Is local enough for quantum field theory?

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and ... work in progress

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JEONGWON HO (working with O-Kab Kwon, Sang-A Park, Sang-Heon Yi)

Contents

1. Algebraic approach to QFTCS

QFT in Minkowski spacetime is invariant under the action of the Poincar´e group, the symmetry group of Minkowski spacetime. ➤ the existence of a unique, Poincare invariant vacuum state

QFT in curved spacetimes, there is generically no distinguished ground state or vacuum vector in a Hilbert space, since there is no natural 'energy' that we should try to minimize.

QFT in curved spacetimes with a naturally defined Hilbert space that **does not contain any distinguished vector**

1. Algebraic approach to QFTCS

"Algebraic Quantum Field Theory" as a good candidate for this

1. Haag, R.: Local Quantum Physics: Fields, Particles, Algebras. Springer-Verlag, Berlin (1992)
2. Haag, R., Narnhofer, H., Stein, U.: On quantum field theory in gravitational background. Commun.Math.Phys. 94, 219 (1984)
3. Kay, B.S., Wald, R.M.: Theorems on the uniqueness and thermal properties of stationary, nonsingular, quasifree states on space-times with a bifurcate Killing horizon. Phys. Rep. 207, 49–136 (1991)
4. Wald, R.M.: Quantum Field Theory on Curved Spacetimes and Black Hole Thermodynamics, The University of Chicago Press (Chicago, 1994)

1. Algebraic approach to QFTCS

"Algebraic Quantum Field Theory" as a good candidate for this

Idea : The mathematical framework of operator algebras permits a very clear and efficient way to precisely formulate the conceptual underpinnings of QFT in curved spacetime - **locality and covariance**.

Formulating QFT in curved spacetime via the algebraic approach in a manner that **does not require one to single out a preferred state** in order to define the theory.

1. Algebraic approach to QFTCS

"Algebraic Quantum Field Theory" as a good candidate for this

Basic Procedure :

(1) Taking the relations satisfied by the quantum fields—such as commutation relations and field equations.

(2) Specifying the complete set of algebraic relations satisfied by the fundamental field and composite fields.

(3) States are defined to be positive linear functions on the algebra of quantum fields.

(4) The GNS construction shows that every state in this sense arises as a vector in a Hilbert space that carries a representation of the field algebra.

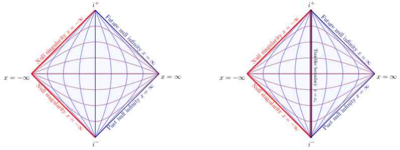
Connecting the algebraic notion of states with usual notions of states in quantum theory

2. Free scalar field in the (1+1)-dim supersymmetric background

- the supersymmetric background : arXiv:2211.05699 Kwon, JH, Park, Yi

$$ds^2 = \frac{1}{(a+e^{-bx})^2}(-dt^2 + dx^2)$$

The above (1+1)-dim SUSY background has a naked null curvature singularity at $x \to -\infty$ for $\mathcal{R} = 2ab^2 e^{-bx}$. This spacetime is geodesically incomplete, but hyperbolic. Thus, wave dynamics on this backgound is well-defined.



(a) (b)

2. Free scalar field in the (1+1)-dim supersymmetric background

- Klein-Gordon equation

Free bosonic Lagrangian
$$\sqrt{-g}\,\mathcal{L}_{\rm free} = \sqrt{-g}\Big[-\frac{1}{2}g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi - \frac{1}{2}m_0^2\phi^2 - \frac{1}{2}\xi\mathcal{R}\phi^2\Big]$$

The Klein-Gordon equation in the supersymmetric background for the free scalar field is given by

$$(\Box + m_0^2 + \xi\mathcal{R})\phi = 0 \qquad \blacktriangleright \qquad \partial_t^2\phi = -A\phi\,, \qquad A = -\partial_x^2 + e^{2\omega}(m_0^2 + \xi\mathcal{R})\,.$$

$$ds^2 = \frac{1}{(a+e^{-bx})^2}(-dt^2 + dx^2) \qquad e^{\omega(x)} = \frac{1}{a+e^{-bx}}\,,$$

