More on Canonical Quantization in 2-Dimensional Curved Spaces

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based on works

[arXiv:2206.13210] EPJP 138 (2023) 3 & [arXiv:2211.05699]
with Jeongwon Ho, O-Kab Kwon and Sang-A Park

[arXiv:2303.05057] JHEP 05 (2023) 045
with Dongsu Bak and Chanju Kim

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- Canonical quantization of Jackiw-TeitelboimI(JT) gravity
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- Discussion: von Neumann algebra and etc

JT Gravity

Jackiw-Teitelboim model with a massive scalar field $\quad (8\pi G = 1, \ell = 1)$

$$I = \frac{1}{2}\int_M d^2x\sqrt{-g}\,\phi\,(R+2) + I_{\mathrm{surf}} + I_m(g,\varphi)\,,$$

where

$$\begin{aligned} I_{\mathrm{surf}} &= \int_{\partial M} du\sqrt{-\gamma_{uu}}\,\phi\,(K-1)\,,\\ I_m &= -\frac{1}{2}\int_M d^2x\sqrt{-g}\,\left(g^{ab}\nabla_a\varphi\nabla_b\varphi + m^2\varphi^2\right)\,. \end{aligned}$$

The equations of motion (The dilaton field ϕ plays the role of the Lagrange multiplier)

$$\begin{aligned} R+2 = 0\,, &\qquad \Longrightarrow \qquad \mathrm{AdS}_2 \ \text{space}\\ \nabla_a\nabla_b\phi - g_{ab}\nabla^2\phi + g_{ab}\phi = -T_{ab}\,,&\\ \nabla^2\varphi - m^2\varphi = 0\,,& \end{aligned}$$

where $T_{ab} = \nabla_a\varphi\nabla_b\varphi - \frac{1}{2}g_{ab}\left(\nabla\varphi\cdot\nabla\varphi + m^2\varphi^2\right)$

Canonical Quantization

Vacuum solution (matter $\varphi = 0$)

$$\mathrm{ds}^2 = \frac{-\mathrm{d}\tau^2 + \mathrm{d}\mu^2}{\cos^2\mu}\,, \qquad \phi = \bar{\phi}\,\mathrm{L}\;\frac{\cos\tau}{\cos\mu}$$

where the dilaton ϕ specifies the horizon:

AdS_2 r/l cutoff boundaries

Along the AdS/CFT correspondence, we need some cutoff. The prescription for metric and dilaton in this case to get the cutoff boundary

$$\mathrm{ds}^2|_{\text{cutoff}} = -\frac{1}{\epsilon^2}\mathrm{du}^2\,, \qquad \phi|_{\text{cutoff}} = \frac{\bar{\phi}}{\epsilon}\,, \qquad \epsilon \ll 1$$

-continued

The boundary dynamics may be identified as a combination of (right and left) Schwarzian theories [Maldacena et al + Jensen + Engelsoy et al]

$$S = \int du\, L_r + \int du\, L_l\,, \qquad L_{r/l} = \frac{C}{2}\left[\left(\frac{\tau''_{r/l}}{\tau'_{r/l}}\right)^2 - \tau'^2_{r/l}\right], \qquad C = \bar{\phi}$$

Adding Lagrange multiplier terms $p_{\tau_{r/l}}(\tau'_{r/l} - e^{\chi_{r/l}}/C)$, one can also obtain Hamiltonians respectively by

$$H_{r/l} = \frac{1}{2C}\left[p^2_{\chi_{r/l}} + 2p_{\tau_{r/l}}\, e^{\chi_{r/l}} + e^{2\chi_{r/l}}\right].$$

The linear dependence of $H_{r/l}$ in $p_{\tau_{r/l}}$ tells us that

$H_{r/l}$ are not bounded from below,

which may be viewed as an indication of instability of the system.

cf. Tentative phase space variables are $(\tau_{\text{rł}}, p_{\tau_{r/l}}, \chi_{r/r}, p_{\chi_{r/l}})$

-continued

This is, of course, the well-known aspect of higher derivative theory.

