2023 01 11 Black Holes, Gravitational Waves, etc.

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1. Trajectories in the Schwarzschild spacetime

the Schwarzschild geometry :

= the geometry of the vacuum spacetime outside a spherical star or outside of the horizon of the black hole

$$ds^2 = -(1-\frac{2M}{r})dt^2 + \frac{1}{1-\frac{2M}{r}}dr^2 + r^2 d\Omega^2 \quad (11.1\text{-}1)$$

Note : solution with 1-parameter, the mass M,

Black holes in Newtonian gravity

escape velocity

$$\frac{1}{2}v^2 = \frac{GM}{R} \qquad (11.1\text{-}2)$$

setting $v = c$ gives

$$R = \frac{2GM}{c^2} \quad \text{"horizon" radius} \quad (11.1\text{-}3)$$

Ex) $M = M_{\odot} \rightsquigarrow R = 3km$

Mean density

$$\bar{\rho} = \frac{M}{\frac{4\pi}{3}R^3} = \frac{3c^6}{32\pi G^3 M^2} \propto M^{-2}$$

ex) $M = 10^8 M_{\odot} \rightsquigarrow \bar{\rho} = \rho_{\text{water}}$

or,

$$\bar{\rho} = (1.8\text{x}10^{16}\ \text{g/cm}^3)\ \text{x} \left(\frac{M_{\odot}}{M}\right)^2$$

cf) Nuclear density $\rho = 2.04\text{x}10^{17}\text{kg/m}^3 = 2.04\text{x}10^{14}\text{g/cm}^3$

Conserved quantities

1) t-independence of the metric $\rightsquigarrow$

the energy= $-p_0$ is constant on the trajectory

Define

- for massive particles $m \neq 0$

$\tilde{E} = -p_0/m$ the energy/mass (specific energy) (11.1-4)

- for photons,

$E = -p_0$

2) ϕ-independence of the metric $\rightsquigarrow$

the angular momentum = p_ϕ is constant on the trajectory

define

particle $\tilde{L} := p_\phi/m$ photon $L = p_\phi$ (11.1-5)

3) Spherical symmetry $\rightsquigarrow$

motion confined to a single plane, equatorial plane,

we choose ($\theta = \frac{\pi}{2}$ =const for the orbit, i.e.,

$0 = \frac{d\theta}{d\lambda} \propto p^\theta$ (11.1-6)

Angular velocity

$$\frac{dt}{d\phi} = \frac{dt/d\tau}{d\phi/d\tau} = \left(\frac{r^3}{M}\right)^{1/2} \quad (11.1\text{-}19)$$

The period (coordinate time)

$$P = \frac{2\pi}{d\phi/dt} = 2\pi\left(\frac{r^3}{M}\right)^{1/2} \quad (11.1\text{-}20)$$

A slightly noncircular orbit - radial oscillation
Newtonian- closed, ellipse
GR : almost closed ellipse for weak gravity: precession
Ex) Mercury 43''/century

Particle orbit equation

$$\left(\frac{dr}{d\tau}\right)^2 = \tilde{E}^2 - \left(1 - \frac{2M}{r}\right)\left(1 + \frac{\tilde{L}^2}{r^2}\right) \quad (11.1\text{-}10)$$

$$\frac{d\phi}{d\tau} = \frac{1}{r^2}\tilde{L} \quad (11.1\text{-}17)$$

$$\left(\frac{dr}{d\phi}\right)^2 = \frac{\tilde{E}^2 - \left(1-\frac{2M}{r}\right)\left(1+\frac{\tilde{L}^2}{r^2}\right)}{\frac{\tilde{L}^2}{r^4}} \quad (11.1\text{-}21)$$

Define

$$u := \frac{1}{r}$$

to obtain

$$\left(\frac{du}{d\phi}\right)^2 = \frac{\tilde{E}^2}{\tilde{L}^2} - (1 - 2Mu)\left(\frac{1}{\tilde{L}^2} + u^2\right) \quad (11.1\text{-}22)$$

Newtonian

$$\underbrace{\left(\frac{du}{d\phi}\right)^2 = \frac{\tilde{E}^2}{\tilde{L}^2} - \frac{1}{\tilde{L}^2}(1 - 2Mu) - u^2}_{\text{Newtonian}} + O(u^3) \quad (11.1\text{-}23)$$

A circular orbit in Newtonian theory

$$u = \frac{M}{\tilde{L}^2}$$

Define

$$y = u - \frac{M}{\tilde{L}^2}$$

Then

$$\left(\frac{dy}{d\phi}\right)^2 = \frac{\tilde{E}^2 - 1}{\tilde{L}^2} + \frac{M^2}{\tilde{L}^4} - y^2 \quad (11.1\text{-}24)$$

Solution (Newtonian)

$$y = \left[\frac{\tilde{E}^2 + \frac{M^2}{\tilde{L}^2} - 1}{\tilde{L}^2}\right]^{1/2} \cos(\phi + B) \quad (11.1\text{-}25)$$

$$\frac{1}{r} = \frac{M}{\tilde{L}^2} + \left[\frac{\tilde{E}^2 + \frac{M^2}{\tilde{L}^2} - 1}{\tilde{L}^2}\right]^{1/2} \cos(\phi + B) \quad (11.1\text{-}26)$$

$$\left(\frac{dy}{d\phi}\right)^2 = \frac{\tilde{E}^2 + \frac{M^2}{\tilde{L}^2} - 1}{\tilde{L}^2} + \frac{2M^4}{\tilde{L}^6} + \frac{6M^3}{\tilde{L}^2}y - \left(1 - \frac{6M^2}{\tilde{L}^2}\right)y^2 \quad (11.1\text{-}27)$$

Solution

$$y = y_0 + A\cos(k\phi + B) \quad (11.1\text{-}28)$$

$$k = \left(1 - \frac{6M^2}{\tilde{L}^2}\right)^{1/2}$$

$$y_0 = \frac{3M^3}{k^2\tilde{L}^2}$$

$$A = \frac{1}{k}\left[\frac{\tilde{E}^2+\frac{M^2}{\tilde{L}^2}-1}{\tilde{L}^2} + \frac{2M^4}{\tilde{L}^6} - {y_0}^2\right]^{1/2}$$

The change in ϕ from one perihelion to the next

$$\Delta\phi = \frac{2\pi}{k} = 2\pi\left(1 - \frac{6M^2}{\tilde{L}^2}\right)^{-1/2} \quad (11.1\text{-}29)$$

For nearly Newtonian orbits

$$\Delta\phi \simeq 2\pi\left(1 + \frac{3M^2}{\tilde{L}^2}\right) \quad (11.1\text{-}30)$$

The perihelion advance, from one orbit to the next, is

$$\Delta\phi = \frac{6\pi M^2}{\tilde{L}^2} \text{ radians per orbit}$$

Orbits about a nonrelativistic star from (11.1-15),

$$\tilde{L}^2 = \frac{Mr}{1-\frac{3M}{r}} \approx Mr$$

So that

$$\Delta\phi \approx 6\pi\frac{M}{r}$$

For Mercury's orbit,

$r = 5.55 \times 10^7$km

$M = 1\, M_\odot = 1.47$km

$(\Delta\phi)_{\text{Mercury}} = 4.99 \times 10^{-7}$radians per orbit

Orbit period=0.24yr

$= 0.43''/yr$

$= 0.43''$/century

,

Binary pulsars

Hulse-Taylor binary pulsar PSR B1913+16 : two neutron stars

mean separation 1.2×10^9m

$\Delta\phi \approx 6\pi\frac{M}{r}$, $M = 1.4\, M_\odot = 2.07$km

$= 3.3 \times 10^{-5}$radians per orbit

$=2°.1$ per year(=$4°.2$ per year) easier to measure than Mercury

Observation : 4.2261° $\pm\, 0.0007$ per year

PSR J0737-3039 double pulsar system

$\Delta\phi \approx 17°$ per year

Post-Newtonian gravity

a correction to Newtonian motion in the limit of

1) weak gravitational fields and

2) slow motion.

Ex) 1) Pericenter shift

2) deflection of light

3) dragging of inertial frames (due to the rotation of the gravity source) (gravitomagnetism)

etc.

Note : The Newtonian (Keplerian) motion of a planet depends only on g_{00}

$\left(\frac{dr}{dt} \ll 1\right)$

g_{00} : responsible for the gravitational redshift

Newtonian gravity can be identified with the gravitational redshift

Newtonian gravity exclusively by the curvature of time

Spatial curvature comes in only at the level of post-Newtonian corrections

1) Pericenter shift

Note : the metric of spacetime describing a Newtonian system

$$\begin{aligned} ds^2 &= g_{\alpha\beta}(x)dx^\alpha dx^\beta \\ &= -(1+2\phi)dt^2 + (1-2\phi)d\vec{x}^2 + \mathcal{O}(\phi^2) \\ &= -(1+2\phi)dt^2 + (1-2\phi)(dr^2 + r^2 d\Omega^2) + \mathcal{O}(\phi^2) \end{aligned}$$

$$= -(1-\frac{2M}{r})dt^2 + (1+\frac{2M}{r})\underbrace{(dr^2 + r^2 d\Omega^2)}_{\text{Isotropic coordinate}} + \mathcal{O}(\phi^2)$$

Schwarzschild metric

$$ds^2 = -(1-\frac{2M}{r})dt^2 + \frac{1}{1-\frac{2M}{r}}dr^2 + r^2 d\Omega^2$$

$$\approx -(1-\frac{2M}{r})dt^2 + (1+\frac{2M}{r})dr^2 + r^2 d\Omega^2 + \mathcal{O}\left(\left(\frac{M}{r}\right)^2\right)$$

Can redefine the radial coordinate
to express the isotropic coordinates.

$$(t, r, \theta, \phi) \longrightarrow (t, \bar{r}, \theta, \phi)$$

For large r, let

$$\bar{r} = r - M$$

then

$$1 + \frac{2M}{r} = 1 + \frac{2M}{\bar{r}+M} = 1 + \frac{2M}{\bar{r}}\frac{1}{1+\frac{M}{\bar{r}}} = 1 + \frac{2M}{\bar{r}} - \frac{2M^2}{\bar{r}^2} + \cdots$$

and

$$r^2 = (\bar{r}+M)^2 = \bar{r}^2(1+\frac{M}{\bar{r}})^2$$

$$\approx \bar{r}^2\left(1+\frac{2M}{\bar{r}}\right)$$

$$\approx -\left(1-\frac{2M}{r}\right)dt^2 + \left(1+\frac{2M}{\bar{r}}\right)\left(d\bar{r}^2 + \bar{r}^2 d\Omega^2\right)$$

Gravitational deflection of light

Schwarzschild metric

$$ds^2 = -(1-\frac{2M}{r})dt^2 + \frac{1}{1-\frac{2M}{r}}dr^2 + r^2 d\Omega^2$$

The equation of the orbit

$$\left(\frac{dr}{d\lambda}\right)^2 = E^2 - \left(1-\frac{2M}{r}\right)\frac{L^2}{r^2}$$

$$\frac{d\phi}{d\lambda} = \frac{1}{r^2}L$$

$$\frac{d\phi}{dr} = \pm\frac{1}{r^2}\left[\frac{1}{b^2} - \frac{1}{r^2}\left(1-\frac{2M}{r}\right)\right]^{-1/2}$$

Where

$$b := \frac{L}{E}$$

$$\frac{d\phi}{du} = \left(\frac{1}{b^2} - u^2 + 2Mu^3\right)^{-1/2}, \quad u = \frac{1}{r}$$

Neglecting the u^3-term,

$r\sin(\phi-\phi_0) = b$ a straight line (Newtonian result)

Assume $Mu \ll 1$

Define $y := u(1-Mu)$,

$$u = y(1+My) + \mathcal{O}(M^2u^2) := u(1-Mu),$$

Then,

$$\frac{d\phi}{dy} = \frac{1+2My}{\left(\frac{1}{b^2}-y^2\right)^{1/2}} + \mathcal{O}(M^2u^2)$$

To give

$$\phi = \phi_0 + \frac{2M}{b} + \sin^{-1}(by) - 2M\left(\frac{1}{b^2}-y^2\right)^{\frac{1}{2}}$$