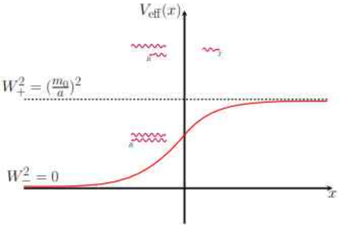
or

$$\Big[-\frac{d^2}{dx^2} + V_{\rm eff}(x)\Big]\phi_\omega(x) = \omega^2\phi_\omega(x)\,, \qquad V_{\rm eff}(x) \equiv m_{\rm eff}^2(x) = \frac{(m_0^2 e^{bx} + 2\xi ab^2)e^{bx}}{(ae^{bx}+1)^2}$$

2. Free scalar field in the (1+1)-dim supersymmetric background

- Klein-Gordon equation

We consider the case of S-shaped potential: $a > 0,\ \xi \geq \frac{1}{4}$



2. Free scalar field in the (1+1)-dim supersymmetric background

- Hypergeometric differential equation

By taking the change of variable $y = ae^{bx} = e^{b(x-x_0)}$ in

$$\left[-\frac{d^2}{dx^2} + V_{\text{eff}}(x)\right]\phi_\omega(x) = \omega^2\phi_\omega(x)\,, \qquad V_{\text{eff}}(x) \equiv m_{\text{eff}}^2(x) = \frac{(m_0^2 e^{bx} + 2\xi ab^2)e^{bx}}{(ae^{bx}+1)^2}$$

and setting $\phi_\omega(y) \equiv y^\alpha(1+y)^\gamma f_\omega(y)$ with $\alpha = \pm i\frac{\omega}{b}$ and $\gamma = \beta$ or $\gamma = 1-\beta$, $\beta \equiv 2\xi = \frac{m_0}{ab}$, then the differential equation satisfied by $f_\omega(y)$ becomes a hypergeometric differential equation in the form of

$$\left[y(1+y)\frac{d^2}{dy^2} + \Big(2\alpha + 1 + (2\gamma + 2\alpha + 1)y\Big)\frac{d}{dy} + \gamma(2\alpha+1) - 2\xi\right]f_\omega(y) = 0\,.$$

2. Free scalar field in the (1+1)-dim supersymmetric background

- Mode Solutions

Then, we can immediately write down solutions to our differential equations in terms of hypergeometric functions as

$$\phi_\omega(y) = (1+y)^\gamma\Big[a_1 y^\alpha F(A,B\,;C\,|-y) + a_2\, y^{\alpha+1-C} F(A-C+1, B-C+1\,; 2-C\,|-y)\Big]\,,$$

and we take $A = \frac{i}{b}(\omega - k) + \beta\,, \qquad B = \frac{i}{b}(\omega + k) + \beta\,, \qquad C = 1 + 2\frac{i}{b}\omega\,,$

where k is defined by

$$k^2 \equiv \omega^2 - (2b\xi)^2\,.$$

in the asymptotic region of $x \to -\infty$ $(y \to 0)$,

$$\phi_\omega(x) \underset{x\to-\infty}{\longrightarrow} a_1\, e^{i\omega(x-x_0)} + a_2\, e^{-i\omega(x-x_0)}$$

In the case of $k^2 > 0,$ by using the linear transformation of hypergeometric function, one can rewrite the solution in the form of

$$\phi_\omega(y) = (1+y)^\beta \Big[b_1 \, y^{\alpha-A} F\Big(A, A-C+1; A-B+1 \Big| -\frac{1}{y}\Big) + b_2 \, y^{\alpha-B} F\Big(B, B-C+1; B-A+1 \Big| -\frac{1}{y}\Big) \Big],$$

where the constants $b_{1,2}$ (with $\beta \in \mathbf{R}$) are related to the previous constants $a_{1,2}$ as follows

