Stochastic dynamics and holography

Jae-Hyuk Oh, Hanyang University

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1 My memory with Prof. Chaiho Rim

- J. H. Oh, J. Park and C. Rim, "Annulus amplitude of FZZT branes revisited," [arXiv:1109.5465 [hep-th]].
- Frequently visited Geo-Gu-Jang to have dinner or lunch during the period of collaboration

2 I do not get into holography yet.

- A chance to have discussion with Prof. Kim Ki-seok in POSTECH(at Field 23.4 in Hanyang)
- Now I am learning related stuffs. This talk is based on "Udo Seifert, Phys. Rev. Lett. **95**.040602" and Kirone Mallick et. al.,"J. Phys. A: Theor. **44**(2011) 095002.

3 Entropy Production of a Single Trajectory

Stochastic dynamics describes irreversible process, where particle's energy is dissipated into its surrounding medium. One might guess if the stochastic process is related to thermodynamic second law, if entropy does not decrease as the system progresses. Even for a single particle injected into a medium, one can discuss this as follows.

Consider a stochastic process of a single particle being given by

$$\dot{x} \equiv \frac{\partial x}{\partial \tau} = -\mu F\left(x, \lambda(\tau)\right) + \zeta(\tau), \tag{1}$$

where $\lambda(\tau)$ is a reparametrization of time τ, F is the force, and ζ is a Gaussian noise. The force F is given by

$$F\left(x, \lambda(\tau)\right) = -\partial_x V\left(x, \lambda(\tau)\right) + f\left(x, \lambda(\tau)\right), \tag{2}$$

where V is a potential and f is an external force. The Gaussian noise ζ satisfies

$$\langle \zeta(\tau)\zeta(\tau')\rangle = 2D\delta(\tau - \tau'), \tag{3}$$

where D is the diffusion constant. Einstein relation, $D = \mu T$.

Expectation value $x(\tau)$ is the particle trajectory. For a operator $\mathcal{O}$, its expectation value is defined by

$$\langle \mathcal{O}(\tau)\rangle = \int dx \mathcal{O}(x, \tau) p(x, \tau), \tag{4}$$

where $p(x, \tau)$ is the probability distribution to find the particle at x at given time τ.

Fokker-Planck equation Equivalently, one can derive Fokker-Planck equation, which describe evolution of the probability $p(x,\tau)$ along the trajectory. This is given by a form of conservation equation for $p(x,\tau)$:

$$\partial_\tau p(x,\tau) = -\partial_x j(x,\tau) = -\partial_x \left\{ \left(\mu F\left(x,\lambda\right) - D\partial_x\right) p(x,\tau) \right\}, \tag{5}$$

The j is the probability current being given by

$$j(x,\tau) = \mu F\left(x,\lambda\right) p(x,\tau) - D\partial_x p(x,\tau). \tag{6}$$

Now consider the entropy expectation value

$$S(\tau) \equiv \langle s(\tau) \rangle = -\int dx p(x,\tau) \log p(x,\tau), \tag{7}$$

where

$$s(\tau) = -\log p(x,\tau). \tag{8}$$

The entropy production is written as

$$\dot{s}(\tau) = -\frac{\partial_\tau p(x,\tau)}{p(x,\tau)} - \frac{\partial_x p(x,\tau)}{p(x,\tau)} \dot{x}(\tau) \tag{9}$$

$$= -\frac{\partial_\tau p(x,\tau)}{p(x,\tau)} + \frac{\dot{x}(\tau)}{p(x,\tau)} \left(\frac{j(x,\tau)}{D} - \frac{\mu F(x,\lambda)}{D} p(x,\tau) \right). \tag{10}$$

We may interpret the last term as a dissipation heat from the particle to the medium, which can be written as

$$T\dot{s}_m(\tau) = F(x,\lambda)\frac{dx}{d\tau} = \frac{dW}{d\tau}, \tag{11}$$

where $\dot{s}_m$ is the entropy increase of the medium, and $T = D/\mu$ is the temperature.