Initial trajectory

$y \to 0$, so $\phi \to \phi_0$: incoming direction

The photon reaches its smallest r when $y = \frac{1}{b}$

(at $\phi = \phi_0 + \frac{2M}{b} + \frac{\pi}{2}$

Total angle passing through $= \pi + \frac{4M}{b}$

The net deflection $\Delta\phi = \frac{4M}{b}$

impact parameter $b = \frac{L}{E} \approx$ the radius of closest approach

Ex)Sun $M = 1M_\odot = 1.47\text{km}$, $R_\odot = 6.96\times10^5\text{km}$

$b = R_\odot$

$(\Delta\phi)_{\odot,\max} = 8.45 \times 10^{-6}\text{rad}$
$= 1''.74$

Ex) Jupiter $M = 1.12 \times 10^{-3}\text{km}$, $R = 6.98 \times 10^{4}\text{km}$

$(\Delta\phi)_{\Psi,\max} = 6.42 \times 10^{-8}\text{rad}$
$= 0''.013$ measured by Hipparcos astrometry satellite

Newtonian argument for light deflection
Cavendish (1784), J.G. Von Söldner (1801)

$$\underbrace{(\Delta\phi)_{\text{Newtonian}}}_{g_{00}} = \frac{2M}{b} = \frac{1}{2}\underbrace{(\Delta\phi)_{\text{Einstein}}}_{g_{00}, g_{rr}}$$

curvature of time   curvature of time and space

Gravitational lensing

Microlensing

2. **Nature of the surface $r = 2M$: horizon**

Coordinate systems & Coordinate singularities
Ex) **2-sphere S^2** with unit radius
Coordinates $\{(\theta, \phi) | 0 \le \theta \le \pi, 0 \le \phi < 2\pi\}$
$ds^2 = d\theta^2 + \sin^2\theta\, d\phi^2$
$g_{\phi\phi} = \sin^2\theta \to 0$ as $\theta \to 0, \pi$
the metric becomes degenerate (or $g^{\phi\phi} \to \infty$)
What is wrong?
Invariant geometrical quantity : circumference of a circle of constant θ.

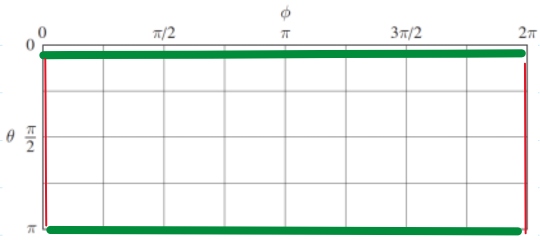
$$\ell = \int_0^{2\pi} \sqrt{g_{\phi\phi}}\, d\phi = \int_0^{2\pi} \sin\theta\, d\phi = 2\pi \sin\theta$$

$\to 0$ as $\theta \to 0, \pi$

Hence, $\theta = 0$ (or π) is a single point independent of ϕ
$\theta = 0$ (or π) : Coordinate singularity

Comment : Lorentzian metric more subtle,
e.g., light-like separation

$S^2 = \{(\theta, \phi) | 0 \le \theta \le \pi, 0 \le \phi < 2\pi\}$



Coordinate singularity
$(\theta = 0, 0 \le \phi < 2\pi)$: a single point
$(\theta = \pi, 0 \le \phi < 2\pi)$: a single point
$\phi = 0$ & $\phi = 2\pi$ should be identified.

The **black hole spacetime manifold**
Schwarzschild coordinate (t, r, θ, ϕ)
The Schwarzschild metric for

(11.1-1) $$ds^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + \frac{1}{1 - \frac{2M}{r}}dr^2 + r^2 d\Omega^2$$

The surface with $r = 2M$ is called the horizon.
At the horizon ($r = 2M$),
$g_{tt} = 0$ $g_{rr} = \infty$
This will be shown to be a coordinate singularity

Outside the horizon ($r > 2M$)

Consider Infalling particles from $r = R$ to $r = 2M$ falling radially ($d\theta = 0 = d\phi$)

1) Proper time elapse

(11.1-10) :

$$\left(\frac{dr}{d\tau}\right)^2 = \tilde{E}^2 - \left(1-\frac{2M}{r}\right)\left(1+\frac{\tilde{L}^2}{r^2}\right)$$

$$m\frac{d\phi}{d\tau} = p^\phi = g^{\phi\phi}p_\phi = m\frac{1}{r^2}\tilde{L},$$

$$d\phi = 0 \rightsquigarrow \tilde{L} = 0$$

$$= \tilde{E}^2 - 1 + \frac{2M}{r}$$

Or,

$$d\tau = -\frac{dr}{\left(\tilde{E}^2-1+\frac{2M}{r}\right)^{1/2}} \quad (11.2\text{-}1)$$

Hence,

$$\Delta\tau = \int d\tau = -\int_R^{2M}\frac{dr}{\left(\tilde{E}^2-1+\frac{2M}{r}\right)^{\frac{1}{2}}} \quad (11.2\text{-}2)$$

If $\tilde{E}^2 > 1$ (unbounded particle), then

$\Delta\tau < \infty$ finite

If $\tilde{E}^2 = 1$ (particle falling from rest at ∞),

$$\Delta\tau = -\int_R^{2M}\sqrt{\frac{r}{2M}}dr = \frac{4M}{3}\left[\left(\frac{r}{2M}\right)^{3/2}\right]_{2M}^{R} < \infty \text{ finite}$$

If $\tilde{E} < 1$, then $r \le r_{max}$ where 1-$\tilde{E}^2 = \frac{2M}{r_{max}}$ and

$\Delta\tau < \infty$ finite

Summary : Any particle can reach the horizon in a finite amount of proper time.
The particle can go inside ($r < 2M$) in a finite proper time

2) Coordinate time elapses

(11.1-18)

$$\frac{dt}{d\tau} \coloneqq U^0 = \frac{p^0}{m} = g^{00}\frac{p_0}{m} = g^{00}(-\tilde{E}) = \frac{\tilde{E}}{1-\frac{2M}{r}} \quad (11.2\text{-}3)$$

$$dt = \frac{\tilde{E}d\tau}{1-\frac{2M}{r}} = -\frac{\tilde{E}dr}{\left(1-\frac{2M}{r}\right)\left(\tilde{E}^2-1+\frac{2M}{r}\right)^{1/2}} \quad (11.2\text{-}4)$$

(Consider the case of $\tilde{E} = 1$ for simplicity)

$$= -\frac{dr}{\left(1-\frac{2M}{r}\right)\left(\frac{2M}{r}\right)^{1/2}} = -\frac{r^{3/2}dr}{(r-2M)(2M)^{1/2}}$$

Near the horizon

$$\epsilon \coloneqq r - 2M$$

$$= \frac{-(\epsilon+2M)^{3/2}d\epsilon}{(2M)^{1/2}\epsilon}$$

$$\Delta t = \int dt = \int_{\epsilon\to 0}\frac{(\epsilon+2M)^{3/2}d\epsilon}{(2M)^{1/2}\epsilon}$$

$$\sim \ln\epsilon \to \infty$$

diverges for general values of $\tilde{E}$

$$\Delta t = \int dt \sim \int_{r\to 2M}\frac{dr}{\left(1-\frac{2M}{r}\right)} \sim \lim_{r\to 2M}\ln(r-2M) \to \infty$$

Summary : A particle reaches the surface $r = 2M$(horizon) only after an infinite coordinate time has elapsed, while can reach in a finite amount of proper time.
In other words, the coordinate time behaves badly.

Inside the horizon ($r < 2M$)

Inside and near the horizon, let

$$\epsilon \coloneqq 2M - r$$

Then, the Schwarzschild metric

$$ds^2 = -\left(1-\frac{2M}{r}\right)dt^2 + \frac{1}{1-\frac{2M}{r}}dr^2 + r^2 d\Omega^2$$

$$= -\left(\frac{r-2M}{r}\right)dt^2 + \frac{r}{r-2M}dr^2 + r^2 d\Omega^2$$

$$= \frac{\epsilon}{2M-\epsilon}dt^2 - \frac{2M-\epsilon}{\epsilon}d\epsilon^2 + (2M-\epsilon)^2 d\Omega^2 \quad (11.2\text{-}5)$$

r-coordinate (line of constant t, θ, ϕ) is timelike:

$$ds^2 = \frac{1}{1-\frac{2M}{r}}dr^2 = -\frac{2M-\epsilon}{\epsilon}d\epsilon^2 < 0 \quad \text{timelike coordinate} \quad (11.2\text{-}6)$$

t-coordinate (line of constant r, θ, ϕ) is spacelike:

$$ds^2 = -\left(1-\frac{2M}{r}\right)dt^2 = \frac{\epsilon}{2M-\epsilon}dt^2 > 0 \quad \text{spacelike coordinate} \quad (11.2\text{-}7)$$

A particle (or a photon) inside the horizon follows timelike world line, going forward in 'time' as seen locally by the particle, which means to decreasing r, and inevitably arrives at $r = 0$.
There is a true curvature singularity ($R_{\alpha\beta\mu\nu} = \infty$) at $r = 0$.

Once a particle crosses the surface $r = 2M$, it cannot be seen by an external observer.

Coordinate systems & Coordinate singularities

Any coordinate systems can be used to describe a manifold.
Coordinate charts (systems) doesn't cover the whole manifolds in general. Each coordinate system may cover a different region.

1) Schwarzschild coordinate

$$\{(t, r, \theta, \phi) | (\theta, \phi) \in S^2, 0 \le r < 2M, r > 2M, 0 \le t < \infty\}$$

The metric is

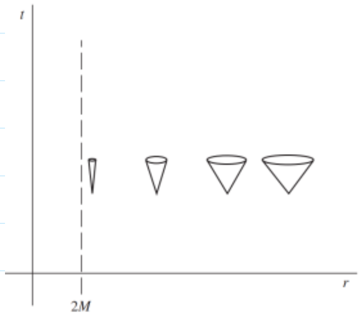
$$ds^2 = -(1-\frac{2M}{r})dt^2 + \frac{1}{1-\frac{2M}{r}}dr^2 + r^2 d\Omega^2$$

Light cones for θ & ϕ constant
(radially ingoing & outgoing null lines)
by solving $ds^2 = 0$

$$0 = ds^2 = -(1-\frac{2M}{r})dt^2 + \frac{1}{1-\frac{2M}{r}}dr^2$$

$$\frac{dt}{dr} = \pm\frac{1}{1-\frac{2M}{r}} \to \begin{cases} \pm 1 & \text{if } r \gg 2M \\ \pm\infty & \text{if } r \to 2M \end{cases} \quad (11.2\text{-}8)$$

Note) Light cones close up near the surface
Note that particle (photon) reaches at $r \to 2M$ at $t \to \infty$
⇝ coordinate singularity



2) Kruskal-Szekeres coordinates

$(r, t) \to (u, v)$

For $r > 2M$

$$u = \left(\frac{r}{2M}-1\right)^{1/2} e^{\frac{r}{4M}} \cosh\frac{t}{4M} \quad (11.2\text{-}9)$$

$$v = \left(\frac{r}{2M}-1\right)^{1/2} e^{\frac{r}{4M}} \sinh\frac{t}{4M}$$

For $r < 2M$

$$u = \left(1-\frac{r}{2M}\right)^{1/2} e^{\frac{r}{4M}} \sinh\frac{t}{4M} \quad (11.2\text{-}10)$$

$$v = \left(1-\frac{r}{2M}\right)^{1/2} e^{\frac{r}{4M}} \cosh\frac{t}{4M}$$

$(u, v) \to (r, t)$ Inverse transform

$r(u, v)$ can be obtained from

$$\left(\frac{r}{2M}-1\right)e^{\frac{r}{2M}} = u^2 - v^2 \quad (11.2\text{-}11)$$

$t(u, v)$ can be obtained from

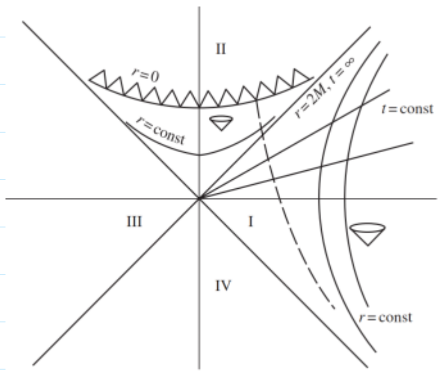
$$\tanh\frac{t}{4M} = \frac{v}{u} \quad (r > 2M) \quad (11.2\text{-}12)$$

$$\coth\frac{t}{4M} = \frac{v}{u} \quad (r < 2M)$$

Note :
The coordinate transformation is singular at $r = 2M$, which is necessary to eliminate the coordinate singularity there.