$$b_1 = \frac{\Gamma(2i\frac{k}{b})\Gamma(1+2i\frac{\omega}{b})[a_1 - R^*_\omega a_2]}{\Gamma(\beta + \frac{i}{b}(\omega+k))\Gamma(1-\beta+\frac{i}{b}(\omega+k))},$$

$$b_2 = \frac{\Gamma(-2i\frac{k}{b})\Gamma(1-2i\frac{\omega}{b})[-R_\omega a_1 + a_2]}{\Gamma(\beta - \frac{i}{b}(\omega+k))\Gamma(1-\beta-\frac{i}{b}(\omega+k))}.$$

Here, $R_\omega \in \mathbf{C}$ is defined by

$$R_\omega \equiv -\frac{\Gamma(C)\Gamma(A-C+1)\Gamma(1-B)}{\Gamma(2-C)\Gamma(A)\Gamma(C-B)} = -\frac{\Gamma\left(1+2i\frac{\omega}{b}\right)\Gamma\left(\beta-\frac{i}{b}(\omega+k)\right)\Gamma\left(1-\beta-\frac{i}{b}(\omega+k)\right)}{\Gamma\left(1-2i\frac{\omega}{b}\right)\Gamma\left(\beta+\frac{i}{b}(\omega-k)\right)\Gamma\left(1-\beta+\frac{i}{b}(\omega-k)\right)}.$$

in the asymptotic region of $x \to \infty$ $(y \to \infty)$

$$\phi_\omega(x) \underset{x\to\infty}{\longrightarrow} b_1 e^{ik(x-x_0)} + b_2 e^{-ik(x-x_0)},$$

3. Canonical Quantizations

Mode solutions in the right and left asymptotic regions

$$\phi_\omega(x) \underset{x\to-\infty}{\longrightarrow} a_1\, e^{i\omega(x-x_0)} + a_2\, e^{-i\omega(x-x_0)} \qquad \phi_\omega(x) \underset{x\to\infty}{\longrightarrow} b_1 e^{ik(x-x_0)} + b_2 e^{-ik(x-x_0)}$$

It is natural to consider two canonical quantization schemes.

In the following, we refer to the quantization scheme using the mode solution presented in the left asymptotic region as L-quantization, while the scheme that employs the mode solution given in the right asymptotic region is referred to as R-quantization.

3. Canonical Quantization : L-quantization

$$\phi_{\mathrm{L}}(\boldsymbol{x}) = \int_0^\infty \frac{d\omega}{\sqrt{2\pi}} \frac{1}{\sqrt{2\omega}} \sum_{i=\pm} \left[a_\omega^{(i)} u_\omega^{(i)}(\boldsymbol{x}) + \left(a_\omega^{(i)}\right)^\dagger \left(u_\omega^{(i)}(\boldsymbol{x})\right)^* \right]$$

$$u_\omega^{(-)}(\boldsymbol{x}) = (1+e^{bx})^{2\xi} F(A, B\,;\,C\,|\,-e^{bx}) e^{-i\omega(t-x)}\,,$$
$$u_\omega^{(+)}(\boldsymbol{x}) = (1+e^{bx})^{2\xi} F(A-C+1, B-C+1\,;\,2-C\,|\,-e^{bx}) e^{-i\omega(t+x)}$$

$$\left(u_\omega^{(-)}(\boldsymbol{x})\right)^* = (1+e^{bx})^{2\xi} F(A-C+1, B-C+1\,;\,2-C\,|\,-e^{bx}) e^{i\omega(t-x)}$$
$$\left(u_\omega^{(+)}(\boldsymbol{x})\right)^* = (1+e^{bx})^{2\xi} F(A, B\,;\,C\,|\,-e^{bx}) e^{i\omega(t+x)}\,.$$

$$[a_\omega^{(i)}, (a_{\omega'}^{(j)})^\dagger] = \delta^{ij}\delta(\omega-\omega')$$