Hence we may define the total entropy production as

$$\dot{S}_{\text{tot}} \equiv \langle \dot{s}_{\text{tot}}(\tau) \rangle \equiv \int dx \dot{s}_{\text{tot}} p(x,\tau) \tag{12}$$

$$= \int dx \frac{\partial_\tau p(x,\tau)}{p(x,\tau)} p(x,\tau) + \int dx \frac{j(x,\tau)}{Dp(x,\tau)} \dot{x} p(x,\tau) \tag{13}$$

$$= \int dx \partial_x j(x,\tau) + \int dx \frac{j(x,\tau)}{D} \frac{j(x,\tau)}{p(x,\tau)} \tag{14}$$

$$= \int dx \frac{j^2(x,\tau)}{Dp(x,\tau)} \geq 0, \tag{15}$$

where we use $j(x,\tau) = p(x,\tau)\dot{x}(\tau)$. The total entropy production is positive semi-definite and the equality holds if it is a reversible process.

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4 Fluctuation Theorem

This entropy production may be related to fluctuation theorem. What we interested in is the ratio of probability of a single particle's path to it time reversed probability.

Time reversal We define the time reversal quantities by

$$\tilde{\lambda}(\tau) = \lambda(t-\tau), \tag{16}$$
$$\tilde{x}(\tau) = x(t-\tau), \tag{17}$$

with initial and final boundary conditions

$$x_0 \equiv x(0) = \tilde{x}_t \equiv \tilde{x}(t), \tag{18}$$
$$x_t \equiv x(t) = \tilde{x}_0 \equiv \tilde{x}(0). \tag{19}$$

Ratio of conditional probabilities to its time reversed one Consider the following quantity:

$$\Delta s_m = \int_0^t d\tau \frac{F(x,\tau)}{T}\dot{x} = \log \frac{p[x(t)|x_0]}{\tilde{p}[\tilde{x}(t)|\tilde{x}_0]}, \tag{20}$$

where $p[x(t)|x_0]$ denotes the probability to find the particle at position x at time $\tau = t$ given that the particle was found at position x_0 at time $\tau = 0$.

Define a quantity R by

$$R[x(\tau),\lambda(\tau);p_0,p_1] \equiv \log \frac{p[x(t)|x_0]p_0(x_0)}{\tilde{p}[\tilde{x}(t)|\tilde{x}_0]p_1(\tilde{x}_0)} = \Delta s_m + \log\frac{p_0(x_0)}{p_1(x_t)} = \Delta s_m + \Delta s. \tag{21}$$

Then the expectation value of e^{-R} is

$$\langle e^{-R}\rangle \equiv \int dx p[x(\tau)]e^{-R} = \sum_{\text{all possible } x(\tau)} p(x(\tau))e^{-R} \tag{22}$$
$$= \sum_{x(\tau),x_0} p[x(\tau)|x_0]\ p_0(x_0)e^{-R} \tag{23}$$
$$= \sum_{\tilde{x}(\tau),\tilde{x}_0} \tilde{p}[\tilde{x}(\tau)|\tilde{x}_0]\ \tilde{p}_0(\tilde{x}_0) \tag{24}$$
$$= 1, \tag{25}$$

where the third line is obtained by the definition of R. This statement is nothing but

- If $x(\tau)$ satisfies Fokker-Plank equation, $p_1(x_t) = p(x,\tau)$, and then

$$\langle e^{-[\Delta s_m + \Delta s]} \rangle = \langle e^{-\Delta s_{\text{tot}}} \rangle = 1 \tag{26}$$

- This also means that $\langle \Delta s_{\text{tot}} \rangle \geq 0$.

5 Appendix

Definition of conditional probability

$$p_{l|k}\left(y_{k+1}, t_{k+1}; \ldots; y_{k+l}, t_{k+l} \mid y_1, t_1; \ldots; y_k, t_k\right) \equiv \frac{p_{k+l}(y_1, t_1; \ldots; y_{k+l}, t_{k+l})}{p_k(y_1, t_1; \ldots; y_k, t_k)}. \tag{27}$$

Definition of noise probability

$$p(\eta) = \frac{\exp\left(-\frac{1}{4D}\int d\tau \eta^2(\tau)\right)}{\int D\eta \exp\left(-\frac{1}{4D}\int d\tau \eta^2(\tau)\right)}. \tag{28}$$

5.1 The ratio of conditional probabilities

Consider an identity of $1 = \int_{x(0)=x_0} d\zeta(\tau)\delta[c-\zeta]$ for an arbitrary function c. Then, we modify this identity as

$$1 = \int_{x(0)=x_0} dx(\tau)\delta\left(\dot{x} - \mu F(x,\lambda) - \zeta\right)\mathcal{J}\left(\frac{\delta\zeta}{\delta x}\right) \tag{29}$$

$$= \int_{x(0)=x_0} dx(\tau)d\bar{x}(\tau)\exp\left\{-\int d\tau \bar{x}\left[\dot{x} - \mu F(x,\lambda) - \zeta\right]\right\}\mathcal{J}\left(\frac{\delta\zeta}{\delta x}\right), \tag{30}$$

where $\mathcal{J}$ denotes the Jacobian factor.