Metric

$$ds^2 = -\frac{32M^3}{r(u,v)}e^{-\frac{r(u,v)}{2M}}(dv^2 - du^2) + r(u,v)^2 d\Omega^2 \ (11.2\text{-}13)$$



Any 45° line is a radial null line
Each point is a two-sphere (θ, ϕ) of events

(u, v) : good coordinates
$(u = 0, v = 0)$: a point
↔ $(t, r = 2M)$ line in a 'bad' coordinate
⇜⇝ $(\theta = 0, \phi)$
'bad' coordinate/coordinate singularity

Region I : $r > 2M$ exterior of the horizon
Region II : $0 \le r < 2M$ interior
Region III : extension, ≈ Region I
Region IV : extension, ≈ Region II

The dashed line may be the path of the surface of the collapsing star. Then
Right Side of line : outside the star
Left Side : should be substituted by the

Metric

$$ds^2 = -\frac{32M^3}{r(u,v)} e^{-\frac{r(u,v)}{2M}} \left(dv^2 - du^2\right) + r(u,v)^2 d\Omega^2 \quad (11.2\text{-}13)$$

Note) The metric (11.2-11) is not singular at $r = 2M$

It is singular at $r = 0$ (the physical singularity)

Radial ($d\theta = 0, d\phi = 0$) null line ($ds^2 = 0$)

$dv = \pm du$ as open as in SR

Note

1) Any 45° line is a radial null line

2) Line of constant r are hyperbolae

$$u^2 - v^2 = \left(\frac{r}{2M} - 1\right) e^{\frac{r}{2M}} \equiv C = const$$

$r \gg 2M$ ($C \gg 0$), run roughly vertical,

$r \to 2M$, asymptotic to the 45° line from the origin ($u = v = 0$)

$r < 2M$ ($C < 0$), roughly horizontal, the same asymptotes

$\therefore$ timelike line cannot remain at constant r.

$r = 0$ ($C = -0$), physical singularity,

not a point but a whole hyperbola.

3) Lines of constant t are straight lines radiating outwards from the origin ($u = v = 0$)

$\frac{v}{u} = T = const$

4) The origin ($u = v = 0$) : single point $\leftrightarrow$line ($-\infty < t < \infty$) at $r = 2M$ in (t, r)-coordinates $\rightsquigarrow$ coordinate singularity

5) The true horizon is the line $r = 2M$, $t = +\infty$, which is a null line, marginal boundary between null rays that cannot get out and those that can.

6) For a distant observer

- it takes an infinite time for the infalling object to reach the horizon.
- time on the infalling clock is slowing down and eventually stopping. (an infinite gravitational redshift)

The dashed line may be the path of the surface of the collapsing star. Then

Right Side of line : outside the star

Left Side : should be substituted by the interior of the star, hence corresponding parts of regions I&II are ignorable (irrelevant) in the star dynamics including regions III &IV.

3. General black holes

Formation of black holes in general

Black hole is a structure only with gravitational interaction

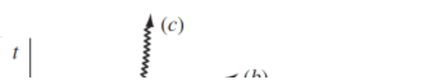
(no other interactions involved)

It Is through the gravitational mass collapse

1. Primordial black hole
2. Black holes through the star collapse

- A black hole is formed through the collapse of a star.
  Initial state : no BH, collapsing star,
  Final state : Black Hole

- A photon from the center of a (spherically) collapsing star
  earlier enough gets out
  (a) hardly feels anything,
  (b) is delayed, and
  (c) is marginally trapped.
  later than (c) is permanently trapped
- Photon (c) represent the horizon at all times, being the boundary between trapped and untrapped.
- There exists the event horizon in the black hole.
- The horizon is growing continuously from zero radius to its full size $2M$ as the collapse proceeds.
- No information can come out of the horizon.



Gravity : attractive→ leading to the collapse

To prevent the collapse,

needs pressure against the collapse

Source of the pressure

thermal pressure

degenerate pressure of fermions

Fate of the stars

after complete consumption of the fuel for the thermal pressure

White Dwarf-ex) fate of the Sun

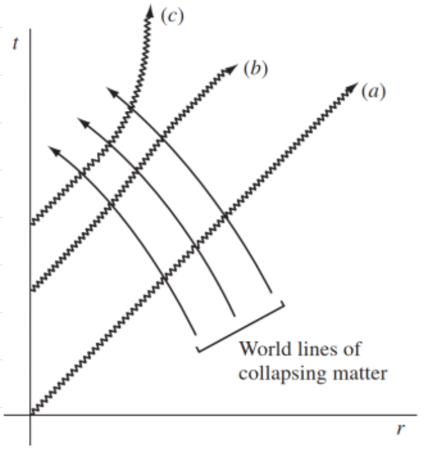
Neutron Stars

Black holes

full size $2M$ as the collapse proceeds.
- No information can come out of the horizon.

• The horizon
- is not a location in space but a boundary in spacetime.
- is a property of the spacetime as a whole, not of space at any one moment of time.

• It is not possible to determine the location of the horizon by looking at a system at one particular time;
- instead we must look at its entire evolution in time, find out which null rays eventually really do escape and which ones are trapped, and then trace out the boundary between them.
- If many years later some further gas falls into the BH with mass M, then we would find that even those null rays at $r = 2M$ would actually be trapped and would not have been part of the true event horizon after all.
- The only way to find the true horizon is to know the entire future evolution of the spacetime and then trace out the boundary between trapped and untrapped regions.



Schematic spacetime diagram of spherical collapse.
Light ray (a) hardly feels anything,
(b)is delayed, and
(c) is marginally trapped.
The horizon is defined as the ray (c),

Locally trapped surface :
- two-dimensional surface at any particular time whose outwardly directed null rays are neither expanding nor contracting at that particular moment.
- Locally trapped surfaces are always inside true horizons in dynamical situations, but very often they are so close to the true horizon that they offer an excellent approximation to it.

General properties of black holes

1. Any black hole will eventually become stationary.
   The stationary vacuum black hole uniquely characterized by two parameters :
   its total mass M & total angular momentum J,
   • Schwarzschild BH: M
   • Kerr Black Hole : M, J

2. The black hole not in the vacuum may be more general
   carrying an electric charge Q. a magnetic monopole charge F, or some other charge such as scalar.
   • Reissner-Northstrom BH : M, Q
     mass M, charge Q
   • Kerr-Newmann BH : M, L, Q
     mass M, angular momentum L, charge Q

3. During the gravitational collapse, all nonspherical parts of the mass distribution -quadrupole, octupole moments, etc.- except for the angular momentum are radiated away in gravitational waves, and a stationary Kerr type BH is left behind.

4. The area theorem (Hawking): BH thermodynamics
   • In any dynamical process involving black holes, the total area of all the horizons, like the entropy, cannot decrease in time
   • Implication : while two black holes can collide and coalesce, a single black hole can never bifurcate spontaneously into two smaller ones.
   • Assumption : the local energy density of matter in spacetime (ρ) is positive.
   • The violation of the area theorem happens by quantum effects, where energies are not always required to be positive.

5. Any horizon will contain within it a singularity where the curvature, and hence the tidal gravitational force, becomes mostly infinite.
   - Infalling geodesics are incomplete and cannot be continued for an infinite amount of proper time or affine distance.

- The existence of these singularities is a serious shortcoming of general relativity, that its predictions have limited validity in time inside horizons.
- This shortcoming may be remedied by a quantum theory of gravity.

6. The so-called naked singularities, that is singularities outside horizons would be far more problematic for general relativity, for it would mean that situations could arise in which general relativity could make no predictions beyond a certain time even for normal regions of spacetime.
- Having singularities in unobservable regions inside horizons is bad enough, but if singularities arose outside horizons, general relativity would be even more flawed.
- The cosmic censorship conjecture (Penrose (1979))
  no naked singularities can arise out of nonsingular initial conditions in asymptotically flat spacetimes.
  One naked singularity seems inescapable in general relativity: the Big Bang of standard cosmology is naked.
  If the universe re-collapses, there will similarly be a Big Crunch in the future of all our world lines.

- A massive black hole with the horizon and the entire spacetime geometry outside it is described fully and exactly by just two numbers, its mass and spin.
- All the complication of the formation process is effaced, forgotten, reduced to two simple numbers. No other macroscopic body is so simple to describe.
- For example, a star may be characterized by its mass, luminosity, and color, but these are just a start. Stars can have magnetic fields, spots, shock waves, winds, differential rotation, and many other large-scale features, to say nothing of the different motions of individual atoms and ions, including nucleosynthesis.
- For a black hole, it is simply not there. There is nothing but mass and spin, no individual structure or variety revealed by microscopic examination of the horizon.
- In fact, the horizon is not even a real surface, it is just a boundary in empty space between trapped and untrapped regions.
- The fact that no information can escape from inside the hole means that no information about what fell in is visible from the outside.
- The only quantities that remain are those that are conserved by the fundamental laws of physics: mass and angular momentum.

Kerr black hole

- axially symmetric, but not spherically symmetric
  (rotational symmetry about the angular momentum axis)
- two-parameter (M & J) solutions

cf) Angular momentum

$$[J] = [\text{mass}][\text{length}] \quad (c = 1)$$
$$= [length]^2 \quad (c = 1 = G)$$

Note : Any $\underbrace{\text{axially symmetric}}_{\phi-\text{indep}}$, $\underbrace{\text{stationary}}_{t-\text{indep}}$ metric
has preferred coordinates t and ϕ such that

$$g_{\alpha\beta,t} = 0 = g_{\alpha\beta,\phi}$$

i.e., $g_{\alpha\beta}(r,\theta)$

the remaining coordinates r and θ are orthogonal

1) to t and ϕ

$$g_{rt} = 0 = g_{\theta t}, \quad g_{r\phi} = 0 = g_{\theta\phi}$$

2) and to each other

$$g_{\theta r} = 0$$

Stationary,
axially symmetric metric

$$\begin{pmatrix} g_{tt}(r,\theta) & 0 & 0 & g_{t\phi}(r,\theta) \\ 0 & g_{rr}(r,\theta) & 0 & 0 \\ 0 & 0 & g_{\theta\theta}(r,\theta) & 0 \\ g_{t\phi}(r,\theta) & 0 & 0 & g_{\phi\phi}(r,\theta) \end{pmatrix}$$

cf) A static,
spherically symmetric metric

$$(g_{\alpha\beta}) = \begin{pmatrix} g_{tt}(r) & 0 & 0 & 0 \\ 0 & g_{rr}(r) & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin^2\theta \end{pmatrix}$$

Define

$a := \frac{J}{M}$ the same dimension as M

Metric in Boyer-Lindquist coordinates

$$ds^2 = -\frac{\Delta - a^2\sin^2\theta}{\rho^2}dt^2 - 2a\frac{2Mr\sin^2\theta}{\rho^2}dtd\phi$$

$$+\frac{\left(r^2+a^2\right)^2-a^2\Delta}{\rho^2}\sin^2\theta\, d\phi^2+\frac{\rho^2}{\Delta}dr^2+\rho^2 d\theta^2$$
(11.3-1)

where

$$\Delta := r^2-2Mr+a^2 \qquad (11.3\text{-}2)$$
$$\rho^2 := r^2+a^2\cos^2\theta \qquad (11.3\text{-}3)$$

Note
1. Surfaces $t=$ const, $r=$ const don't have the 2-sphere metric
2. $0 \le a \le M$,
   $a=0$ ⇝ Schwarzschild BH,
   $a=M$ ⇝ extremal BH (maximal angular momentum)
3. ∃ an off-diagonal metric term
$$g_{t\phi}=-a\frac{2Mr\sin^2\theta}{\rho^2}$$