We define the vacuum in L-quantization, $|0\rangle_{\mathrm{L}}$, as the state annihilated by $a_\omega^{(\mp)}$ as

$$a_\omega^{(\mp)}|0\rangle_{\mathrm{L}} = 0$$

The Fock space constructed by $a_\omega^{(\mp)}$ and $(a_\omega^{(\mp)})^\dagger$ is denoted by $\mathscr{F}_{\mathrm{L}}$. At the left asymptotic region, $u_\omega^{(\mp)}(\boldsymbol{x})$'s reduce to

$$u_\omega^{(\mp)}(\boldsymbol{x}) \underset{x\to-\infty}{\longrightarrow} e^{-i\omega(t\mp x)}$$

and in L-quantization scheme, the form of scalar field at the left asymptotic region ($x \to -\infty$) approaches to

$$\phi_{\mathrm{L}}(\boldsymbol{x}) \underset{x\to-\infty}{\simeq} \int_0^\infty \frac{d\omega}{\sqrt{2\pi}}\frac{1}{\sqrt{2\omega}}\left[a_\omega^{(+)}e^{-i\omega(t+x)} + a_\omega^{(-)}e^{-i\omega(t-x)} + \left(a_\omega^{(+)}\right)^\dagger e^{i\omega(t+x)} + \left(a_\omega^{(-)}\right)^\dagger e^{i\omega(t-x)}\right].$$

L-quantization describes the system in terms of massless particles, in the x → −∞ region, but it is not warranted in the whole region.

3. Canonical Quantization : R-quantization

$$\phi_{\mathrm{R}}(\boldsymbol{x}) = \int_0^\infty \frac{dk}{\sqrt{2\pi}} \frac{1}{\sqrt{2\omega}} \sum_{i=\pm} \left[b_k^{(i)} v_k^{(i)}(\boldsymbol{x}) + (b_k^{(i)})^\dagger (v_k^{(i)}(\boldsymbol{x}))^* \right] \qquad \omega = \sqrt{k^2 + b^2\beta^2}$$

$$v_k^{(-)}(\boldsymbol{x}) = (1+e^{-bx})^{2\xi} F\left(A, A-C+1;\, A-B+1 \,\middle|\, -e^{-bx}\right) e^{-i(\omega t - kx)}\,,$$
$$v_k^{(+)}(\boldsymbol{x}) = (1+e^{-bx})^{2\xi} F\left(B, B-C+1;\, B-A+1 \,\middle|\, -e^{-bx}\right) e^{-i(\omega t + kx)}\,.$$

$$\left(v_k^{(-)}((\boldsymbol{x})\right)^* = (1+e^{-bx})^{2\xi} F(B, B-C+1;\, B-A+1 \,|\, -e^{-bx}) e^{i(\omega t - kx)}\,,$$
$$\left(v_k^{(+)}((\boldsymbol{x})\right)^* = (1+e^{-bx})^{2\xi} F(A, A-C+1;\, A-B+1 \,|\, -e^{-bx}) e^{i(\omega t + kx)}\,.$$

$$[b_k^{(i)}, (b_{k'}^{(j)})^\dagger] = \delta^{ij}\delta(k-k')\,.$$

Vacuum in R-quantization $\quad b_k^{(\mp)}|0\rangle_{\mathrm{R}} = 0$

The Fock space constructed by $b_k^{(\mp)}$ and $\left(b_k^{(\mp)}\right)^\dagger$ is denoted by $\mathscr{F}_{\mathrm{R}}$.