Conditional probability The conditional probability to find the particle at $x = x_1$ under the condition that it was at $x = x_0$ is given by

$$p(x_1|x_0) \equiv \int \mathcal{D}\zeta \exp\left[-\frac{1}{4D}\int d\tau \zeta^2(\tau)\right] \delta(x(t_f) - x_1) \times 1 \tag{31}$$

$$= \int \mathcal{D}\zeta \exp\left[-\frac{1}{4D}\int d\tau \zeta^2(\tau)\right] \delta(x(t_f) - x_1) \tag{32}$$

$$\times \int_{x(0)=x_0} dx(\tau) d\bar{x}(\tau) \exp\left\{-\int d\tau \bar{x}\left[\dot{x} - \mu F(x,\lambda) - \zeta\right]\right\} \mathcal{J}\left(\frac{\delta\zeta}{\delta x}\right).$$

Perform the Gaussian integral with the noise field ζ first as

$$\int \mathcal{D}\zeta \exp\left[-\frac{1}{4D}\int d\tau \left(\zeta^2(\tau) + 4D\bar{x}\zeta\right)\right] = \int \mathcal{D}\zeta \exp\left[-\frac{1}{4D}\int d\tau \left(\zeta(\tau) + 2D\bar{x}\right)^2\right] e^{D\int d\tau \bar{x}^2}. \tag{33}$$

We also use

$$\frac{\delta\zeta}{\delta x(\tau')} = \frac{d}{d\tau}\left(\delta(\tau - \tau')\right) - \mu\frac{\delta F(x,\lambda)}{\delta x}, \tag{34}$$

With all these, we have (32) becomes

$$p(x_1|x_0) = \int_{x(0)=x_0} [dx(\tau)] d\bar{x}(\tau) \exp\left\{-\int d\tau \bar{x}\left[\dot{x} - \mu F(x,\lambda) - D\bar{x}\right] - \mu\frac{\delta F}{\delta x}\right\} \delta(x(t_f) - x_1), \tag{35}$$

where $[dx(\tau)] \equiv \prod_{x(\tau)} \{\delta(x(\tau) - x^s(\tau))\} dx(\tau)$ for a given trajectory of $x^s(\tau)$.

Now, we consider the following transform $\bar{x}$ into

$$\bar{x} \longrightarrow -\bar{x} - \frac{\mu F}{D}. \tag{36}$$

Then, the terms at the square bracket in the exponent of (35) transform as

$$\bar{x}\left[\dot{x} - \mu F(x,\lambda) - D\bar{x}\right] \longrightarrow \bar{x}\left[-\dot{x} - \frac{\mu F}{D} - D\bar{x}\right] - \frac{\mu F}{D}\dot{x}. \tag{37}$$

Now let us apply time reversal transform as $t \longrightarrow t_f - t$. Then, $\dot{x} \longrightarrow -\dot{\tilde{x}}$, and so we have

$$\tilde{p}(x_1|x_0) = \int_{\tilde{x}(0)=x_1}^{\tilde{x}(t_f)=x_0} [d\tilde{x}(\tau)] d\bar{x}(\tau) \exp\left[-\int d\tau \bar{x}\left(\dot{\tilde{x}} - \frac{\mu F}{D} - \bar{x}\right)\right] e^{\int_0^t d\tau \frac{\mu F}{D}\dot{x}} \tag{38}$$

$$= \tilde{p}(x_0|x_1) \exp\left\{\int_0^t d\tau \frac{\mu F}{D}\dot{x}\right\}$$

$$= \tilde{p}(\tilde{x}_1|\tilde{x}_0) \exp\int_0^t d\tau \frac{F}{T}\dot{x}$$

The second equality is hold for any given(single) path, $x(\tau)$ and so $\tilde{x}(\tau)$.