However, in the present case, there would be a gauge symmetry $\widetilde{SL}(2, \mathbf{R})$, which ensures the total Hamiltonian becomes positive on physical Hilbert space. For a math-oriented audience, we are doing the symplectic quotient $\mathcal{M}//\widetilde{SL}(2,\mathrm{R})$

Concretely, the AdS_2 space has an $SL(2, R)$ symmetry under the isometric coordinate transformations that are generated by Killing vectors

$$\begin{aligned}\xi_1 &= -\partial_\tau\,,\\ \xi_2 &= -\cos\tau\sin\mu\,\partial_\tau - \sin\tau\cos\mu\,\partial_\mu\,,\\ \xi_3 &= -\sin\tau\sin\mu\,\partial_\tau + \cos\tau\cos\mu\,\partial_\mu\,.\end{aligned}$$

By the standard Noether procedure, the corresponding (quantum) $SL(2,R)$ generators may be constructed as [Jafferis + Kolchmeyer]

$$\begin{aligned}J_1^{r/l} &= p_{\tau_{r/l}}\,,\\ J_2^{r/l} &= \pm e^{\chi_{r/l}}\cos\tau_{r/l} \mp \sin\tau_{r/l}\, p_{\chi_{r/l}} \pm \cos\tau_{r/l}\, p_{\tau_{r/l}} \pm \frac{i}{2}\sin\tau_{r/l}\,,\\ J_3^{r/l} &= \pm e^{\chi_{r/l}}\sin\tau_{r/l} \pm \cos\tau_{r/l}\, p_{\chi_{r/l}} \pm \sin\tau_{r/l}\, p_{\tau_{r/l}} \mp \frac{i}{2}\cos\tau_{r/l}\,,\end{aligned}$$

-continued

It is then straightforward to check that

$$2\mathbf{C}\mathrm{H}_{r/l} = \eta^{ij} J_i^{r/l} J_j^{r/l} - \frac{1}{4}\,, \qquad \eta^{ij} = \mathrm{diag}(-1,\ 1,\ 1)\,,$$

which corresponds to the quadratic Casimir of $SL(2,\mathbf{R})$ and so ensures the $SL(2,\mathbf{R})$ invariance of the Hamiltonians.

Now, let us consider the gauge constraint:

$\widetilde{SL}(2,\mathbf{R})$ gauge symmetry is generated by

$$\widetilde{J}_i = J_i^r + J_i^l\,, \qquad i = 1, 2, 3$$

which leaves the full geometry, including the cutoff boundaries, invariant.

After quantization, physical states should have zero gauge charges.

-continued

Now, one can perform a canonical quantization of the Schwarzian theory and can construct the Hilbert space: gravity part

Matter part: Along the AdS/CFT correspondence, we impose the vanishing boundary condition for the bulk scalar as

$$\varphi\big|_{\mathsf{r/l}} = \mathcal{O}(\cos^{\Delta} \mu_{\mathsf{r/l}}) = \mathcal{O}(\epsilon^{\Delta})$$

where Δ denotes the dimension of the operator dual to the bulk matter field.

The corresponding bulk matter charges may be evaluated as

$$\mathsf{J}_{\mathsf{i}}^{\mathsf{m}} = \int_{-\pi/2}^{\pi/2} \mathsf{d}\mu\ \mathsf{T}_{\tau \mathsf{a}}\, \xi_{\mathsf{i}}^{\mathsf{a}}\,,$$

One can check that these are conserved and satisfy the $\mathsf{SL}(2,\mathbf{R})$ algebra $[\mathsf{J}_{\mathsf{i}}^{\mathsf{m}}, \mathsf{J}_{\mathsf{j}}^{\mathsf{m}}] = \mathsf{i}\epsilon_{\mathsf{ijk}}\eta^{\mathsf{kl}}\mathsf{J}_{\mathsf{l}}^{\mathsf{m}}$ even at the quantum level.

