Modular Average and Weyl Anomaly in Two-Dimensional Schwarzian Theory

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History

CS Formulation
Witten 1988

Liouville Theory
Coussaert, Henneaux, van Driel 1995

Monster CFT
Witten 2007

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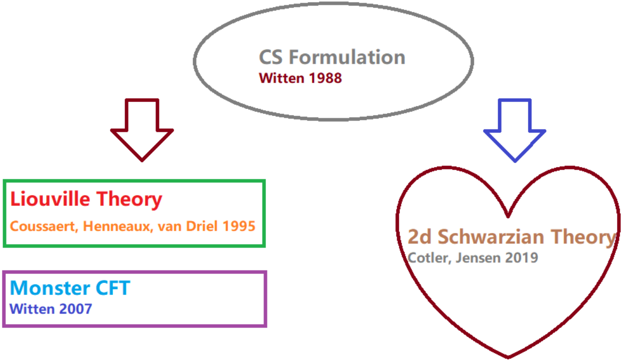
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2d Schwarzian Theory
Cotler, Jensen 2019

Gauge Formulation of Einstein Gravity Theory

- SL(2)×SL(2) Chern-Simons formulation

$$\frac{k}{2\pi}\int d^3x\ \mathrm{Tr}\Big(A_tF_{r\theta}-\frac{1}{2}(A_r\partial_tA_\theta-A_\theta\partial_tA_r)\Big)$$
$$-\frac{k}{2\pi}\int d^3x\ \mathrm{Tr}\Big(\bar{A}_t\bar{F}_{r\theta}-\frac{1}{2}(\bar{A}_r\partial_t\bar{A}_\theta-\bar{A}_\theta\partial_t\bar{A}_r)\Big)$$
$$-\frac{k}{4\pi}\int dtd\theta\ \mathrm{Tr}\Big(\frac{E_t^+}{E_\theta^+}A_\theta^2\Big)+\frac{k}{4\pi}\int dtd\theta\ \mathrm{Tr}\Big(\frac{E_t^-}{E_\theta^-}\bar{A}_\theta^2\Big)$$

- $k\equiv l/(4G_3)$, $1/l^2\equiv-\Lambda$, where Λ is a cosmological constant
- bulk metric $g_{\mu\nu}=2\mathrm{Tr}(e_\mu e_\nu)$, where $A-\bar{A}\sim e$
- boundary conditions for the torus:

$$(E_\theta^+A_t-E_t^+A_\theta)|_{r\to\infty}=0;\ (E_\theta^-\bar{A}_t-E_t^-\bar{A}_\theta)|_{r\to\infty}=0 \tag{1}$$

Boundary Theory for Torus

- two-dimensional Schwarzian theory [Cotler, Jensen 2019]

$$
\begin{aligned}
& S_{\mathrm{Gb}} \\
= \quad & \frac{k}{2\pi}\int dt d\theta\, \left(\frac{3}{2}\frac{(D_-\partial_\theta\mathcal{F})(\partial_\theta^2\mathcal{F})}{(\partial_\theta\mathcal{F})^2}-\frac{D_-\partial_\theta^2\mathcal{F}}{\partial_\theta\mathcal{F}}\right) \\
& -\frac{k}{2\pi}\int dt d\theta\, \left(\frac{3}{2}\frac{(D_+\partial_\theta\bar{\mathcal{F}})(\partial_\theta^2\bar{\mathcal{F}})}{(\partial_\theta\bar{F})^2}-\frac{D_+\partial_\theta^2\bar{\mathcal{F}}}{\partial_\theta\bar{\mathcal{F}}}\right),
\end{aligned}
\quad (2)
$$

where

$$
\mathcal{F}\equiv\frac{F}{E_\theta^+}, \qquad \bar{\mathcal{F}}\equiv\frac{\bar{F}}{E_\theta^-}; \quad (3)
$$

$$
D_+=\frac{1}{2}\partial_t-\frac{1}{2}\frac{E_t^-}{E_\theta^-}\partial_\theta, \qquad D_-=\frac{1}{2}\partial_t-\frac{1}{2}\frac{E_t^+}{E_\theta^+}\partial_\theta \quad (4)
$$

- measure $\int d\mathcal{F}d\bar{\mathcal{F}}\ \left(1/(\partial_\theta\mathcal{F}\partial_\theta\bar{\mathcal{F}})\right)$

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$$-\frac{k}{4\pi}\int dtd\theta\ \mathrm{Tr}(A_\theta A_\theta + \bar{A}_\theta \bar{A}_\theta) + \frac{k}{8\pi}\int dtd\theta\ (A_\theta^2 A_\theta^2 + \bar{A}_\theta^2 \bar{A}_\theta^2) \quad (5)$$

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- implement the averaging of a modular group to compute the Rényi-2 mutual information for disjoint two-intervals
- non-perturbative effect kills the phase transition

