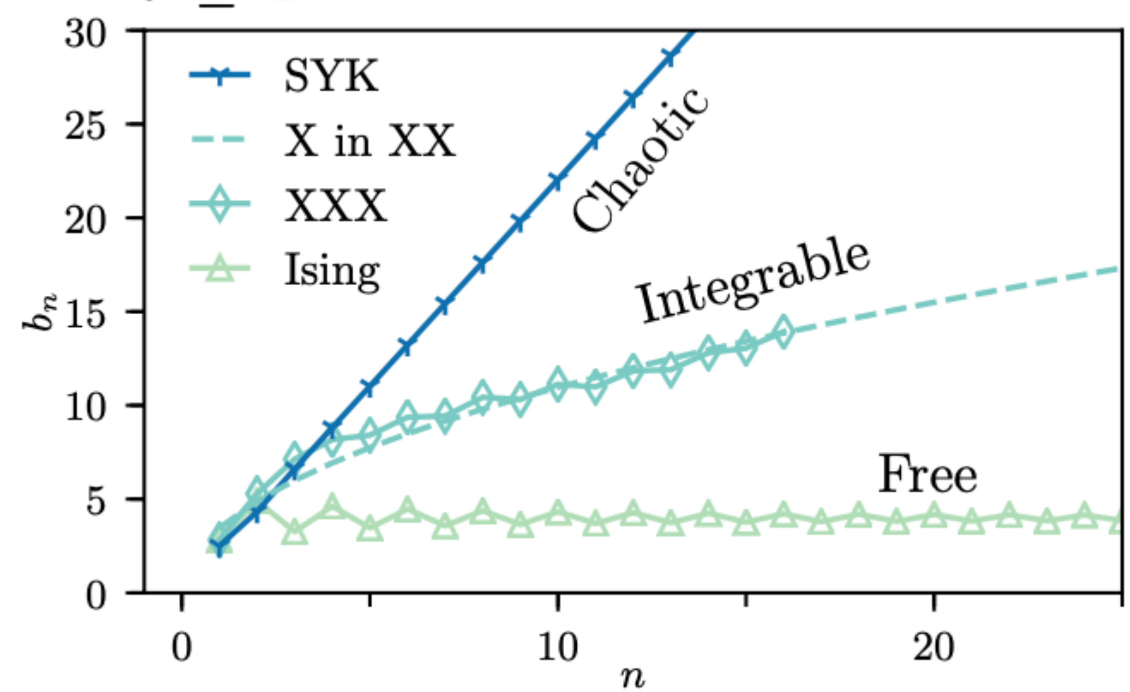
$$f^W(\omega) = N(\beta,\Lambda,\delta)\frac{\omega}{\sinh(\frac{\beta\omega}{2})}\boxed{\exp(-|\omega/\Lambda|^{1/\delta})}$$



$$b_n \sim n^\delta \Longleftrightarrow f^W(\omega) \sim \exp(-|\omega/\omega_0|^{1/\delta})$$



$\delta \leq 1$



$\beta\Lambda = 40.$

- Is it possible to extract the chaos-info from a C(t) or a power spectrum?
- More scales: compact space, interaction, other spins, open systems etc
- Holographic counterpart?
- State (spread) complexity?
- Observations, conjectures, mathematical justification