At the right asymptotic region, $v^{(\mp)}(\boldsymbol{x})$'s reduce to

$$v^{(\mp)}(\boldsymbol{x}) \underset{x\to\infty}{\longrightarrow} e^{-i(\omega t \mp kx)}$$

Canonically quantized form of the scalar field at the right asymptotic region (x → ∞) reduces to

$$\phi_{\mathrm{R}}(\boldsymbol{x}) \underset{x\to\infty}{\simeq} \int_0^\infty \frac{dk}{\sqrt{2\pi}} \frac{1}{\sqrt{2\omega}} \left[b_k^{(+)} e^{-i(\omega t + kx)} + b_k^{(-)} e^{-i(\omega t - kx)} + \left(b_k^{(+)}\right)^\dagger e^{i(\omega t + kx)} + \left(b_k^{(-)}\right)^\dagger e^{i(\omega t - kx)} \right]$$

The above equation shows that in contrast to L-quantization, R-quantization gives rise to particles with a mass of $b\beta = m_0/a = 2\xi b$, which can be inferred from $\omega = \sqrt{k^2 + b^2\beta^2}$. This means that the (local) 'right' observers cannot detect massless particles of the energy, $\omega < b\beta$, which can be observed in the 'left' region.

4. Conclusions and Speculations : **The L-quantization scheme is inequivalent to the R-quantization scheme.**

Why would this happen?

Absence of propagating modes in the right asymptotic region for the range of $0 \leq \omega \leq b\beta$.

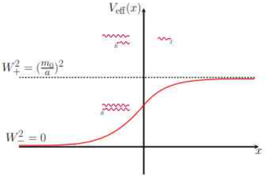
In the $x \to \infty$ region (R-quantization scheme)

: **Exponentially decaying parts** under the range of $0 \leq \omega \leq b\beta$

In the $x \to -\infty$ region (L-quantization scheme)

: $a_\omega^{(+)}$ and $a_\omega^{(-)}$ are **not independent but related** by the relation

$a_\omega^{(-)} = \tilde{R}_\omega^* a_\omega^{(+)}$ within the range of $0 \leq \omega \leq b\beta$,



4. Conclusions and Speculations : **The L-quantization scheme is inequivalent to the R-quantization scheme.**

Mathematical description

Mode functions $u_\omega^{(i)}$ in L-quantization scheme and Mode functions $v_k^{(i)}$ in R-quantization scheme are related by the transformations of hypergeometric functions.

Coefficients of the mode functions are related through SU(1,1) transformation.

$$\tilde{a}_i \equiv a_i\sqrt{\omega}\,, \quad \tilde{b}_i \equiv b_i\sqrt{k}\,, \qquad \begin{pmatrix} \tilde{b}_1 \\ \tilde{b}_2 \end{pmatrix} = \begin{pmatrix} p_k & q_k \\ q_k^* & p_k^* \end{pmatrix} \begin{pmatrix} \tilde{a}_1 \\ \tilde{a}_2 \end{pmatrix}, \qquad |p_k|^2 - |q_k|^2 = 1\,,$$

However, it fails to preserve the canonical commutation relation among the operators $b_k^{(\mp)}$ transformed from $a_\omega^{(+)}$ and $a_\omega^{(-)}$.

The condition for the equivalence of Fock spaces and the preservation of canonical commutation relations is described by the Shale theorm. D. Shale, "LINEAR SYMMETRIES OF FREE BOSON FIELDS," Transactions of the American Mathematical Society 103 (1962): 149-167.

In our setup, the Shale theorem's condition is not met because the SU(1, 1) transformation is incomplete and non-invertible due to the absence of propagating modes in the right asymptotic region for the range of $0 \leq \omega \leq b\beta$.

Consequently, both L-, R-quantization schemes are distinct, and neither vacuum $|0\rangle_{\mathrm{L}}$ and $|0\rangle_{\mathrm{R}}$, **is preferred.**

4. Conclusions and Speculations : Algebraic approach to QFTCS

In the algebraic formulation, a physically important class of quantum states are given by

Gaussian Hadamard states not by a Preferred Vacuum State.

■ Contrary to the ordinary vacuum state, the Hadamard state is not unique for a given background spacetime but forms a class in general.

■ In the case of a Gaussian Hadamard state,

- one can obtain a Fock space representation of the algebra of quantum fields.
- one can identify the Gaussian Hadamard state with the Fock space vacuum.

■ In this way, one can see that some well-known vacuums belong to the Hadamard class.