4. $-g_{tt}=\frac{\Delta-a^2\sin^2\theta}{\rho^2}=\frac{r^2-2Mr+a^2\cos^2\theta}{\rho^2}=\frac{r^2-2Mr+a^2\cos^2\theta}{r^2+a^2\cos^2\theta}$
   numerator $=\rho^2-2Mr=r^2-2Mr+a^2\cos^2\theta$
   $g_{tt}=0$ (or $r^2-2Mr+a^2\cos^2\theta=0$) occurs at
   $r=r_{0+}, r_{0-}$
   where
   $r_{0\pm}=M\pm\sqrt{M^2-a^2\cos^2\theta}$ outer(inner) ergosphere

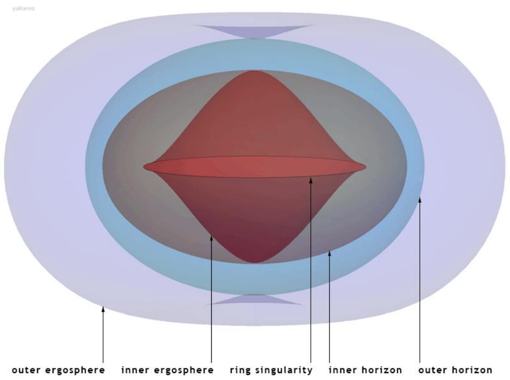
	singularity		inner ergosph		outer ergosph	
r :	0		r_{0-}		r_{0+}	
g_{tt} :		$-$	0	$+$	0	$-$

5. $g_{rr}=\frac{\rho^2}{\Delta}=\frac{\rho^2}{r^2-2Mr+a^2}=\frac{r^2+a^2\cos^2\theta}{r^2-2Mr+a^2}$
   $g_{rr}\to\infty$ if $\Delta(r=r_{h\pm})=r^2-2Mr+a^2|_{r_{h\pm}}=0$
   i.e.,
   $r_{h\pm}=M\pm\sqrt{M^2-a^2}$ outer(r_{h+})/inner(r_{h-}) horizon

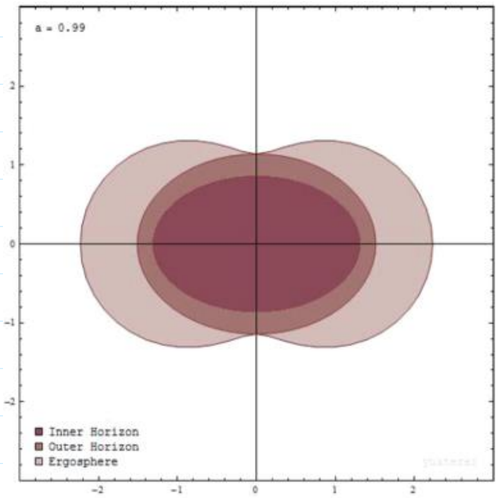
	singularity		inner horizon		outer horizon	
r :	0		r_{h-}		r_{h+}	
g_{rr} :		$+$	∞	$-$	∞	$+$

To summarize, in a Kerr solution,
the (outer) ergosphere r_0 or $r_{\text{ergosphere}}$, where
$g_{tt}(r_0)=0$,
$r_0 := r_{\text{ergosphere}}$
$= M+\sqrt{M^2-a^2\cos^2\theta}$ (11.3-4)
dependent on the angle θ
minimum$=M+\sqrt{M^2-a^2}$ at $\theta=0,\pi$ (poles)
maximum$=2M$ at $\theta=\frac{\pi}{2}$ (equator)
the (outer) horizon r_+ or r_{horizon} , where
$g_{rr}(r_+)=\infty$, i.e., $\Delta=0$
$r_+ := r_{\text{horizon}}$
$= M+\sqrt{M^2-a^2}$ (11.3-5)
independent of the angle θ
angular momentum a dependence gives
$$\underbrace{M}_{a=1} \le r_{\text{horizon}} \le \underbrace{2M}_{a=0}$$

Note : $r_{\text{ergosphere}} \ge r_{\text{horizon}}$, , equality at the poles
the Kerr ergosphere lies outside the horizon
except at the poles where they coincide.

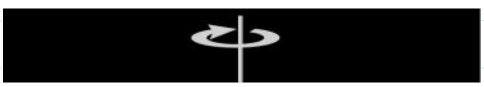


Location of the horizons, ergospheres and the ring singularity of the Kerr spacetime in Cartesian Kerr-Schild coordinates.[13]
출처: <https://en.wikipedia.org/wiki/Kerr_metric>



In the ergosphere (shown here in light gray), the component g_{tt} is positive, i.e., acts like a purely spatial metric component. Consequently, timelike or lightlike worldlines within this region must co-rotate with the inner mass. Cartesian Kerr–Schild coordinates, equatorial perspective.[1]
출처: <https://en.wikipedia.org/wiki/Ergosphere>

Ergoregion



Ergoregion

Consider photon emission in the equatorial plane ($\theta=\frac{\pi}{2}$)
at a given r, going in the $\pm\phi$-direction

$(ds^2=g_{\alpha\beta}(x)dx^\alpha dx^\beta=0,\ dr=0,\ d\theta=0)\rightsquigarrow$

$$0=g_{tt}dt^2+2g_{t\phi}dtd\phi+g_{\phi\phi}d\phi^2$$

$$\therefore\quad \frac{d\phi}{dt}=-\frac{g_{t\phi}}{g_{\phi\phi}}\pm\left[\left(\frac{g_{t\phi}}{g_{\phi\phi}}\right)^2-\frac{g_{tt}}{g_{\phi\phi}}\right]^{1/2}\qquad(11.3\text{-}9)$$

Two solutions $\underbrace{\frac{d\phi_+}{dt}>0}_{\text{forwards}}$ and $\underbrace{\frac{d\phi_-}{dt}\le 0}_{\text{backwards}}$

Note) $g_{t\phi}\le 0$, while $g_{tt}\le 0$ (("outside") $g_{tt}\ge 0$ ("inside")

Def) The ergosphere (the static limit) is the surface where

$g_{tt}|_{r_0}=0$

This occurs at

$$r_0\coloneqq r_{\text{ergosphere}}=M+\sqrt{M^2-a^2\cos^2\theta}$$

On the ergo sphere ($g_{tt}=0$),
we get,

- photon sent 'forwards' (the same direction as BH rot.)
  $\frac{d\phi_+}{dt}=-2\frac{g_{t\phi}}{g_{\phi\phi}}>0$, (the same sign as the parameter a.)
- photon sent 'backwards'
  $\frac{d\phi_-}{dt}=0$,
  * the photon sent 'backwards' initially doesn't move
  * The dragging of orbits is so strong that
    this photon cannot move opposite the rotation

Note :

1) A particle moves slower than photons hence rotates with the hole indep of ang. mom in the opposite sign.
2) The ergosphere (surface with $g_{tt}=0$) lies outside the horizon.
3) Inside the ergosphere, no particle can remain at fixed r,θ,ϕ, and also $g_{tt}>0$. hence all particles and photons must rotate with the hole.
4) Ergoregions : $g_{tt}>0$.
   There may be toroidal ergoregions. Their boundaries are then ergotoroids. They can exist in solutions which have no horizon at all. The stars are extremely compact (relativistic) and very rapidly rotating. It seems unlikely that real neutron stars would have ergoregions.

The Kerr horizon

the ergosphere at $g_{tt}=0$,
the horizon at $g_{rr}=\infty$, i.e., $\Delta=0$

$$r_+:=r_{\text{horizon}}=M+\sqrt{M^2-a^2}$$

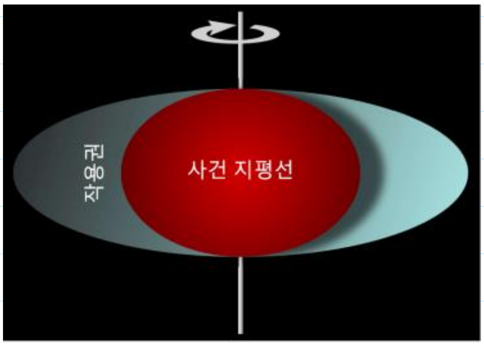
Note : the Kerr ergosphere lies outside the horizon
except at the poles

The horizon is a surface of constant r and t

$\rightarrow dr=0=dt$

The line element on the horizon,

$$d\ell^2=\frac{(r^2+a^2)^2-a^2\Delta}{\rho^2}\sin^2\theta\,d\phi^2+\rho^2d\theta^2$$
$$=g^h_{\phi\phi}d\phi^2+g^h_{\theta\theta}d\theta^2$$



커 계량에서는 두 가지 표면의 특이점을 가진 것으로 보인다. 내부의 표면은 사건 지평선이고, 외부의 표면은 정지 한계라고 불리는 편구면(oblate spheroid)이다. 작용권은 두 표면 사이에 있다.

출처: <https://ko.wikipedia.org/wiki/%EC%9E%91%EC%9A%A9%EA%B6%8C>

The proper Area

$$A(r) = \int \sqrt{\det g^h_{\alpha\beta}}\, d\phi d\theta = \int_0^{2\pi} d\phi \int_0^{\pi} d\theta \sqrt{g^h_{\phi\phi} g^h_{\theta\theta}}$$
$$= \underbrace{\int_0^{2\pi} d\phi \int_0^{\pi} d\theta \sin\theta}_{4\pi} \sqrt{(r^2+a^2)^2 - a^2\Delta}$$
$$= 4\pi\sqrt{(r^2+a^2)^2 - a^2\Delta} \quad (11.3\text{-}10)$$

The proper area of the Horizon ($\Delta = 0$)

$$A(\text{Horizon}) = 4\pi\left(r_+{}^2 + a^2\right) \quad (11.3\text{-}11)$$

1. Real black holes in astronomy

Note) The unique stationary (time-independent) solution for a black hole is the Kerr metric,
with just two intrinsic degrees of freedom: its mass and spin.
This is unlike any other system such as each star with 10^{57} particles which has an enormous number of physical degrees of freedom.

Blak hole observation

Class	Approx. mass	Approx. radius
Supermassive black hole	10^5–10^{10} M☉	0.001–400 AU
Intermediate-mass black hole	10^3 M☉	10^3 km ≈ R_{Earth}
Stellar black hole	10 M☉	30 km
Micro black hole	up to M_{Moon}	up to 0.1 mm

Black hole classifications

출처: <https://en.wikipedia.org/wiki/Black_hole#Detection_of_gravitational_waves_from_merging_black_holes>

Identifying systems containing black holes is
- indirect evidence, on their effects on nearby gas and stars.
- The small size of black holes require to make an observation either
- with very high angular resolution to see matter near the horizon

or
- by using photons of very high energy, which originate from strongly compressed and heated matter near the horizon.
- Direct evidence : Gravitational wave detectors became sufficiently sensitive to detect radiation from black holes.

Direct observation of the black holes
Gravitational wave detection from the black hole merger.

1. Black holes of stellar mass
2. An isolated black hole, formed by the collapse of a massive star,
   would be very difficult to identify.
   It might accrete a small amount of gas as it moves through the interstellar medium, but this gas would not emit much X-radiation before being swallowed.
   No such candidates have been identified.

Ii) Black hole candidates, in binary systems in our Galaxy
- All known stellar-mass black holes are in binary systems, using orbiting X-ray observatories since the 1970s
- masses around 10 $M_\odot$, arisen from the gravitational collapse of a massive star
- Companion star is so large that it begins dumping gas on to the hole.
  Being in a binary system, the gas has angular momentum, and so it forms a disk around the black hole.
  But within this disk there is friction, possibly caused by turbulence or by magnetic fields.
  Friction leads material to spiral inwards through the disk, giving up angular momentum and energy.
  Some of this energy goes into the internal energy of the gas, heating it up to temperatures in excess of 10^6 K, so that the peak of its emission spectrum is at X-ray wavelengths
- Many such X-ray binary systems are now known. Not all of them contain black holes:
  a neutron star is compact enough so that gas accreting on to it will also reach X-ray temperatures.
- Distinguishing black holes from neutron stars in these systems by two means: mass and pulsations.
  (1) If the accreting object pulsates in X-rays at a very steady rate, then it is a pulsar and it cannot be a black hole: black holes cannot hold on to a magnetic field and make it rotate with the hole's rotation.