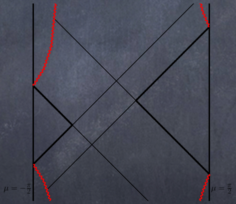
-continued

When the matter field is turned on, the gauge generators are given by

$$\widetilde{J}_i = J_i^r + J_i^l + J_i^m$$

Side Remark: When the scalar field φ is turned on,

the left/right side causal wedges are deformed and there is no left/right symmetry in the configuration (classical picture and large deformation) [Bak+Kim+S.-H.Yi]



deformed

However, we will focus on the symmetric configuration, to simplify the description: small fluctuation of

the matter field.

-continued

Explicit solution of the bulk scalar matter ($\Delta = \frac{1}{2} + \sqrt{\frac{1}{4} + m^2}$)

$$\varphi = \sum_{n=0}^{\infty} c_n N_n (\cos\mu)^{\Delta} C_n^{\Delta}(\sin\mu) e^{-i(n+\Delta)\tau} + \text{c.c.}$$

where C_n^{Δ} is the so-called Gegenbauer function

and the normalization constant N_n is fixed by A Klein-Gordon inner product as [Spradlin+Strominger]

$$N_n = 2^{n-1}\Gamma(\Delta)\sqrt{\frac{\Gamma(n+1)}{\pi\Gamma(n+2\Delta)}}$$

It is straightforward to perform canonical quantization of the bulk matter scalar field by

$$[\varphi(\tau,\mu), \pi_{\varphi}(\tau,\mu')] = i\,\delta(\mu-\mu') \qquad \Longleftrightarrow \qquad [\hat{c}_n, \hat{c}_m^{\dagger}] = \delta_{nm}$$

Vacuum $|0\rangle$ is defined by $c_n|0\rangle = 0$.

-continued

Canonical quantization of the matter scalar field is done by using global coordinates (τ, μ)

One can say that the matter (global) AdS_2 vacuum state corresponds to the Poincaré invariant vacuum in Minkowski space

and the matter Hilbert space can be constructed from this global AdS vacuum.

This construction can be thought to be related to the fact that AdS space is one of the maximal symmetric space (large isometry group)

-continued

In summary,

Canonical quantization of JT gravity + a single scalar matter:

Hilbert space $\mathcal{H}$ is composed of the tensor product of the gravity part and the matter part

$$\mathcal{H} = \mathcal{H}_{\text{grav}} \otimes \mathcal{H}_{\text{matt}}$$

However, the wave function Ψ in the Hilbert space should be constrained by the gauge constraint $\widetilde{J}^{i}\,\Psi = 0$.

For technical details See [Pennington + Witten]

SYK model

Let us review SYK model very briefly (I am not an expert on the SYK model)

Essentially, it is a solvable condensed model that describes a strongly coupled system, known to be dual to JT gravity:

Each Majorana fermion consisting of N fermions ($\psi^i,\ i=1,2,\cdots,N$) is interacting with other $q-1$ fermions by random couplings.

Concretely, its Hamiltonian is given by

$$H = i^{\frac{q}{2}} \sum_{1\le i_1, i_2, \cdots, i_q} J_{i_1,i_2,\cdots,i_q} \psi^{i_1}\psi^{i_2}\cdots\psi^{i_q}\,, \qquad \{\psi^i,\psi^j\} = \delta^{ij}\,,$$

where the coupling $J_{i_1,i_2,\cdots,i_q}$ has ensemble average

$$E\Big(J_{i_1,i_2,\cdots,i_q}\ J_{j_1,j_2,\cdots,j_q}\Big) = \frac{2^q}{2q}\frac{(q-1)!}{N^{q-1}}\mathcal{J}\delta_{i_1j_1}\delta_{i_2j_2}\cdots\delta_{i_qj_q}$$

Thermofield double of a finite quantum system:

purification of thermal system by doubling the Hilbert space

-continued

SYK model to JT gravity is composed of two parts: right and left (r/l)

or its low energy CFTs on the boundary has the structure

$$\mathrm{CFT}_l \otimes \mathrm{CFT}_r$$

However, it turns out that there is no factorization property of states in the Hilbert space of JT gravity.