■ The Hadamard method encompasses the usual Fock space canonical quantization and implements appropriately relevant requirements such as general covariance of stress tensor, while it connects unitarily inequivalent representations of the algebras of observables.

4. Conclusions and Speculations : Algebraic approach to QFTCS

Hadamard state is defined as an algebraic state satisfying the Hadamard conditions.

- The short distance singularity structure of the n-point functions of the Hadamard state on curved spacetime should be given by that of the n-point functions of the vacuum state in the Minkowski spacetime.
- **The ultra-high energy mode of quantum fields resides essentially in the ground state.**
- The singular structure of the n-point functions should be of positive frequency type.

4. Conclusions and Speculations : Algebraic approach to QFTCS

The Gaussian Hadamard state, ω_H is defined by the renormalized two point function of scalar field φ as

$$\omega_{\mathrm{H}}\big(\phi(\mathbf{x})\phi(\mathbf{x}')\big) = F(\mathbf{x},\mathbf{x}') - H(\mathbf{x},\mathbf{x}')$$

- Hadamard function, $F(\mathbf{x},\mathbf{x}')$ is an unrenormalized two point function.
- Hadamard parametrix, $H(\mathbf{x},\mathbf{x}')$ is a local covariant function of the half of squared geodesic length, $\sigma(x, x')$ between two points x and x′, written in terms of the metric and the curvature and takes the same form for any Hadamard states.

4. Conclusions and Speculations : Algebraic approach to QFTCS

$$\omega_{\mathrm{H}}\big(\phi(\mathbf{x})\phi(\mathbf{x}')\big) = F(\mathbf{x},\mathbf{x}') - H(\mathbf{x},\mathbf{x}')$$

$$H(\mathbf{x},\mathbf{x}') = \alpha_D \frac{U(\mathbf{x},\mathbf{x}')}{\sigma^{\frac{D}{2}-1}(\mathbf{x},\mathbf{x}')} + \beta_D V(\mathbf{x},\mathbf{x}') \ln \mu^2 \sigma(\mathbf{x},\mathbf{x}') \quad \text{for even } D\,,$$

$$H(\mathbf{x},\mathbf{x}') = \alpha_D \frac{U(\mathbf{x},\mathbf{x}')}{\sigma^{\frac{D}{2}-1}(\mathbf{x},\mathbf{x}')} \quad \text{for odd } D\,,$$

where α_D, β_D are numerical constants depending on the dimension D and μ is a certain mass scale introduced from the dimensional reason.

Symmetric bi-scalars U(x, x′) and V(x, x′), which are regular for x′ → x, are universal geometrical objects independent of any Hadamard states.

Local Hadamard condition : the two-point function's coincident limit has a universal divergence structure given by the Minkowski case.

4. Conclusions and Speculations : **Algebraic approach to QFTCS**

In the context of algebraic approach, **these vacua are (local) quasi-free states** satisfying the local Hadamard condition which justifies the naming **(local) "vacua"** for $|0\rangle_{\mathrm{L}}$ and $|0\rangle_{\mathrm{R}}$.

In a similar vein, it is natural to interpret the Fock spaces, $\mathscr{F}_{\mathrm{L}}$ and $\mathscr{F}_{\mathrm{R}}$, as **local Hilbert spaces, rather than global ones.**

Even though (local) $\mathscr{F}_{\mathrm{L}}$ and $\mathscr{F}_{\mathrm{R}}$, are not unitarily equivalent, **they share the same algebraic relation among the field operators.**

Is it possible for two vacua $|0\rangle_{\rm L}$ and $|0\rangle_{\rm R}$, satisfying local Hadamard condition, to be extended to global Hadamard states, respectively?

"If G(x, x') has the Hadamard singularity structure in an open neighborhood of a Cauchy surface, then it does so everywhere, i.e., Cauchy evolution preserves the Hadamard singularity structure."

S. A. Fulling, M. Sweeny, and R. M. Wald Commun. Math. Phys. 63 (1978). 257

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Thank You