(2) Most systems do not show such pulsations, however.
In these cases, estimate the mass of the accreting object from observations of the velocity and orbital radius of the companion star (obtained by monitoring the Doppler shift of its spectral lines) and from an estimate of the companion's mass (again from its spectrum).
If the accreting object has an estimated mass much more than about $5M_{\odot}$, then it is believed likely to be a black hole.
This is because the maximum mass of a neutron star cannot be much more than $3M_{\odot}$ and is likely much smaller.
Mass turns out to be a good discriminator: There does seem to be a mass gap, therefore, between black holes and neutron stars, at least for objects formed in binaries.
Ex) Cyg X-1, mass is around $10M_{\odot}$.

The largest stellar black-hole mass ever estimated is $70M_{\odot}$, for the black hole in the binary system M33 X-7.
in the galaxy M33, which at 1 Mpc is about twice as far from our Galaxy as the Andromeda Galaxy (M31) is.
It is an eclipsing system, which constrains the inclination angle enough to make the mass estimate more secure.
stellar evolution occasionally produces black holes in supernova explosions,

- Most gamma-ray bursts are associated with black-hole formations from very rapidly rotating progenitor stars.

- How do we know that these massive objects are black holes? The answer is that any other explanation is less plausible.
  They cannot be neutron stars: no equation of state that is causal (i.e. has a sound speed less than the speed of light) can support more than about $3M_{\odot}$.
- It would be possible to invent some kind of exotic matter (sometimes called bosonic matter) that might just make a massive compact object that does not collapse.

Directo observation

- detected the signature of black holes in gravitational waves, such as their ringdown radiation.

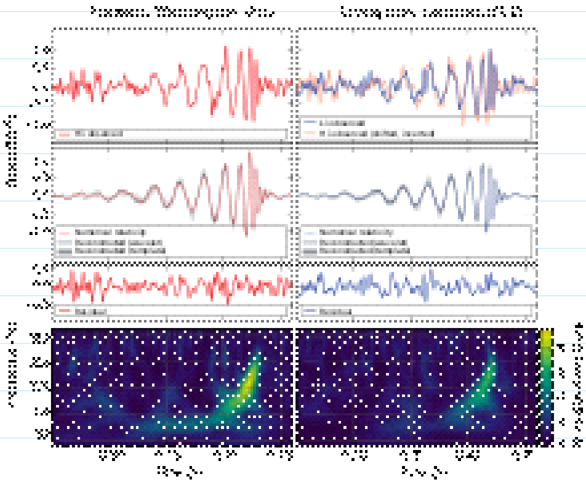
Detection of gravitational waves from merging black holes

On 14 September 2015 the LIGO gravitational wave observatory made the first-ever successful direct observation of gravitational waves.[11][165] The signal was consistent with theoretical predictions for the gravitational waves produced by the merger of two black holes: one with about 36 solar masses, and the other around 29 solar masses.[11][166] This observation provides the most concrete evidence for the existence of black holes to date. For instance, the gravitational wave signal suggests that the separation of the two objects prior to the merger was just 350 km (or roughly four times the Schwarzschild radius corresponding to the inferred masses). The objects must therefore have been extremely compact, leaving black holes as the most plausible interpretation.[11]

More importantly, the signal observed by LIGO also included the start of the post-merger ringdown, the signal produced as the newly formed compact object settles down to a stationary state. Arguably, the ringdown is the most direct way of observing a black hole.[167] From the LIGO signal it is possible to extract the frequency and damping time of the dominant mode of the ringdown. From these it is possible to infer the mass and angular momentum of the final object, which match independent predictions from numerical simulations of the merger.[168]

출처: <https://en.wikipedia.org/wiki/Black_hole#Detection_of_gravitational_waves_from_merging_black_holes>

LIGO measurement of the gravitational waves at the Livingston (right) and Hanford (left) detectors, compared with the theoretical predicted values	
Distance	410+160 −180 Mpc[1]
Redshift	0.093+0.030 −0.036[1]
Total energy output	3.0+0.5 −0.5 $M_{\odot}$ × c^2[2][note 1]
Other designations	GW150914
Related media on Wikimedia Commons	
[edit on Wikidata]	



GW150914

출처: <https://en.wikipedia.org/wiki/First_observation_of_gravitational_waves>

1. supermassive black holes in the centers of galaxies
   since the 1990s
   with masses ranging from 10^6 to $10^7 M_\odot$.

It came as a big surprise when astronomers found that almost all galaxies that are close enough to allow the identification
of a central black hole have turned out to contain one.
Indeed, one of the most secure black hole identifications is the 4.3 × 106M black hole in the center of our own Galaxy (Genzel and Karas 2007)!

1. The supermassive black hole in the center of the MilkyWay
- Mass = 4.3 ×$10^6 M_\odot$, the best evidence & the closest one to us:
- discovered by repeated high-resolution measurements of the positions of stars in the very center of the Galaxy.
- using infrared light, because in visible light the center is obscured by interstellar dust.
- Over a period of ten years some of the stars were observed to move by very considerable distances, and not just on straight lines, but rather on clearly elliptical orbits.
- All the orbits had a common gravitating center, but the center was dark.
- The stars' spectra revealed their radial velocities, so with three-dimensional velocities and a good idea of how far away the galactic center is, it is possible to estimate the mass of the central object from each orbit.
- All the orbits are consistent, and point to a dark object of 4.3 × $10^6 M_\odot$ directly at the center.
- Remarkably, one of the orbiting stars approaches to within about 120 AU of the gravitating center, and attains a speed of some 5000 km s−1, more than 1% of the speed of light!
- Other than a black hole, no known or plausible matter system could contain such a large amount of mass in such a small volume, without itself collapsing rapidly to a black hole.

Proper motions of stars orbiting Sagittarius A*

The proper motions of stars near the center of our own Milky Way provide strong observational evidence that these stars are orbiting a supermassive black hole.[170] Since 1995, astronomers have tracked the motions of 90 stars orbiting an invisible object coincident with the radio source Sagittarius A*. By fitting their motions to Keplerian orbits, the astronomers were able to infer, in 1998, that a 2.6×10⁶ M☉ object must be contained in a volume with a radius of 0.02 light-years to cause the motions of those stars.[171] Since then, one of the stars—called S2—has completed a full orbit. From the orbital data, astronomers were able to refine the calculations of the mass to 4.3×10⁶ M☉ and a radius of less than 0.002 light-years for the object causing the orbital motion of those stars.[170] The upper limit on the object's size is still too large to test whether it is smaller than its Schwarzschild radius; nevertheless, these observations strongly suggest that the central object is a supermassive black hole as there are no other plausible scenarios for confining so much invisible mass into such a small volume.[171] Additionally, there is some observational evidence that this object might possess an event horizon, a feature unique to black holes.[172]

출처: <https://en.wikipedia.org/wiki/Black_hole#Detection_of_gravitational_waves_from_merging_black_holes>

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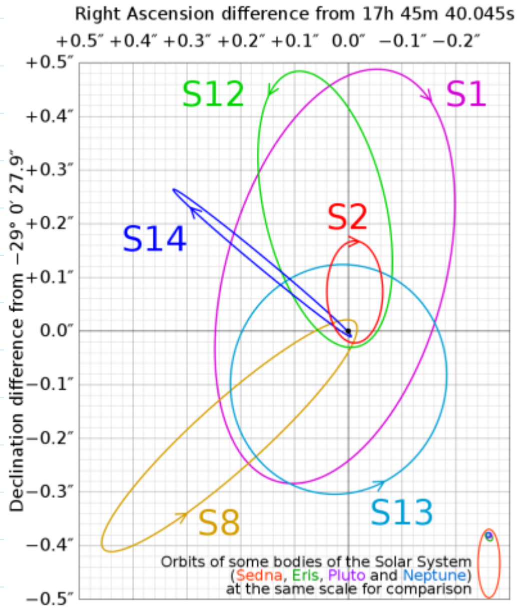
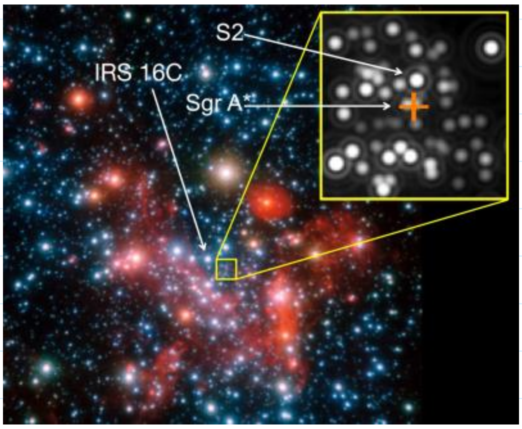
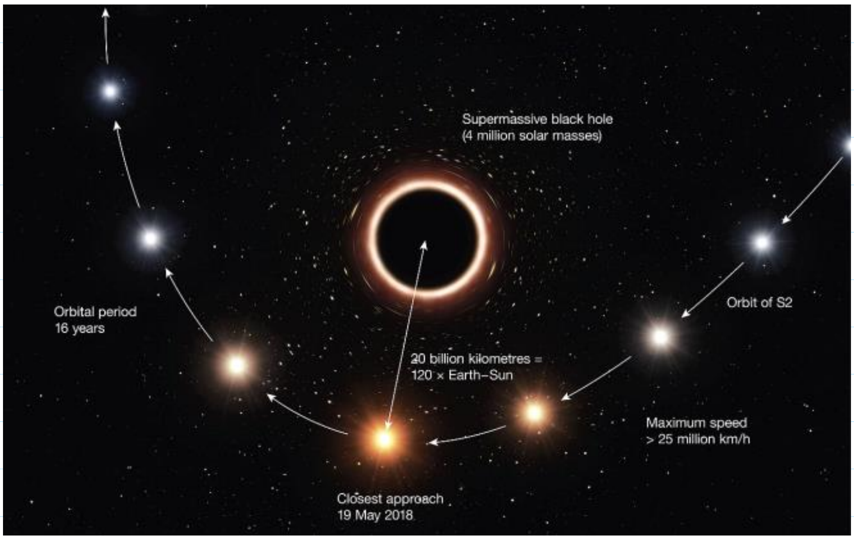


Inferred orbits of S2 and five other stars around supermassive black hole candidate Sgr A* at the Milky Way galactic centre[30]

출처: <https://en.wikipedia.org/wiki/S2_(star)>





ii) Other black holes of similar masses in external galaxies, including in our nearest neighbor M31.

- It seems that black holes are associated with galaxy formation, but the nature of this association is not clear: did the holes come first, or did they form as part of the process of galaxy formation?
- It is also not clear whether the holes formed with large mass, say 10^5–$10^6 M_\odot$, or whether they started out with

smaller masses (perhaps 10^4–$10^5 M_\odot$) and grew later.

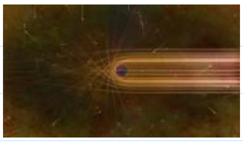
- And if they grew, it is not clear whether they grew by accreting gas and stars or by merging with other black holes.
- This last question may be answered by the LISA satellite, which will detect mergers over a wide mass range throughout the universe.

- the quasar phenomenon is created by gas accreting on to much more massive black holes, typically $10^9 M_\odot$. the only quasar model that has survived decades of observation in many wave bands is the black hole model. Since quasars were much more plentiful in the early universe than they are today, it seems that these ultra-massive black holes had to form very early, while their more modest counterparts like that, in the Milky Way might have taken longer.
- This suggests that the black holes in quasars did not form by the growth of holes like our own; this is another of the unanswered questions about supermassive black holes.
- a good relationship between the mass of the central black hole and the velocity dispersion (random velocities) in the galactic bulge. The more massive the hole, the higher the velocities. This relationship seems to have a simple form all the way from 10^6 to $10^{10} M_\odot$. It might be a clue to how the holes formed.
- galaxies frequently merge, and this ought to bring at least some of their black holes to merge as well, producing strong gravitational waves in the LISA band. In fact, current models for galaxy formation suggest that all galaxies are themselves the products of repeated mergers with smaller clusters of stars, and so it is possible that, in the course of the formation and growth of galaxies, the central black holes grew larger and larger by merging with incoming black holes.
- As remarked before, LISA should decide this issue, but already there is a growing body of evidence for mergers of black holes. A number of galaxies with two distinct bright cores are known, and remarkable evidence was announced for the ejection of a supermassive black hole at something like 1% of the speed of light from the center of a galaxy. Speeds like this can only be achieved as a result of the 'kick' that a final black hole gets as a result of the merger (see below).