At first glance, it seems that there is a big mismatch between the boundary theory and the bulk theory

factorization issues in JT theory [Harlow + Jafferis]

c.f. There are similar issues in higher dimensional AdS black holes

[Marolf + Wall]

We have provided the concrete expression of canonical quantization of JT gravity + a massless scalar field [Bak + Kim + S.-H. Yi]

Commnets

Recent progress by Witten,

(which is based on the pioneering works by [Leutheusser + Liu])

is focusing on the understanding of the field algebra structure.

Though the bulk state is not factorizable, the left/right algebra $\mathcal{A}_{\mathrm{r/l}}$ on causal (or entanglement) wedges are mutual commutant and form an appropriate von Neumann algebra factor, respectively.

For instance, the bulk algebra type III_1, which is argued to hold for any QFT in the local spacetime region, is emergent from the boundary type I algebra in the large N limit (see [Leutheusser + Liu, Witten et al])

In the context of JT gravity with bulk scalar matter, it has been argued that the bulk algebra is type II_1 factor, which can be equivalently regarded as the algebra on the cut-off boundary. [Penington+Witten]

Inhomogeneous field theory (IFT)

In the case of JT gravity, one can perform canonical quantization without much difficulty (Lorentzian method).

This result shows us that the Euclidean path integral approach is consistent with the Lorentzian results.

However, this success of the canonical quantization method seems to be based on the existence of the symmetry invariant (global) vacuum state.

Now, let us consider an example which reveals the insufficiency of the canonical approach,

which is already discussed in Dr. Ho's talk.

- continued

From the perspective of effective field theory, mass and coupling parameters may have spacetime-dependence, resulting in all or part of Poincaré symmetry breaking. In the following, let us call such theory as inhomogeneous field theory (IFT).

One may think that this spacetime dependence originates from other dynamical fields by taking their non-dynamical limits. In this regard, there are various examples in string/M-theory,

For instance, Janus (or spatially modulated) deformed theory of $\mathcal{N} = 4$ SYM and $\mathcal{N} = 6$ ABJM [Bak+Y.Kim+C.Kim+Kwon+ K.Kim+$\cdots$]

In the supersymmetric context, the promotion of mass and coupling parameters to superfield has been very useful, as emphasized by [Seiberg]

In the condensed matter/neutron physics or cosmology, the position (or time) dependence of parameters has also been investigated. However, it doesn't seem to be a general framework in QFT.

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One of the important issues in IFT is

- What is an appropriate method of quantization?

One may guess that the standard canonical quantization might be sufficient.

Though there is no essential fault in the canonical quantization,

the absence of Poincaré symmetry interrupts the blinded application of canonical quantization.

One of the difficulties is the absence of the Poincaré-invariant vacuum (or absence of large symmetry group), which makes the Fock space construction obscured. [W. Kim]

This situation is reminiscent of quantum field theory on curved spacetime (FTCS), where *algebraic* method of quantization has suggested as an appropriate one (AQFT). [Haag, Fredenhagen, Kay, Wald, etc]

Algebraic method

Basically, algebraic approach is based on the (local) operator algebra and then relevant Hilbert space is constructed from the algebra.

Algebra is determined by commutation relations, classical EOM, and various physical considerations.

Without preferred Poincaré invariant vacuum

canonical quantization should be treated more carefully.

Slogan

Hilbert space is a secondary concept, whereas the algebra of field operators is the primary object of consideration.

-continued

Canonical quantization of a scalar field ϕ: $\phi(t,x) = \sum_i \left(a_i u_i + a_i^\dagger u_i^*\right)$
where we need to find a complete orthonormal basis of solutions to EOM

satisfying the Klein-Gordon inner product relations

$$\langle u_i, u_j \rangle = \delta_{ij}\,, \quad \langle u_i^*, u_j \rangle = 0\,, \quad \langle u_i^*, u_j^* \rangle = -\delta_{ij}$$

The absence of Poincaré invariant vacuum corresponds to the non-uniqueness of the basis $\{u_i\}$.