Rényi-2 Mutual Information

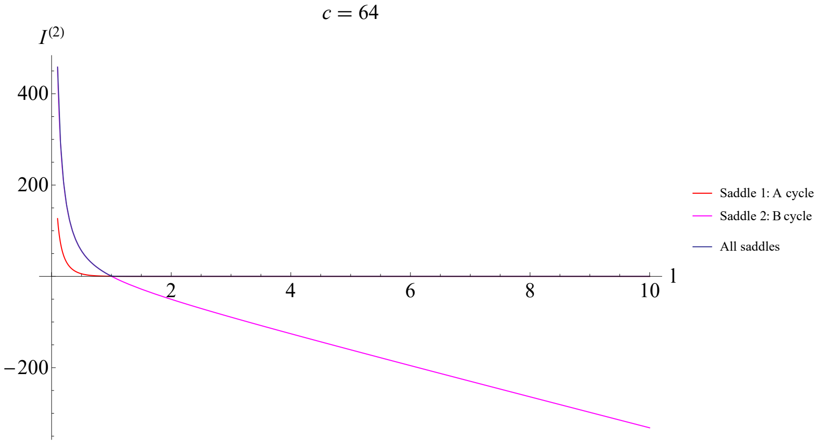

Figure: We show the Rényi-2 mutual information from summing all saddle points, A-cycle, and B-cycle with $c = 64$. Not all saddle points decay to zero when $l \equiv -i\tau \to \infty$ (like B-cycle), where τ is a complex structure.

Discussion and Conclusion

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- non-perturbative effect kills the phase transition
- boundary theory=resummation of perturbative gravity

Thank you!

Logarithmic Partition Function

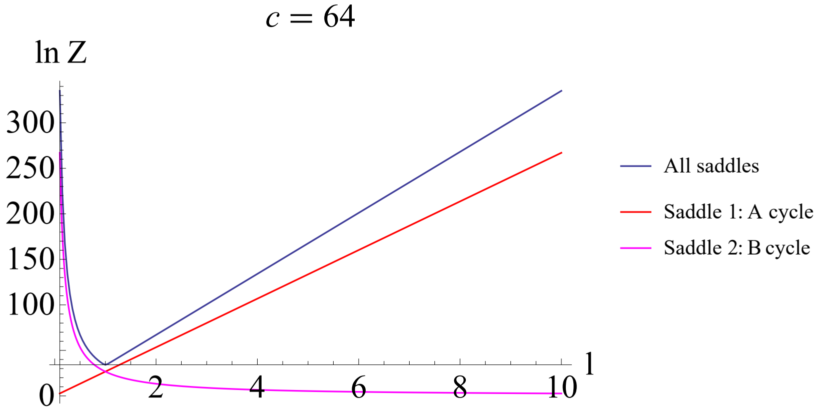

Figure: The logarithmic partition function is continuous for l.

First-Order Derivative

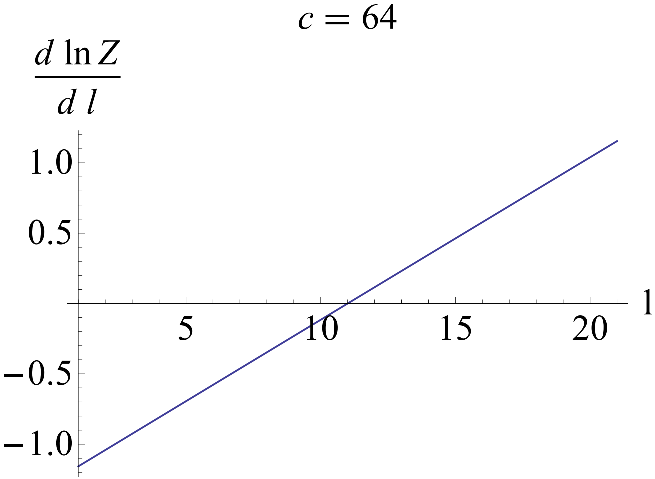

Figure: The first-order derivative of the logarithmic partition function is a continuous function for l.

Modular Averaging

- partition function of 2d Schwarzian theory on a torus is [Cotler, Jensen 2019]

$$Z_T(\tau) = |q|^{-\frac{c}{12}} \frac{1}{\prod_{n=2}^{\infty} |1-q^n|^2}, \;\; q \equiv e^{2\pi i \tau}, \tag{6}$$

where τ is a complex structure

- summation is over the SL(2, $\mathbb{Z}$) group (or modular group) in the boundary theory, which is equivalent to the path integration for all asymptotic AdS$_3$ boundary [Maloney, Witten 2010]

$$Z_M(\tau) = \sum_{c_1, d_1;\ (c_1, d_1)=1} Z_T\left(\frac{a_1\tau + b_1}{c_1\tau + d_1}\right), \tag{7}$$

where $a_1 d_1 - b_1 c_1 = 1, \;\; a_1, b_1, c_1, d_1 \in \mathbb{Z}$

Rényi-2 Mutual Information

- Rényi entropy of order q is

$$R^{(q)} \equiv \frac{\ln \mathrm{Tr}\rho^q}{1-q} = \frac{\ln Z^{(q)} - q \ln Z^{(1)}}{1-q}, \tag{8}$$

where ρ is a reduced density matrix, and $Z^{(q)}$ is a q-sheet partition function

- Rényi mutual information of order q is

$$I^{(q)}_{[u_1,v_1],[u_2,v_2]} = R^{(q)}_{[u_1,v_1]} + R^{(q)}_{[u_2,v_2]} - R^{(q)}_{[u_1,v_1]\cup[u_2,v_2]} \tag{9}$$

- Rényi-2 mutual information is [Headrick 2010]

$$I^{(2)} = \ln Z_T(\tau) + c \ln\left(\frac{\theta_2(\tau)}{2\eta(\tau)}\right) \tag{10}$$

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- modular averaging is equivalent to replacing Z_T with Z_M