Messier 87 galaxy – home of the first imaged black hole

On 10 April 2019

radio waves.



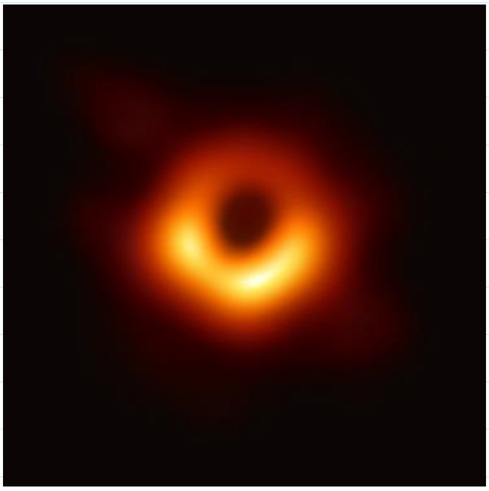
This artist's impression depicts the paths of photons in the vicinity of a black hole. The gravitational bending and capture of light by the event horizon is the cause of the shadow captured by the Event Horizon Telescope.

The Event Horizon Telescope (EHT), is an active program that directly observes the immediate environment of the event horizon of black holes, such as the black hole at the centre of the Milky Way. In April 2017, EHT began observation of the black hole in the center of Messier 87.[155] "In all, eight radio observatories on six mountains and four continents observed the galaxy in Virgo on and off for 10 days in April 2017" to provide the data yielding the image two years later in April 2019.[156] After two years of data processing, EHT released the first direct image of a black hole, specifically the supermassive black hole that lies in the center of the aforementioned galaxy.[157][158] What is visible is not the black hole, which shows as black because of the loss of all light within this dark region, rather it is the gases at the edge of the event horizon, which are displayed as orange or red, that define the black hole.[159]

The brightening of this material in the 'bottom' half of the processed EHT image is thought to be caused by Doppler beaming, whereby material approaching the viewer at relativistic speeds is perceived as brighter than material moving away. In the case of a black hole, this phenomenon implies that the visible material is rotating at relativistic speeds (>1,000 km/s), the only speeds at which it is possible to centrifugally balance the immense gravitational attraction of the singularity, and thereby remain in orbit above the event horizon.

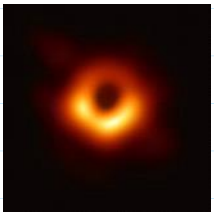
출처: <https://en.wikipedia.org/wiki/Black_hole#Detection_of_gravitational_waves_from_merging_black_holes>

출처: <https://en.wikipedia.org/wiki/Black_hole#Detection_of_gravitational_waves_from_merging_black_holes>

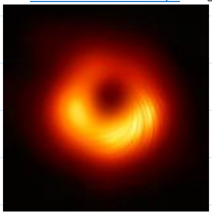


The supermassive black hole at the core of supergiant elliptical galaxy Messier 87, with a mass about 7 billion times that of the Sun,[18] as depicted in the first false-colour image in radio waves released by the Event Horizon Telescope (10 April 2019).[19][14][20][21] Visible are the crescent-shaped emission ring and central shadow,[22] which are gravitationally magnified views of the black hole's photon ring and the photon capture zone of its event horizon. The crescent shape arises from the black hole's rotation and relativistic beaming; the shadow is about 2.6 times the diameter of the event horizon.[14]

출처: <https://en.wikipedia.org/wiki/Black_hole>



The Event Horizon Telescope image of the core of M87 using 1.3 mm radio waves. The central dark spot is the shadow of M87* and is larger than the black hole's event horizon.



A view of the M87* supermassive black hole in polarised light, taken by Event Horizon Telescope and revealed on 24 March 2021. The direction of lines atop the total intensity mark the direction of the electromagnetic wave electric vector oscillations.

출처: <https://en.wikipedia.org/wiki/Messier_87>

1. Intermediate-mass black holes
   - black holes with masses around 100–$10^4 M_\odot$
   - Some bright X-ray sources may be large black holes, and perhaps there are black holes in this mass range in globular clusters.
   - These objects might have formed in the first generation of star formation, when clouds made up of pure hydrogen and helium first collapsed. These first stars were probably a lot more massive than the stars, like our Sun, that formed later from gas clouds that had been polluted by the heavier elements made by the first generation of stars.

Numerical simulations suggest that some of these first stars (called Population III stars) could have rapidly collapsed to black holes.
- unpublished data from the European Space Agency suggest the existence of a black hole with mass $4 \times 10^4 M_\odot$ in the cluster Omega Centauri. Observations of similar objects may well reveal more such black holes.

1. Dynamical black holes

- When a black hole is formed, any initial asymmetry (such as quadrupole moments) must be radiated away in gravitational waves, until finally only the mass and angular momentum are left behind. This generally happens quickly: studies of linear perturbations of black holes show that black holes have a characteristic spectrum of oscillations, but that they typically damp out (ring down) exponentially after only a few cycles. The Kerr metric takes over very quickly.
- Binary systems involving black holes will eventually merge, and black holes in the centers of galaxies can merge with other massive holes when galaxies merge. When black holes are involved the full metric has a singularity where its components diverge; this has somehow to be removed from the numerical domain.
- One of the most interesting aspects of black hole mergers is the so-called 'kick'. When there is no particular symmetry in the initial system, then the emitted gravitational radiation will emerge asymmetrically, so that the waves will carry away a net linear momentum in some direction. The result will be that the final black hole recoils in the opposite direction. The velocity of the recoil, being dimensionless, does not depend on the overall mass scale of the system, just on dimensionless initial data: the ratio of the masses of the initial holes, the dimensionless spin parameters of the holes (a/M), and the directions of the spins. Normally these recoil velocities are of order a few hundred km s—1, which could be enough to expel the black hole from the center of a star cluster or even a spiral galaxy. Even more remarkably, recoil velocities exceeding 10% of the speed of light are inferred from simulations for some coalescences.

1. **Quantum mechanical emission of radiation by black holes**
Hawking process, Hawking radiation

- black holes radiate energy continuously! (1974 Stephen Hawking) due to quantum property of electromagnetic fields near a black hole.
- Photons , according to the uncertainty principle, cannot be localized to arbitrary precision. Near the horizon this markedly changes the behavior of 'real' photons from what we have already described for idealized null particles.

- Hawking's prediction based on the calculation (Hawking 1975) of quantum field theory can be derived very simply from a 'plausibility argument'. One form of the uncertainty principle is $\Delta E \Delta t \geq \hbar/2$, where ΔE is the minimum uncertainty in a particle's energy which resides in a quantum mechanical state for a time Δt.
- According to quantum field theory, ordinary space is filled with 'vacuum fluctuations' in electromagnetic fields, which consist of pairs of photons being produced at one event and recombining at another.
- Such pairs violate conservation of energy, but if they last less than $\Delta t = \hbar/2\Delta E$, where ΔE is the amount of violation, they violate no physical law.
- Thus, in the large scale, energy conservation holds rigorously, while, on a small scale, it is always being violated.
- Consider a fluctuation which produces two photons, one of energy E and the other with energy —E. In flat spacetime the negative-energy photon would not be able to propagate freely, so it would necessarily recombine with the positive-energy one within a time $\hbar/2\Delta E$.
- But if produced just outside the horizon, it has a chance of crossing the horizon before the time $\hbar/2\Delta E$ elapses; once inside the horizon it can propagate freely, as we shall now show.
- Consider the Schwarzschild metric for simplicity, and recall from our discussion of orbits in the Kerr metric that negative energy is normally excluded because it corresponds to a particle that propagates backwards in time. Inside the event horizon, an observer going forwards in time is one going toward decreasing r.
- For simplicity let us choose one on a trajectory for which $p_0 = 0 = U^0 = 0$. Then U^r is the only nonzero component of $\mathbb{U}$, and by the normalization condition $\mathbb{U} \cdot \mathbb{U} = -1$ we find U^r:

$$U^r = -\left(\frac{2M}{r} - 1\right)^{\frac{1}{2}}, \qquad r < 2M$$

negative because the observer is ingoing. Any photon orbit is allowed for which $-\mathbb{P} \cdot \mathbb{U} > 0$

- Consider a zero angular-momentum photon, moving radially inside the horizon. it clearly has E = $\pm p^r$. Then its energy relative to the observer is

$$-\mathbb{P} \cdot \mathbb{U} = -p^r U^r g_{rr} = -\left(\frac{2M}{r} - 1\right)^{-\frac{1}{2}} p^r$$

This is positive if and only if the photon is also ingoing: $p^r < 0$. But it sets no restriction at all on E. Photons may travel on null geodesics inside the horizon, which have either sign of E, as long as $p^r < 0$. (E is a spatial momentum component there.)
Since a fluctuation near the horizon can put the negative-energy photon into a realizeable trajectory, the positive-energy photon is allowed to escape to infinity.

- Let us see what we can say about its energy. We first look at the fluctuations in a freely falling inertial frame, which is the one for which spacetime is locally flat and in which the fluctuations should look normal. A frame that is momentarily at rest at coordinate 2M + ε will immediately begin falling inwards, following the trajectory of a particle with $\tilde{L} = 0$ and $\tilde{E} = [1 - 2M/(2M + \varepsilon)]^{1/2} \approx (\varepsilon/2M)^{1/2}$,
- It reaches the horizon after a proper-time lapse τ obtained by integrating Eq. (11.59):

$$\Delta\tau = -\int_{2M+\epsilon}^{2M} \left(\frac{2M}{r} - \frac{2M}{2M+\epsilon}\right)^{-\frac{1}{2}} dr = 2(2M\epsilon)^{\frac{1}{2}}$$

This should be equal to $\hbar/2\mathcal{E}$ result in

$$\mathcal{E} = \frac{1}{4}\hbar(2M\epsilon)^{-\frac{1}{2}}$$

This is the energy of the outgoing photon, the one which reaches infinity, as calculated on the local inertial frame. To find its energy when it gets to infinity we recall that

$$\mathcal{E} = -\mathbb{P}\cdot\mathbb{U} \text{ with } -U_0 = \tilde{E} = (\epsilon/2M)^{1/2}$$

$$E = \mathcal{E}(\epsilon/2M)^{1/2} = \frac{\hbar}{8M} \qquad (11.104)$$

- **Rigorous calculation**

the photons which come out have the spectrum characteristic of a black body with a temperature

$$\boldsymbol{T_H = \frac{\hbar}{8\pi k M}}$$

The energy of the photon at the peak of the black body spectrum (Wien's displacement law)

$$E = 4.965kT = \frac{1.580\hbar}{8M}$$

fairly close to our crude result, Eq. (11.104). Our argument does not show that the photons should have a black-body spectrum; but the fact that the spectrum originates in random fluctuations, plus the fact that the black hole is, classically, a perfect absorber, makes this result plausible as well.

- the negative-energy photons in the Hawking effect are not the same as the negative-energy photons that we discussed in the Penrose process above.
- The Penrose process works only inside an ergoregion, and uses negative-energy orbits that are outside the horizon of the black hole.
- The Hawking result is more profound: it operates even for a nonspinning black hole and connects negative-energy photons inside the horizon with positive-energy counterparts outside. It operates in the Kerr metric as well, but again it happens across the horizon, not the ergosphere.
- The Hawking effect does not lead to an unstable runaway, the way the Penrose process does for a star with an ergoregion. This is because Hawking's negative-energy photon is already inside the horizon and does not create any further positive-energy photons outside.
- So the Hawking radiation is a steady thermal radiation, by ever-present quantum fluctuations near the horizon.
- Notice that the Hawking temperature of the hole is proportional to M–1.