Fock vacuum $|0\rangle_u$ is defined by $a_i|0\rangle_u = 0$

Using another basis $\{w_i\}$ $\phi(t,x) = \sum_i \left(b_i w_i + b_i^\dagger w_i^*\right)$ which allows another vacuum $|0\rangle_w$ defined by $b_i|0\rangle_w = 0$.

Hilbert space constructed from $|0\rangle_u$ may be unrelated to the one from $|0\rangle_w$, since the transformation from $\{u_i\}$ to $\{w_i\}$ is not a unitary one.

Rindler Example

In the canonical quantization approach,

the Fock space in Minkowski space $\mathcal{M}$ constructed from $|0\rangle_{\mathcal{M}}$ is not unitary equivalent to the one in Rindler space $\mathcal{R}$ constructed from $|0\rangle_{\mathcal{R}}$. Furthermore, $|0\rangle_{\mathcal{M}}$ should be regarded as a mixed state not a pure state in Rindler space

vs

In the algebraic approach, an *algebraic state* is introduced

as a normalized positive linear functional that acts on field algebra.

This allows us to set a mixed state (operator) in equal footing with a pure state in some sense.

This approach explains various interesting phenomena

in curved spacetime such as Unruh effect satisfactorily.

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Based on the similarity between IFT and FTCS, it is very tempting to adopt an algebraic method to IFT

Our proposal:

$\boxed{\text{The algebraic method is an appropriate method of quantization in IFT}}$

This proposal seems quite natural and may be thought as implicitly known. However, it would be meaningful to say it explicitly and consider concrete examples.

Our concrete example is

A certain IFT model can be related to field theory on $(1+1)$-dimensional curved background

-continued

I will not go into the detailed construction of IFT and its relation to field theory on curved space (FTCS),

but just provide a specific example, which can also be interpreted as an IFT model.

$$\sqrt{-g}\,\mathcal{L}_{\text{free}} = \sqrt{-g}\left[-\frac{1}{2}g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi - \frac{1}{2}m_0^2\phi^2 - \frac{1}{2}\xi\mathcal{R}\phi^2\right].$$

$$ds^2 = e^{2\omega(x)}(-dt^2 + dx^2)\,,$$

where

$$e^{\omega(x)} = \frac{1}{a + e^{-bx}}\,, \qquad ab = \frac{m_0}{2\xi}$$

Without loss of generality, one can take $b > 0$ and the relation among a, b and m_0 are dictated by supersymmetry.

Interestingly, the rigid background described by the above metric allows various field theories like ϕ^6, Sine-Gordon, Liouville, etc.

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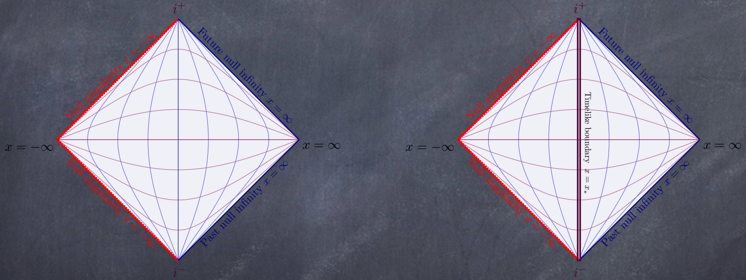
Our supersymmetric background metric $e^{2\omega(x)} = \frac{1}{(a+e^{-bx})^2}$ has curvature singularity at $x = -\infty$, which is naked and null. [Perose Diagram]



(a) a>0 **(b)** a <0

The red zigzag line denotes naked null singularity ($\mathcal{R} = 2ab^2e^{-bx}$).

-continued

From the FTCS viewpoint, one may object the background spacetime in which field theory lives, because of the singularity.

However, scalar FT on singularity region corresponds to a massless FT from the IFT viewpoint, which cause no essential problems (except for the well-known infrared divergence).

The IFT interpretation tells us that the singularity would be *mild* one

in the sense that the scalar wave propagation would be well-defined.

-continued

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in the sense that the scalar wave propagation would be well-defined.