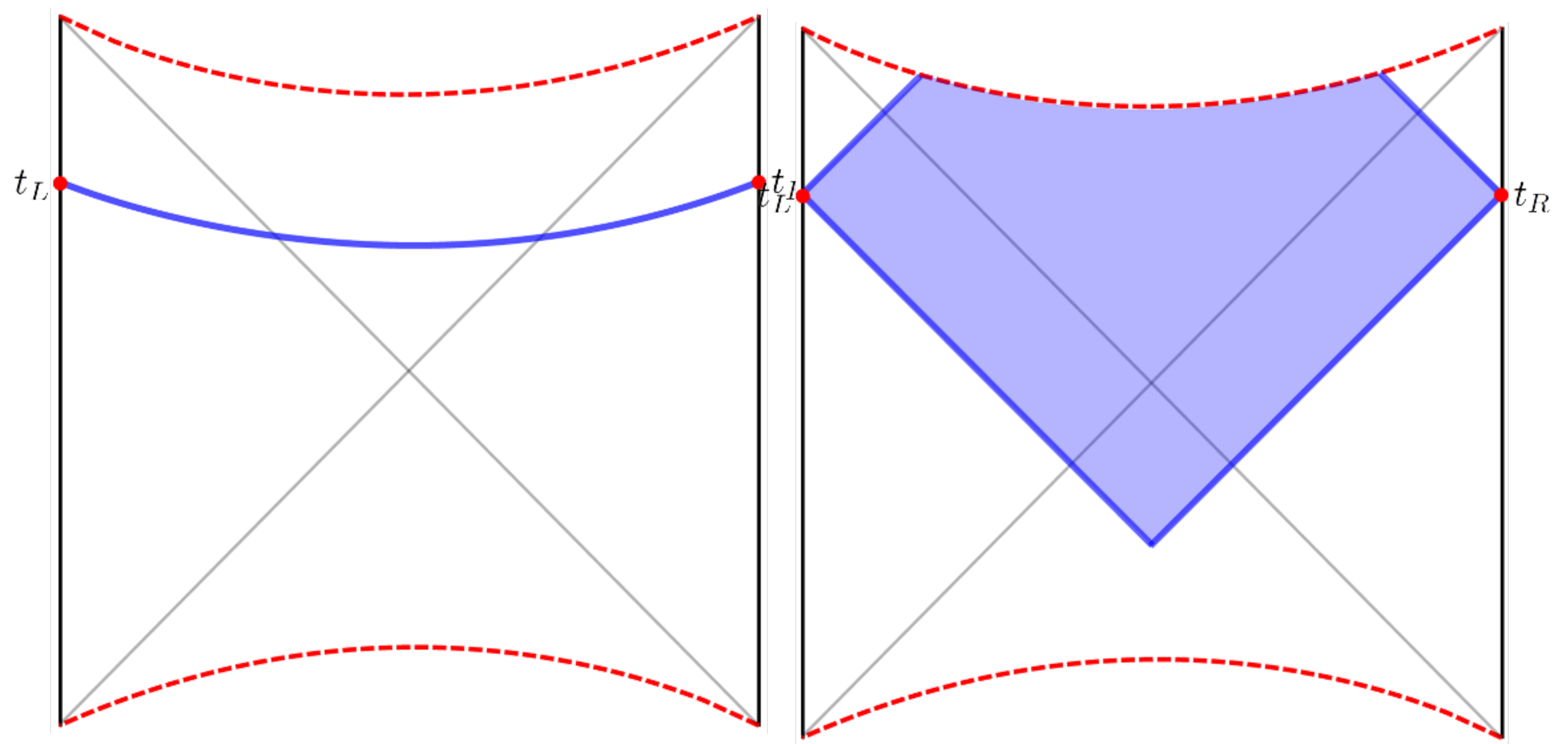
Holographic conjecture

CV (complexity-volume)

CA (complexity-action)

[Susskind: 1402.5674
Stanford and Susskind: 1406.2678]

[Brown, Roberts, Susskind Swingle and Zhao:
1509.07876, 1512.04993]



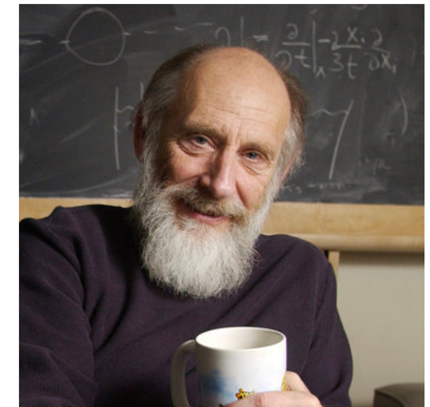
$$\mathcal{C}_V = \max_{\partial\Sigma = t_L \cup t_R} \left[\frac{V(\Sigma)}{G_N \ell} \right]$$

- Equation of motion
- Free scale: ambiguity

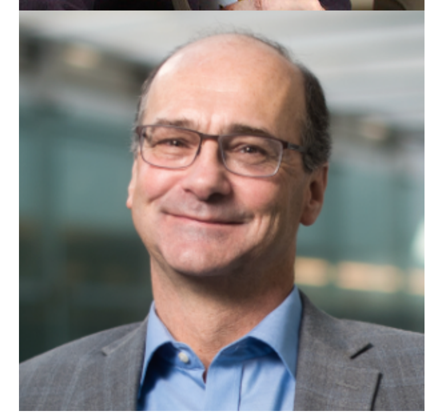
$$\mathcal{C}_A = \frac{I_{\mathrm{WDW}}}{\pi \hbar}$$

- Boundary terms
- Singularity

Fig. from [Jefferson, Myers: 1707.08570]



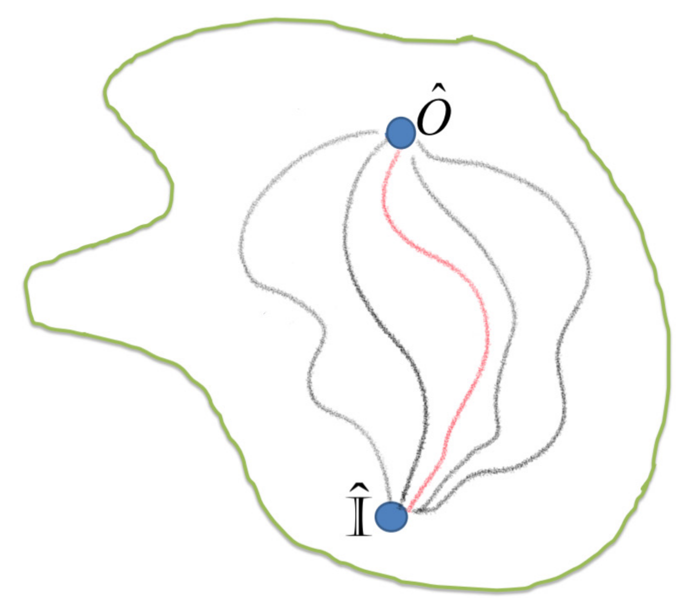
2014
Entanglement is not enough!
Black hole interior?



2017
Complexity in Field theory?

2020
decay of activities

Complexity = geodesic length?



$$F^2 \sim \mathrm{Tr} H H^\dagger \sim H^a H^b \mathrm{Tr} T_a T_b \sim H^a H^b g_{ab}$$

$$F^2 \sim \mathrm{Tr} H M H^\dagger \sim H^a H^b \mathrm{Tr} T_a M T_b \sim H^a H^b g^M_{ab}$$