- The rate of radiation from a black body is proportional to AT4, where A is the area of the body, in this case of the horizon, which is proportional to M2 (see Eq. (11.85)). So the luminosity of the hole is proportional to M–2. This energy must come from the mass of the hole (every negative-energy photon falling into it decreases M), so we have

Black body radiation of Photons with a temperature

$$\frac{\mathrm{dM}}{\mathrm{dt}} \sim \frac{1}{M^2}$$

$$M^2 dM \sim dt$$

The bigger the hole the longer it lives, and the cooler its temperature. The numbers work out that a hole of mass 10^12 kg has a lifetime of 10^10 yr, about the age of the universe. Thus

Life time $\tau \sim M^3$

$$\left(\frac{\tau}{10^{10}\mathrm{yr}}\right) = \left(\frac{M}{10^{12}\mathrm{kg}}\right)^3$$

Since a solar mass is about 1030 kg, black holes formed from stellar collapse are essentially unaffected by this radiation, which has a temperature of about 10–7 K. On the other hand, it is possible for holes of 1012 kg to form in the very early universe. To see the observable

Ex) $M = M_{\odot} \sim 10^{30}\mathrm{kg}$

$T_H = 10^{-7}$K

effect of their 'evaporation', let us calculate the energy radiated in the last second by setting
τ = 1 s = (3 × 107)—1 yr in Eq. (11.109). We get
M ≈ 106 kg ~ 1023 J. (11.110)
So for a brief second it would have a luminosity about 0.1% of the Sun's luminosity, but in spectrum it would be very different. Its temperature would be 1011 K, emitting primarily in γ -rays! We might be tempted to explain the gamma-ray bursts mentioned earlier in this chapter as primordial black hole evaporations, but the observed gamma bursts are in fact billions of times more luminous. A primordial black-hole evaporation would probably be visible only if it happened in our own Galaxy. No such events have been identified.
It must be pointed out that all derivations of Hawking's result are valid only if the typical photon has E M, since they involve treating the spacetime of the black hole as a fixed background in which we solve the equations of quantum mechanics, unaffected to first order by the propagation of these photons. This approximation fails for M ≈ h/M, or for black holes of mass
MPl = h1/2 = 1.6 × 10—35 m = 2.2 × 10—8 kg. (11.111)
This is called the Planck mass, since it is a mass derived only from Planck's constant (and c and G). To treat quantum effects involving such holes, we need a consistent theory of quantum gravity, which is one of the most active areas of research in relativity today. All we can say here is that the search has not yet proved fully successful, but Hawking's calculation appears to have been one of the most fruitful steps.
The Hawking effect has provided a remarkable unification of gravity and thermodynamics. Consider Hawking's area theorem, which we may write as

$$0 = \frac{d}{dr}\tilde{V}^2(r) = \frac{d}{dr}\left[\left(1-\frac{2M}{r}\right)\left(1+\frac{\tilde{L}^2}{r^2}\right)\right] = \frac{2M}{r^2}\left(1-\frac{\tilde{L}^2}{2M}\frac{1}{r}+\frac{3\tilde{L}^2}{r^2}\right) = \frac{2M}{r^4}\left(r^2-\frac{\tilde{L}^2}{2M}r+3\tilde{L}^2\right)$$

$$r_\pm = \frac{\tilde{L}^2}{2M}[1 \pm \left(1-\frac{12M^2}{\tilde{L}^2}\right)^{1/2} \quad (11.1\text{-}13)$$

$$d\tau = -\frac{dr}{\left(\tilde{E}^2-1+\frac{2M}{r}\right)^{1/2}} \quad (11.2\text{-}1)$$

Hence,

$$\Delta\tau = \int d\tau = -\int_R^{2M}\frac{dr}{\left(\tilde{E}^2-1+\frac{2M}{r}\right)^{\frac{1}{2}}} \quad (11.2\text{-}2)$$

If $\tilde{E}^2 > 1$ (unbounded particle), then

$\Delta\tau < \infty$ finite

If $\tilde{E}^2 = 1$ (particle falling from rest at ∞),

$$\Delta\tau = -\int_R^{2M}\sqrt{\frac{r}{2M}}dr = \frac{4M}{3}\left[\left(\frac{r}{2M}\right)^{3/2}\right]_{2M}^{R} < \infty \text{ finite}$$

$M_{Pl} = h^{\frac{1}{2}} = 1.6 \times 10^{-35}$m = 2.2×10^{-8}kg

Planck mass

$M_{Pl} = \sqrt{\frac{\hbar c}{G}} = 1.2209 \times 10^{19}\text{GeV}/c^2 = 1.9561 \times 10^9 \text{J}/c^2 = 2.176 \times 10^{-8}$kg

Planck length

$\ell_{Pl} = \sqrt{\frac{\hbar G}{c^3}} = 1.616 \times 10^{-44}$sec

Planck time

$t_{Pl} = \sqrt{\frac{\hbar G}{c^5}} = 5.391 \times 10^{-35}$m

Planck temperature

$T_{Pl} = \sqrt{\frac{\hbar c^5}{G{k_B}^2}} = 1.4168 \times 10^{32}\mathrm{K}$

Gravity + thermodynamics
The 1st law
$dE = T_H dS$

Entropy
$S = \frac{kA}{4\hbar}$
Cf) Boltzmann constant

Values of k[1]	Units
1.380649×10^{-23}	J·K ^{1}
$8.617333262145\times10^{-5}$	eV·K ^{1}
1.380649×10^{-16}	erg·K ^{1}

출처: <https://en.wikipedia.org/wiki/Boltzmann_constant>

Measurement	Unit	SI value of unit
Energy	eV	$1.602176634\times10^{\ 19}$ J
Mass	eV/c^2	$1.782662\times10^{\ 36}$ kg
Momentum	eV/c	$5.344286\times10^{\ 28}$ kg·m/s
Temperature	eV/k_B	1.160451812×10^4 K
Time	$\hbar$/eV	$6.582119\times10^{\ 16}$ s
Distance	$\hbar c$/eV	$1.97327\times10^{\ 7}$ m

출처: <https://en.wikipedia.org/wiki/Electronvolt>

Outside the horizon ($r > 2M$)

Consider Infalling particles from $r = R$ to $r = 2M$ falling radially ($d\theta = 0 = d\phi$)

1) Proper time elapse
(11.1-10) :

$$\left(\frac{dr}{d\tau}\right)^2 = \tilde{E}^2 - \left(1 - \frac{2M}{r}\right)\left(1 + \frac{\tilde{L}^2}{r^2}\right)$$

$$m\frac{d\phi}{d\tau} = p^\phi = g^{\phi\phi}p_\phi = m\frac{1}{r^2}\tilde{L},$$

$$d\phi = 0 \rightsquigarrow \tilde{L} = 0$$

$$= \tilde{E}^2 - 1 + \frac{2M}{r}$$

Or,

$$d\tau = -\frac{dr}{\left(\tilde{E}^2 - 1 + \frac{2M}{r}\right)^{1/2}} \quad (11.2\text{-}1)$$

Hence,

$$\Delta\tau = \int d\tau = -\int_R^{2M}\frac{dr}{\left(\tilde{E}^2 - 1 + \frac{2M}{r}\right)^{\frac{1}{2}}} \quad (11.2\text{-}2)$$

If $\tilde{E}^2 > 1$ (unbounded particle), then
$\Delta\tau < \infty$ finite
If $\tilde{E}^2 = 1$ (particle falling from rest at ∞),

$$\Delta\tau = -\int_R^{2M}\sqrt{\frac{r}{2M}}dr = \frac{4M}{3}\left[\left(\frac{r}{2M}\right)^{3/2}\right]_{2M}^{R} < \infty \text{ finite}$$

If $\tilde{E} < 1$, then $r \le r_{max}$ where 1-$\tilde{E}^2 = \frac{2M}{r_{max}}$ and

$\Delta\tau < \infty$ finite

Summary : Any particle can reach the horizon in a finite amount of proper time.
The particle can go inside ($r < 2M$) in a finite proper time

2) Coordinate time elapses

(11.1-18)

$$\frac{dt}{d\tau} \coloneqq U^0 = \frac{p^0}{m} = g^{00}\frac{p_0}{m} = g^{00}(-\tilde{E}) = \frac{\tilde{E}}{1-\frac{2M}{r}} \quad (11.2\text{-}3)$$

$$dt = \frac{\tilde{E}d\tau}{1-\frac{2M}{r}} = -\frac{\tilde{E}dr}{\left(1-\frac{2M}{r}\right)\left(\tilde{E}^2-1+\frac{2M}{r}\right)^{1/2}} \quad (11.2\text{-}4)$$

(Consider the case of $\tilde{E} = 1$ for simplicity)

$$= -\frac{dr}{\left(1-\frac{2M}{r}\right)\left(\frac{2M}{r}\right)^{1/2}} = -\frac{r^{3/2}dr}{(r-2M)(2M)^{1/2}}$$

Near the horizon

$\epsilon \coloneqq r - 2M$

$$= \frac{-(\epsilon+2M)^{3/2}d\epsilon}{(2M)^{1/2}\epsilon}$$

$$\Delta t = \int dt = \int_{\epsilon\to 0}\frac{(\epsilon+2M)^{3/2}d\epsilon}{(2M)^{1/2}\epsilon}$$

$\sim \ln\epsilon \to \infty$

diverges for general values of $\tilde{E}$

$$\Delta t = \int dt \sim \int_{r\to 2M}\frac{dr}{\left(1-\frac{2M}{r}\right)} \sim \lim_{r\to 2M}\ln(r-2M) \to \infty$$

Summary : A particle reaches the surface $r = 2M$(horizon) only after an infinite coordinate time has elapsed, while can reach in a finite amount of proper time.
In other words, the coordinate time behaves badly.

Inside the horizon ($r < 2M$)

Inside and near the horizon, let

$\epsilon \coloneqq 2M - r$

Then, the Schwarzschild metric

$$ds^2 = -(1-\frac{2M}{r})dt^2 + \frac{1}{1-\frac{2M}{r}}dr^2 + r^2d\Omega^2$$

$$= -(\frac{r-2M}{r})dt^2 + \frac{r}{r-2M}dr^2 + r^2d\Omega^2$$

$$= \frac{\epsilon}{2M-\epsilon}dt^2 - \frac{2M-\epsilon}{\epsilon}d\epsilon^2 + (2M-\epsilon)^2d\Omega^2 \quad (11.2\text{-}5)$$

r-coordinate (line of constant t, θ, ϕ) is timelike:

$$ds^2 = \frac{1}{1-\frac{2M}{r}}dr^2 = -\frac{2M-\epsilon}{\epsilon}d\epsilon^2 < 0 \quad \text{timelike coordinate} \quad (11.2\text{-}6)$$

t-coordinate (line of constant r, θ, ϕ) is spacelike:

$$ds^2 = -\left(1-\frac{2M}{r}\right)dt^2 = \frac{\epsilon}{2M-\epsilon}dt^2 > 0 \quad \text{spacelike coordinate} \quad (11.2\text{-}7)$$

A particle (or a photon) inside the horizon follows timelike world line, going forward in 'time' as seen locally by the particle, which means to decreasing r, and inevitably arrives at $r = 0$.
There is a true curvature singularity ($R_{\alpha\beta\mu\nu} = \infty$) at $r = 0$.

Once a particle crosses the surface $r = 2M$, it cannot be seen by an external observer.

Gravity : attractive→ leading to the collapse
To prevent the collapse,
needs pressure against the collapse
Source of the pressure
thermal pressure
degenerate pressure of fermions

Fate of the stars

after complete consumption of the fuel for the thermal pressure
White Dwarf-ex) fate of the Sun
Neutron Stars
Black holes

1. Gravitational waves

Weak gravity in the Minkowski spacetime

Wave equation (in the Lorentz gauge $\bar{h}^{\alpha\beta}{}_{,\beta} = 0$)

$$\left(-\frac{\partial^2}{\partial t^2} + \nabla^2\right)\bar{h}^{\alpha\beta} = 0 \quad \text{(9.1-1a)}$$

Or

$$\eta^{\mu\nu}\bar{h}^{\alpha\beta}{}_{,\mu\nu} = 0 \quad \text{(9.1-1b)}$$

Solution ansatz (plane wave)

$$\bar{h}^{\alpha\beta} = A^{\alpha\beta}e^{ik_\alpha x^\alpha} \quad \text{(9.1-2)}$$

$\{k_\alpha\}$: 1-form,
$A^{\alpha\beta}$: polarization tensor, (complex) constant components

Note that

$$\bar{h}^{\alpha\beta}{}_{,\mu} = ik_\mu\bar{h}^{\alpha\beta} \quad \text{(9.1-3)}$$

The Lorentz gauge condition $\bar{h}^{\alpha\beta}{}_{,\beta} = 0$ then requires

$$A^{\alpha\beta}k_\beta = 0 \text{ (9.1-4)}$$

the polarization tensor is orthogonal to the wave vector.