Indeed, one can show that the scalar wave propagation is well-defined on our background. This is also anticipated from the Penrose diagram and the null singularity nature, but it is quite interesting that the whole picture is consistent with the existence of SUSY or the spacetime is a supersymmetric background.

For this purpose, one will focus on 'free' scalar FT. And then, we will see the appearance of SQM in this setup.

Free scalar FTCS and Free scalar IFT

Bosonic parts of Lagrangian of free SFTCS and free SIFT are given by

$$\sqrt{-g}\,\mathcal{L}^{\text{free}}_{\text{SFTCS, b}} = \sqrt{-g}\left[-\frac{1}{2}g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi - \frac{1}{2}m_0^2\phi^2 - \frac{1}{2}\xi\mathcal{R}\phi^2\right].$$

$$\mathcal{L}^{\text{free}}_{\text{SIFT, b}} = -\frac{1}{2}\eta^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}\left(m^2(x) + m'(x)\right)\phi^2 .$$

Klein-Gordon equation is given by $(-\Box + m_0^2 + \xi\mathcal{R})\phi = 0$ or equivalently $(-\partial^2 + m_{\text{eff}}^2)\phi = 0$ which can be rewritten as

$$\partial_t^2\phi = -A\phi\,, \qquad A = -\partial_x^2 + m_{\text{eff}}^2(x)\,, \qquad m_{\text{eff}}^2 \equiv e^{2\omega}(m_0^2 + \xi\mathcal{R}) = m^2 + m' .$$

Frequency mode ϕ_ω satisfies

$$A\ \phi_\omega(x) = \left[-\frac{d^2}{dx^2} + V_{\text{eff}}(x)\right]\phi_\omega(x) = \omega^2\phi_\omega(x)\,, \qquad V_{\text{eff}}(x) \equiv m^2(x) + m'(x)\,,$$

The well-posedness of initial value problem can be phrased as the statement about the symmetric operator A.

Self-adjointness

If the self-adjoint extension, A_E of the symmetric operator A exists,

a satisfactory dynamical evolution could be defined at least for initial data $(\phi_0, \dot{\phi}_0)$ in $C_0^\infty(\Sigma) \times C_0^\infty(\Sigma)$ by

$$\phi_t = \cos(A_E^{1/2} t)\phi_0 + A_E^{-1/2} \sin(A_E^{1/2} t)\dot{\phi}_0 \,, \qquad (\phi_0, \dot{\phi}_0) \in \mathcal{H} \times \mathcal{H} \,.$$

One can show that A_E is uniquely determined in our case, so that the spacetime is globally hyperbolic.

In other words, the symmetric operator A is essentially self-adjoint which means that the extension, A_E is unique and our naked null singularity is *mild*.

To check the essential self-adjointness of the symmetric operator A, SQM plays an interesting role.

-continued

First, one may notice that

$$V_{\rm eff} = m^2 + m' = W_{\rm QM}^2 - \frac{dW_{\rm QM}}{dx}\,, \qquad W_{\rm QM}(x) = -m(x)\,,$$

where W_{QM} is the so-called superpotential in SQM. The symmetric operator A can be interpreted as a Hamiltonian in SQM.

Indeed, $A = D_- D_+$ in our supersymmetric background, and $m(x)$ is written as $(\beta = 2\xi)$

$$m(x) = G^{-1}(x)\frac{dG(x)}{dx}\,, \qquad G(x) \equiv (1 + ae^{bx})^{\beta}\,, \;\; \text{or} \;\; (1 + ae^{bx})^{1-\beta}\,,$$

Now, we can apply various machinery in SQM in our case.

Rosen-Morse / Eckart potential

SQM potential related to our supersymmetric background is

$$V_{\rm eff}(x) \equiv m^2_{\rm eff}(x) = \frac{(m_0^2 e^{bx} + 2\xi a b^2) e^{bx}}{(a e^{bx}+1)^2} = W^2_{\rm QM} - \frac{dW_{\rm QM}}{dx}$$

where W_{QM} is given by (recalling that $2ab\xi = m_0$)

$$W_{\rm QM}(x) = -m(x) = -\frac{m_0\, e^{bx}}{1+ae^{bx}} = \begin{cases} -b\xi\left[1+\tanh\frac{b}{2}(x-x_0)\right], & a>0\,, \\ -b\xi\left[1+\coth\frac{b}{2}(x-x_*)\right], & a<0\,. \end{cases}$$

Here, x_0 and x_* are defined by $e^{-bx_0} \equiv a$ and $e^{-bx_*} = -a$, repectively.

By looking at the tables of solvable SQM, one can identify the above potentials as special cases of Rosen-Morse and Eckart potentials.

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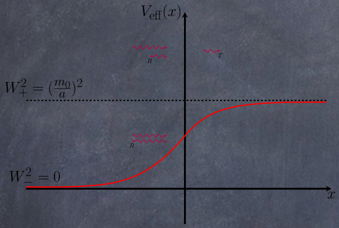

Figure Rosen-Morse
$a > 0$, $\xi \geq \frac{1}{4}$

V_{eff} has a S-shaped graph without any minimum/maximum values, and approaches to zero and $(m_0/a)^2$ as $x \to -\infty$ and $x \to +\infty$, respectively.

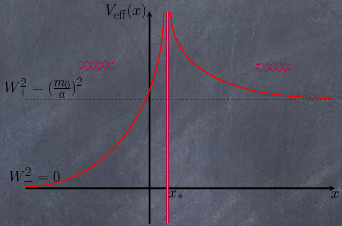

Figure Eckart $a < 0$, $\xi < 0$

V_{eff} diverges to $+\infty$ at x_*, and approaches to zero and $(m_0/a)^2$ as $x \to -\infty$ and $x \to +\infty$, respectively.

Canonical Qauntization

The scalar field has mode solution,

which be expressed as two separate plane waves in the right and left asymptotic regions, respectively.

Mode expansion and canonical quantization $(i, j = \pm)$

appropriate in the left asymptotic region:

$$\phi_L(x) = \int_0^\infty \frac{d\omega}{\sqrt{2\pi}} \frac{1}{\sqrt{2\omega}} \sum_{i=\pm} \left[a_\omega^{(i)} u_\omega^{(i)}(x) + (a_\omega^{(i)})^\dagger (u_\omega^{(i)}(x))^* \right], \qquad [a_\omega^{(i)}, (a_{\omega'}^{(j)})^\dagger] = \delta^{ij}\delta(\omega - \omega'),$$

$$u_\omega^{(-)}(x) = (1 + e^{bx})^{2\xi} F(A, B\,;\, C \,|\, -e^{bx}) e^{-i\omega(t-x)},$$

$$u_\omega^{(+)}(x) = (1 + e^{bx})^{2\xi} F(A - C + 1, B - C + 1\,;\, 2 - C \,|\, -e^{bx}) e^{-i\omega(t+x)},$$

appropriate in the right asymptotic region:

$$\phi_R(x) = \int_0^\infty \frac{dk}{\sqrt{2\pi}} \frac{1}{\sqrt{2\omega}} \sum_{i=\pm} \left[b_k^{(i)} v_k^{(i)}(x) + (b_k^{(i)})^\dagger (v_k^{(i)}(x))^* \right], \qquad [b_k^{(i)}, (b_{k'}^{(j)})^\dagger] = \delta^{ij}\delta(k - k'),$$

$$v_k^{(-)}(x) = (1 + e^{-bx})^{2\xi} F\left(A, A - C + 1\,;\, A - B + 1 \,\middle|\, -e^{-bx}\right) e^{-i(\omega t - kx)},$$

$$v_k^{(+)}(x) = (1 + e^{-bx})^{2\xi} F\left(B, B - C + 1\,;\, B - A + 1 \,\middle|\, -e^{-bx}\right) e^{-i(\omega t + kx)}.$$