Plugging the ansatz (9.1-2) into wave equation (9.1-1b), we get

$$\eta^{\mu\nu}\bar{h}^{\alpha\beta}{}_{,\mu\nu} = -\eta^{\mu\nu}k_\mu k_\nu\bar{h}^{\alpha\beta} = 0 \quad \text{(9.1-5)}$$

The equality holds only if

$$\eta^{\mu\nu}k_\mu k_\nu = k^\nu k_\nu = 0 \quad \text{(9.1-6)}$$

$\{k_\alpha\}$: a null one-form (or $(k^\alpha) = (\omega, \vec{k})$: a null vector)

Plane wave : $\bar{h}^{\alpha\beta}$ is constant on a hypersurface on which $k_\alpha x^\alpha = k_0 t + \vec{k}\cdot\vec{x}$ constant

Ex) A photon moving in the direction of the null vector $\mathbb{k}$

curve : $x^\mu(\lambda) = k^\mu\lambda + l^\mu$ (9.1-7)

Hence, $k_\mu x^\mu(\lambda) = \underbrace{k_\mu k^\mu}_{0}\lambda + k_\mu l^\mu = k_\mu l^\mu$=const.

The transverse-traceless gauge

Lorentz gauge condition still allows any vector solving

$$\left(-\frac{\partial^2}{\partial t^2} + \nabla^2\right)\xi_\alpha = 0 \quad \text{(9.1-8)}$$

Choose a solution

$$\xi_\alpha = B_\alpha e^{ik_\alpha x^\alpha} \quad \text{(9.1-9)}$$

Change in the metric,

$$\bar{h}^{(\text{NEW})}_{\alpha\beta} = \bar{h}^{(\text{OLD})}_{\alpha\beta} - \xi_{\alpha,\beta} - \xi_{\beta,\alpha} + \eta_{\alpha\beta}\xi^\mu{}_{,\mu} \quad \text{(9.1-10)}$$

$$A^{(\text{NEW})}_{\alpha\beta} = A^{(\text{OLD})}_{\alpha\beta} - iB_\alpha k_\beta - iB_\beta k_\alpha + i\eta_{\alpha\beta}B^\mu k_\mu \quad \text{(9.1-11)}$$

B_α can be chosen to impose two further restrictions on $A^{(\text{NEW})}_{\alpha\beta}$

$$A^\alpha{}_\alpha = 0 \text{ (9.1-12)}$$

And

$$A_{\alpha\beta}U^\beta = 0 \quad \text{(9.1-13)}$$

$\mathbb{U}$ some fixed 4-velocity vector (time-like unit vector)

Transverse-traceless (TT) gauge condition

$A^{\alpha\beta}k_\beta = 0$ (9.1-4)
$A^\alpha{}_\alpha = 0$ (9.1-12)
$A_{\alpha\beta}U^\beta = 0$ (9.1-13)

Note :
1) $\bar{h}^{\mathrm{TT}}_{\alpha\beta} = h^{\mathrm{TT}}_{\alpha\beta}$ (9.1-14)
2) can work in the frame through the Lorentz transform to choose
$U^\beta = \delta^\beta{}_0$ (9.1-15)
⇝ (9.1-13) gives
$A_{\alpha 0} = 0 \quad \forall\alpha$ (9.1-16)
We now choose so that the wave is traveling in the z-direction
$(k^\alpha) = (\omega, 0, 0, \omega)$ (9.1-17)
Then (9.1-4) $A^{\alpha\beta}k_\beta = 0$ with (9.1-16) $A_{\alpha 0} = 0$ gives
$A_{\alpha z} = 0 \quad \forall\alpha$
Therefore, only
$A_{xx}, A_{yy}\ A_{xy} = A_{yx}$
are nonzero.
The traceless condition (9.1-12) $A^\alpha{}_\alpha = 0$ implies
$A_{xx} = -A_{yy}$

To summarize,

$$\left(A^{\mathrm{TT}}_{\alpha\beta}\right) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & A_{xx} & A_{xy} & 0 \\ 0 & A_{xy} & -A_{xx} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Only two independent constants,
A^{TT}_{xx} and A^{TT}_{xy}

The effect of waves on free particles
Choose the Lorentz transform so that the particle is initially at rest.
Choose the TT gauge with $\mathbb{U}$ identified with the 4-velocity U^α of the particle.
Free particle motion along the geodesic equation

$$\frac{d}{dt}U^\alpha + \Gamma^\alpha_{\mu\nu}U^\mu U^\nu = 0$$

Initial acceleration

$$\left(\frac{d}{dt}U^\alpha\right)_0 = -\Gamma^\alpha_{00}$$

$$= -\frac{1}{2}\eta^{\alpha\beta}\left(h_{\beta 0,0} + h_{0\beta,0} - h_{00,\beta}\right), \quad \text{Note: } h^{\mathrm{TT}}_{0\beta} = 0$$

$$= 0$$

Can argue that the acceleration is zero afterwards, too.
This statement has no invariant geometrical meaning.

Two nearby particles
one at the origin initially at rest, and no change later
another at $x = \epsilon$, $y = z = 0$ initially at rest, and no change later
the coordinate distance, which is coordinate dependent, is t-indep constant,

Proper distance between them = a coordinate independent number

$$\Delta l \equiv \int |ds^2|^{\frac{1}{2}} = \int \sqrt{\left|g_{\alpha\beta}(x)dx^\alpha dx^\beta\right|}$$

$$= \int_0^\epsilon \sqrt{|g_{xx}|}\,dx \approx g_{xx}(x=0)|^{\frac{1}{2}}\epsilon$$

$$\approx [1 + \tfrac{1}{2}h^{\mathrm{TT}}_{xx}(x=0)]\epsilon$$

Proper distance change with time

Note

1. The change in the distance $\propto \epsilon$
   ⇝ huge scale is preferred in the detection
   ~km (ground-based detectors) ex) LIGO, VIRGO, KAGRA,
   ~Mkm (in space) ex) LISA, Tianchin, Tiankung,
2. The effect $\propto h_{ij}^{TT}$, small $\lesssim 10^{-21}$

Tidal accelerations : Gravitational wave forces
Two freely falling particles with the connecting vector ξ^α
If work in a TT-coordinate, particles remaining at rest makes ξ^α = constant
Work in the local inertial frame,
where
coordinate distances = proper distances

$$\nabla_{\mathbb{U}}\xi^\alpha = \frac{d}{dt}\xi^\alpha + \Gamma^\alpha_{\beta\alpha}\xi^\beta U^\alpha$$

$$\nabla_{\mathbb{U}}\nabla_{\mathbb{U}}\xi^\alpha = (\frac{d}{dt} + \underbrace{\Gamma}_{0})\left(\frac{d}{dt}\xi^\alpha + \Gamma^\alpha_{\beta\mu}\xi^\beta U^\mu\right)$$

$$= \frac{d^2\xi^\alpha}{dt^2}|_A + \Gamma^\alpha{}_{\beta\mu,\nu}(A)\xi^\beta U^\mu U^\nu$$

$$=(\Gamma^\alpha{}_{\beta\mu,\nu}(A) - \Gamma^\alpha{}_{\mu\nu,\beta}(A))\xi^\beta U^\mu U^\nu$$

$$= R^\alpha{}_{\mu\nu\beta}U^\mu U^\nu \xi^\beta$$

$$\frac{d^2}{dt^2}\xi^\alpha = R^\alpha{}_{\mu\nu\beta}U^\mu U^\nu \xi^\beta$$

Take

$\mathbb{U} = (1,0,0,0)$, $\boldsymbol{\xi}$ =$(0,\epsilon,0,0)$

Then, to 1st order in $h_{\mu\nu}$,

$$\frac{d^2}{dt^2}\xi^\alpha = \frac{\partial^2}{\partial {x^0}^2}\xi^\alpha = \epsilon R^\alpha{}_{00x} = -\epsilon R^\alpha{}_{0x0}$$

Riemann tensor is measurable by the proper distance changes between nearby geodesics.

Measuring the stretching of space

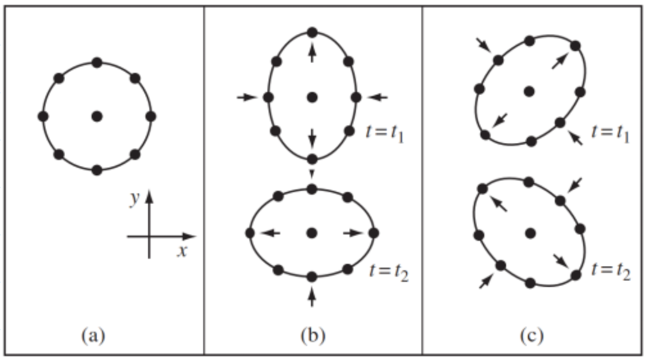
$$\frac{\partial^2}{\partial t^2}\xi^i = -R^i{}_{0j0}\xi^j + \frac{F_B^i}{m_B} - \frac{F_A^i}{m_A}$$

Polarization of gravitational waves
Two particles
Initially separated by ϵ in the x-direction

$$\frac{\partial^2}{\partial t^2}\xi^x = \frac{1}{2}\epsilon\frac{\partial^2}{\partial t^2}h_{xx}^{TT} \qquad \frac{\partial^2}{\partial t^2}\xi^y = \frac{1}{2}\epsilon\frac{\partial^2}{\partial t^2}h_{xy}^{TT}$$

Initially separated by ϵ in the y-direction

$$\frac{\partial^2}{\partial t^2}\xi^x = \frac{1}{2}\epsilon\frac{\partial^2}{\partial t^2}h_{xy}^{TT} \qquad \frac{\partial^2}{\partial t^2}\xi^y = -\frac{1}{2}\epsilon\frac{\partial^2}{\partial t^2}h_{xx}^{TT}$$



(a) a circle of free particles before a wave traveling in the z-direction reaches them

(b) + polarization $h_{xx}^{TT} \neq 0$ $h_{xy}^{TT} = 0$
(c) × polarization $h_{xx}^{TT} = 0$ $h_{xy}^{TT} \neq 0$
Note :
1) Two independent linear polarization
⇜ $h_{\alpha\beta}$: spin 2 gravitonmassless
2) 45° relative one another
Cf) E&M : 1) Two pol ⇜ A_α: spin 1 photon massless , 2) 90°

3. The generation of gravitational waves

$$\left(-\frac{\partial^2}{\partial t^2} + \nabla^2\right)\bar{h}_{\mu\nu} = -16\pi T_{\mu\nu}$$

$T_{\mu\nu} = S_{\mu\nu}(x^i)e^{-i\Omega t}$
Look for a solution
$\bar{h}_{\mu\nu} = B_{\mu\nu}(x^i)e^{-i\Omega t}$

$$(\nabla^2 + \Omega^2)B_{\mu\nu} = -16\pi S_{\mu\nu}$$

$$B_{\mu\nu} = \frac{A_{\mu\nu}}{r}e^{i\Omega r} + \frac{Z_{\mu\nu}}{r}e^{-i\Omega r}$$

Quadrupole moment tensor of the mass distribution
$I^{lm} := \int T^{00} x^l x^m \, d^3x$
$= D^{lm} \, e^{-i\Omega t}$
We have
$\bar{h}_{jk} = -\frac{2\Omega^2 D_{jk}}{r} e^{i\Omega(r-t)}$

$$h_{xx}^{TT} = -\Omega^2(\mathcal{I}_{xx} - \mathcal{I}_{yy})\frac{e^{i\Omega r}}{r}$$
$$h_{xy}^{TT} = -2\Omega^2 \mathcal{I}_{xy}\frac{e^{i\Omega r}}{r}$$
where,
$\mathcal{I}_{jk} := I_{jk} - \frac{1}{3}\delta_{jk}I_l^l$
reduced (trace-free) quadrupole moment tensor