-continued

At the left asymptotic region ($x \to -\infty$), the scalar field reduces to a massless one as

$$\phi(x) \underset{x\to-\infty}{\simeq} \int_0^\infty \frac{d\omega}{\sqrt{2\pi}} \frac{1}{\sqrt{2\omega}} \left[a_\omega^\mp e^{-i\omega(t\mp x)} + a_\omega^{\mp\dagger} e^{i\omega(t\mp x)} \right], \qquad [a_\omega^\mp, a_{\omega'}^{\mp\dagger}] = \delta(\omega-\omega')\,.$$

One may define the 'left' vacuum $|0\rangle_L$ as the state annihilated by a_ω as

$$a_\omega^\mp |0\rangle_L = 0\,.$$

On the other hand, at the right asymptotic region ($x \to \infty$), the scalar field reduces to a massive one in the quantized form of

$$\phi(x) \underset{x\to\infty}{\simeq} \int_0^\infty \frac{dk}{\sqrt{2\pi}} \frac{1}{\sqrt{2\omega}} \left[b_k^\mp e^{-i(\omega t\mp kx)} + b_k^{\mp\dagger} e^{i(\omega t\mp kx)} \right], \qquad [b_k^\mp, b_{k'}^{\mp\dagger}] = \delta(k-k')\,.$$

where $k = \sqrt{\omega^2 - b^2\beta^2}$. This quantization scheme is valid only for $\omega^2 \geq b^2\beta^2$. Just like the 'left' vacuum, one may introduce the 'right' vacuum as

$$b_k^\mp |0\rangle_R = 0\,.$$

Algebraic approach

Left / Right vacuum is a local vacuum but cannot be a global one by construction (*i.e.* defined for the whole range of x)

Algebraic viewpoint: The 'left/right' vacuum is not preferred compared to 'right/left' vacuum in IFT.

Those vacuums are just appropriate ones at the left and right regions just like FTCS interpretation.

Comments:

Contrary to Minkowski/Rindler case, the whole spacetime is covered by a single coordinate and no coordinate transformation is involved here.

In fact, there is an old cousin of our model in (1+1)-dimensional cosmology. [Birrell + Davies]

Two point function

In the algebraic approach to FTCS, a physical state is specified by the so-called Hadamard condition.

Once again, the algebraic state acts on the field operator.

Furthermore, one can characterize its action by the two-point function: Gaussian state (for free field).

One of the important physical condition: the short distance singularity is given by the Minkowski expression

$$\omega(\phi(\mathbf{x})\phi(\mathbf{y})) \underset{\mathbf{x}\to\mathbf{y}}{\sim} \frac{\star}{\sigma} + \star \ln \sigma + \text{regular}$$

where $\sigma(\mathbf{x},\mathbf{y})$ is the signed squared geodesic distance between $\mathbf{x}$ and $\mathbf{y}$.

Roughly speaking in 2-D,

$$\sigma(\mathbf{x},\mathbf{y}) \sim -(t-t')^2 + (x-x')^2\,, \qquad \mathbf{x} = (t,x) \quad \mathbf{y} = (t',x')\,.$$

-continued

This condition is called as the Hadamard condition and the state satisfying this condition as the Hadamard state,

whose singularity structure is universal and determined by the geometrical data in FTCS.

In the community of FTCS, there is a consensus that this condition defines the physical states.

I believe that a similar condition or its generalization should be realized in IFT to specify the allowed physical sate.

Maybe, this is a future direction.

cf. In this regard, the mathematical concept "wavefront set" may be suitable to describe the singularity structure of two point function

since two point function is, strictly specking, a bi-distribution.

Discussion

- JT gravity and Canonical quantization
  - JT gravity is an interesting gravity model
  - 2d JT gravitty is solvable

    related SYK and matrix model

- IFT and Canonical quantization
  - A concrete (supersymmetric) example
  - Canonical quantization is insufficient to capture the whole physics
  - *Global* state in IFT ???

-continued

- Some open questions:
  - More advanced algebraic understanding
  - JT gravity with matter: factorization issues are not resolved yet
  - Witten's works: other directions? de Sitter ...
  - etc

- Condensed matter (experimental) realization of IFT
  - Unruh-like effect?
  - Mass change in IFT?